LOOP SPACE HAMILTONIANS AND FIELD THEORY OF NON-CRITICAL STRINGS

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ABSTRACT

We consider the loop space representation of multi-matrix models. Explaining the origin of a time variable through stochastic quantization we make contact with recent proposals of Ishibashi and Kawai. We demonstrate how collective field theory with its loop space interactions generates a field theory of non-critical strings.

1. Introduction

Recent studies of matrix models have given beautiful solutions of non-critical string theory in low dimensions $0 \leq d \leq 1$) [1]. A significant new insight is contained in the appearance of W_k type symmetries which are most clearly exhibited in the Schwinger-Dyson approach [2]. At $d = 1$ one has a Hamiltonian problem with physical time representing $c = 1$ matter and a field theoretical formulation with a W_∞ spectrum-generating algebra provided by collective field theory [3].

Field-theoretic formulations are also of interest for $d < 1$. Following initial considerations of [4], Kostov constructed a non-polynomial field theory based on the interacting loop gas [5]. Recently a Hamiltonian method was also suggested for $d < 1$ [6] with a time variable inferred from studies of $2d$ continuum gravity [7]. For $d = 0$ one recognizes that the Hamiltonian is identical in structure to the one matrix collective field hamiltonian. It contains simple processes of joining and splitting of strings, given by a cubic interaction, and a linear tadpole term.

In what follows we will consider the matrix model approach further, concentrating on chain integrals. We will present an interpretation of the time variable as originating from stochastic quantization [8,9]. Here the time is introduced to give the evolution of probability density and the correlation functions are then obtained in the limit $t \to \infty$. One is then lead to Fokker Planck Hamiltonians whose specific form is

$$H = -(\frac{\partial}{\partial X} - \frac{\partial S}{\partial X})\frac{\partial}{\partial X} \tag{1}$$

Representing these Hamiltonians in loop space through standard collective field formalism [10] leads us to field theories of non-critical strings. The main feature of such loop space Hamiltonians is a cubic interaction and a consistency condition coming from a requirement of hermiticity. Tha latter leads to a system of equations obeyed by the partition function. These are known in matrix models as τ function conditions.

In general, starting with two matrices the loop space becomes very large. For specific critical potentials however we can consistently truncate loop space to a finite number of string fields. The consistency of this trunctaion is assured by closure of the corresponding algebra.

The content of the paper is as follows. In Section 2 we summarize the basics of the Schwinger-Dyson equations and collective field representation. In Section 3 we describe (through stochastic quantization) the origin of time variables in the problem of $d < 1$ matrix models, and introduce the corresponding Hamiltonians. Their loop space representation is then discussed. It is shown in Section 4 how a hermiticity requirement implies the constraint equations on the partition function. In Section 5 we discuss in detail the 1-matrix example. Particular attention is given to taking the continuum limit and establishing contact with [6]. Extensions to Neveu-Schwarz-Ramond and Kac-Moody algebras are also suggested. In Section 6 we consider the problem of loop space Hamiltonians in the two-matrix case. It is shown how a simple truncation leads to field theories with a finite number of fields.

2. Schwinger-Dyson Equations and Collective Field Integral

Consider a partition function given by the integral

$$Z = \int dM_1 \cdots dM_k e^{-S} \tag{2}$$

with hermitian $N \times N$ matrices M_i, $i = 1, \cdots, k$ and an action

$$S = -c \sum_i Tr(M_i M_{i+1}) + \sum_i V_i(M_i)$$

The correlation functions in general obey the Schwinger-Dyson equations which follow from

$$\int [dM] \frac{d}{dM} \left(e^{-S(M)} F(M) \right) = 0 \tag{3}$$

The $U(N)$ invariant observables are given by traces ($\phi_n = \text{Tr}(M^n)$) for a single

matrix and

$$\phi_C = \text{Tr}(M_1^{n_1} M_2^{n_2} \cdots) \qquad (4)$$

in general. The loop space index C in general denotes the sequence of matrices in trace $C = \{n_1, n_2, \cdots\}$. In Yang-Mills theories they are Wilson loops $W(C) = Tr\, e^{\int A}$. Here too we can use other parametrizations of invariant variables, for instance $\Phi(l) = Tr(e^{-Ml})$, and generalizations thereof.

The invariant S-D equations are obtained by contracting $U(N)$ indices and are

$$\sum_\alpha \langle -\frac{\partial S}{\partial M_\alpha} F^\alpha + \frac{\partial F^\alpha}{\partial M_\alpha} \rangle = 0 \qquad (5)$$

Collective field theory represents a reformulation of the theory (in this case the matrix integral) in terms of invariant loop variables ϕ_C. The partition function is written as an integral

$$Z = \int [d\phi_C]\, J(\phi) e^{-S} \qquad (6)$$

Here the new element is a nontrivial Jacobian coming from a change of variables $M_i = t_\alpha M^\alpha(i) \to \phi_C$. The Jacobian was discussed in general in [10], it is specified by a set of differential equations

$$\left[\Omega(C,C') \frac{d}{d\phi_{C'}} + \omega(\phi_C) + \frac{d\Omega(C,C')}{d\phi_{C'}} \right] J = 0 \qquad (7)$$

Here the repeated indices (C') are summed over. The main ingredients in these are the quantities Ω and ω. They are linear and quadratic (in the field ϕ) respectively. One has

$$\Omega(C,C') \equiv \frac{\partial \phi_C}{\partial M_{\hat\alpha}} \frac{\partial \phi_{C'}}{\partial M_{\hat\alpha}} \qquad (8)$$

where $\hat\alpha = (\alpha, i)$ denotes a sum over both $U(N)$ and other matrix indices. Using

an identity

$$\sum_\alpha Tr(At_\alpha) Tr(Bt_\alpha) = Tr(AB) \qquad (9)$$

for a complete set of $U(N)$ generators one finds that

$$\Omega(C,C') = \sum \phi_{C+C'} \qquad (10)$$

representing a joining of loops C and C'. The sum is over all possible distinct joinings. Similarly

$$\omega(C) \equiv -\frac{\partial^2 \phi_C}{\partial M_{\hat\alpha} \partial M_{\hat\alpha}} = \sum \phi_{C'} \phi_{C''} \qquad (11)$$

represents the process of splitting (of a loop C into C', C''). Again, in general there can be several different ways to split a loop. With the Jacobian specified by the loop joining and splitting processes (eqs 10-11). we can show that the collective field theory representation generates the correct matrix model Schwinger-Dyson equations. Consider the following total derivative in the loop space integral representation:

$$\int [d\phi] \frac{d}{d\phi_{C'}} \left(\Omega(C,C') J e^{-s} \prod_i f_i \right) = 0 \qquad (12)$$

Using the identity for the Jacobian this is shown to give

$$\langle \left[\Omega(C,C') \frac{d}{d\phi_{C'}} - \omega(C) - \Omega(C,S) \right] \prod_i f_i \rangle = 0 \qquad (13)$$

with

$$\Omega(C,S) = \frac{\partial \phi_C}{\partial M_{\hat\alpha}} \frac{\partial S}{\partial M_{\hat\alpha}} \qquad (14)$$

we recognize in the above the S-D equation for $F_\alpha = \frac{\partial \phi_C}{\partial M_\alpha} \prod f_i$.

Clearly the collective Jacobian J is determined in such a way that the correct S-D equations are generated. We comment that the above demonstration extends to higher genuses the arguments given in [10] where the planar limit was considered.

To illustrate the above with the single matrix example one has

$$\phi_n = \text{Tr}(M^n)$$
$$\Omega(n, n') = nn'\phi_{n+n'-2}$$
$$\omega(n) = -n \sum_{n'=0}^{n-2} \phi_{n'}\phi_{n-n'-2}$$

(15)

with the resulting S-D equation

$$\left\langle \left[\sum_{m=n}^{\infty} \phi_{n+m} m \frac{d}{d\phi_m} + \sum_{m=0}^{n} \phi_m \phi_{n-m} + \phi_{n+m} m j_m \right] F \right\rangle = 0$$

(16)

for the general action $S = j_n \phi_n$.

3. Hamiltonians

The relevance of Hamiltonians in evaluating euclidean correlation functions comes in general through the stochastic quantization. For a typical field theory $\varphi(x)$ the time dependent Langevin equation is given by

$$\frac{\partial}{\partial t}\varphi(x,t) = -\frac{\partial S}{\partial \varphi} + \eta$$

(17)

where η is the random variable. The correlation functions are then obtained in the limit $t \to \infty$

$$\langle F(\varphi) \rangle = \lim_{t \to \infty} \int [d\varphi(x)] F(\varphi) P_t$$

(18)

where the time evolution (of the probability distribution) is given by

$$\frac{\partial}{\partial t} P_t = -H_{FP} P_t$$

(19)

with the Fokker-Planck Hamiltonian

$$H_{FP} = -\int \left(\frac{d}{d\varphi(x)} - \frac{\delta S}{\delta \varphi(x)} \right) \frac{d}{d\varphi(x)}$$

(20)

This Hamiltonian has the obvious property that it is made Hermitian by a similarity

transformation

$$e^{-1/2S} H_{FP} e^{1/2S} = -\int \left(\frac{d}{d\varphi} - \frac{1}{2}S^{(1)} \right) \left(\frac{d}{d\varphi} + \frac{1}{2}S^{(1)} \right)$$

(21)

This hermitian form guarantees that the Hamiltonian has a unique ground state given by the wavefunction

$$\psi_0 = e^{-1/2S}$$

(22)

The usefulness of stochastic quantization for loop equations in Yang-Mills theoryhas been pointed out by Marchesini [8] (it was considered in detail in the one matrix case in [9] with the purpose of numerical solutions of S-D equations)

For the specific case of matrix models the formulation of the Hamiltonian goes as follows. One has

$$H_{FP} = -\left(\frac{\partial}{\partial M_{\hat{\alpha}}} - \frac{\partial S}{\partial M_{\hat{\alpha}}} \right) \frac{\partial}{\partial M_{\hat{\alpha}}}$$

(23)

as a nonhermitian matrix model hamiltonian. Here $\hat{\alpha} = (\alpha, i)$ represents a summation over $U(N)$ and other matrix indices. The change to loop space variables ϕ_c is then done, in a standard collective field theory fashion. The corresponding loop space Hamiltonian reads

$$H = -\left(\Omega(C, C') \frac{d}{d\phi_{C'}} - \omega(C) - \Omega(C, S) \right) \frac{d}{d\phi_C}$$

(24)

with a summation over the repeated loop indices C, C' and the loop joining and splitting quantities Ω and ω were given earlier. The additional term $\Omega(C, S) \equiv \frac{\partial \phi_c}{\partial M} \frac{\partial S}{\partial M}$ introduces the specific matrix model action whose correlation functions are under consideration. The above Hamiltonian is cubic in the field Φ_c and its

conjugate $\Pi_C = \frac{d}{d\Phi_C}$ with the standard canonical commutation relations

$$[\phi_C, \Pi_{C'}] = -\delta_{C,C'} \qquad (25)$$

It is useful to write the Hamiltonian in the notation

$$H = -\sum_C \hat{O}_C \frac{d}{d\phi_C} \qquad (26)$$

with the differential operator

$$\hat{O}_C = \sum \Omega(C, C')\Pi_{C'} - \tilde{\omega}(C) \qquad (27)$$

where in the $\tilde{\omega}$ we have added the term coming from the action.

In general the closure of the algebra generated by the operators $\hat{O}_c$ will be of central relevance. It represents a central consistency condition in the collective field formalism

4 Hermiticity Condition and the partition-function

The loop space hamiltonian given above is seemingly nonhermitian, this feature coming from the nonhermiticity of the standard collective hamiltonian plus the additional nonhermitian term introduced by the action S. It is a consistency requirement for the present Hamiltonian approach that there should be a similarity transformation making the hamiltonian hermitian and implying the existence of unique ground state. This (unique) ground state wavefunction will be argued to correspond to the τ-function of matrix models. The Virasoro and the higher W constraints then follow from the Hermiticity requirement.

The manifestly hermitian Hamiltonian comes from the transformation [10]

$$\hat{H} = J^{1/2} H J^{-1/2} \qquad (28)$$

with the Jacobian obeying the equation

$$O_c^\dagger J \equiv \left(-\frac{d}{d\phi_{c'}}\Omega(c, c') - \tilde{\omega}(c) \right) J(\phi) = 0 \qquad (29)$$

Using this equation one has a manifestly hermitian form

$$\hat{H} = \sum_{c,c'} \tilde{O}_c \Omega^{-1}(c, c') \tilde{O}_{c'}^\dagger \qquad (30)$$

with $\tilde{O}$ and $\tilde{O}^\dagger$ being Hermitean conjugates and

$$\tilde{O}_c^\dagger = \frac{1}{2}\{\Omega(c, c'), \frac{d}{d\phi_{c'}}\} + \frac{1}{2}\tilde{\omega}(c) \qquad (31)$$

It is these operators that in the continuum limit of a particular model give the conditions obeyed by the τ-function. This is seen as follows: Recall that the partition function is given by

$$Z = \int [d\phi] \, J(\phi) e^{-j_c\phi_c - S} \qquad (32)$$

where we have in the action general couplings $j_c\phi_c$. Since any terms in S only redefines the couplings one essentially has

$$Z(j) = \tilde{J}(j) = \int [d\phi] J(\phi) e^{-j\cdot\phi} \qquad (33)$$

which says that the partition function is Laplace transform of the Jacobian. From the Hermiticity requirement it then follows that the square root of the Jacobian obeys the constraint equations:

$$\tilde{O}_c^\dagger J^{1/2} = 0 \qquad (34)$$

Denoting $J^{1/2} = \tau(j)$ we then have

$$\{\frac{1}{2}\sum\{\frac{d}{d\Phi_{C'}}, \Omega(C, C')\} + \frac{1}{2}\tilde{\omega}(C)\}\tau = 0 \qquad (35)$$

This is the statement that the τ-functions (or the square root of the partition functions) obeys a set of differential equations given by the operators $\tilde{O}_c$. As a

concluding feature of this general discussion let us comment further on the algebra of operators $\tilde{O}_c^\dagger$. It is a consistency condition for the integrability of the above equations that the operators O_c should close in the sense that

$$\left[\tilde{O}_C^\dagger, \tilde{O}_{C'}^\dagger\right] = f_{CC'C''}(\phi)\tilde{O}_{C''}^\dagger \tag{36}$$

Here in general one allows for field dependent structure constants. Also in general the operators O_c given by the collective construction when commuted in principle can lead to further generators and in the above we mean the complete set. One can in turn define generalized collective Hamiltonians based on closed algebras. We shall give simple examples of the latter at the end of next section, where we present some simple extensions to the supersymetric case and the case of $SL(2,\mathbf{R})$ Kac Moody algebra.

5. Simple Examples

The one matrix collective field Hamiltonian has been extensively discussed in the literature. We begin our discussion with this example, however, in order to demonstrate in detail the steps involved in taking its continum limit to $c = 0$ theory. Previously this Hamiltonian was used for studying the $c = 1$ theory in which case a simple M^2 potential is used. The Hamiltonian appropriate for the $c = 0$ integral with the action

$$S = Tr\left(\frac{1}{2}\mu M^2 - \frac{1}{3}M^3 + \cdots\right) \tag{37}$$

reads

$$H = -Tr\left(\frac{\partial}{\partial M} - \frac{\partial S}{\partial M}\right)\frac{\partial}{\partial M} \tag{38}$$

Changing to the (Loop Space) Collective Field representation with

$$\phi_n = Tr\left(M^n\right) \ , \ \Pi_n = \frac{\partial}{\partial\phi_n} \ ; \ n \geq 0 \tag{39}$$

The Hamiltonian becomes

$$H = -\sum_{n\geq 0}\left(\sum_{m\geq 0}\phi_{n+m-2}m\frac{\partial}{\partial\phi_m} + \sum_{r=0}^{n-2}\phi_r\phi_{n-r} - \Omega\left(S,\phi_n\right)\right)n\frac{\partial}{\partial\phi_n} \tag{40}$$

with

$$\Omega(S,\phi_n) = n\left(\mu\phi_{n+1} - \phi_{n+2} + \cdots\right) \tag{41}$$

The source (S independent) term is a cubic interaction. It is written as

$$H_3 = -\sum_n O_n n\Pi_n \tag{42}$$

with

$$O_{n+2} = \sum_{n=0}^\infty \phi_{n+m}m\Pi_m + \sum_r \phi_r\phi_{m-r} \tag{43}$$

and we have a Virasoro algebra $O_{n+2} = L_n$:

$$[L_n, L_m] = (n-m)L_{n+m} \tag{44}$$

The Hamiltonian couples the Virasoro generators to the conjugate field Π_n. For evaluating the continuum limit it is useful to switch to z-representation with

$$\phi(z) = Tr\frac{1}{z-M} = \int_0^\infty dL e^{-Lz}\phi_L = \int_0^\infty dL e^{-Lz}Tr(e^{LM}) \tag{45}$$

so that

$$\phi(z) = \sum_{n\geq 0} z^{-n-1}\phi_n$$

$$\partial\Pi(z) = \sum_{n\geq 0} z^{n-1}n\Pi_n \tag{46}$$

with

$$O(z) = \sum_n z^{-n-2} O_n \qquad (47)$$

gives

$$O(z) = \int dz' \frac{\phi(z') - \phi(z)}{z - z'} \partial_{z'} \Pi + \phi^2(z) \equiv (\phi \partial_z \Pi)(z) + \phi^2(z) \qquad (48)$$

The bracket notation is to state that there are only z^{-n} components. The Hamiltonian is now

$$H = -\int dz \left[\phi(z) \partial_z \Pi(z) + \phi^2(z) + \left(z^2 - \mu z \right) \phi + (\mu - z) + c_0 \right] \partial_z \Pi(z). \qquad (49)$$

The scaling limit is defined with

$$\begin{aligned} \mu &= \mu_c + a^2 \Lambda \\ z &= z_a + a\zeta \end{aligned} \qquad (50)$$

and a shift

$$\phi(z) = \frac{1}{2}(z\mu - z^2) + a^{3/2} \Phi(\zeta) \qquad (51)$$

After the shift the linear ϕ-term in the operator $O(z)$ gets canceled and the complete potential term reads

$$a^3 \left(\Phi^2(\zeta) - \frac{1}{a^3} \left[\frac{1}{4} \left(\mu z - z^2 \right)^2 + (z - \mu) + c_0 \right] \right) = a^3 \left(\Phi^2(\zeta) - \Phi_0^2(z) \right) \qquad (52)$$

where we have recognized the planar value of Φ^2. For the conjugate field we have

$$\Pi(z) = a^{-5/2} \Pi(\zeta) \qquad (53)$$

To summarize in terms of the basic length a, the dimensions are $[\zeta] = a^{-1}$, $[\Phi(\zeta)] = a^{3/2}$ and $[\Pi(\zeta)] = a^{-5/2}$. If one transforms

$$\Phi(\zeta) = \int_0^\infty dl \, e^{-l\zeta} \Phi(l) \qquad (54)$$

to the loop space $\Phi(l), \Pi(l)$ fields, the dimensions are:

$$[l] = a \quad [\Phi(l)] = a^{-5/2} \quad [\Pi(l)] = a^{3/2} \qquad (55)$$

Let us now turn to the kinetic term. First we exhibit N by scaling $z \to \sqrt{N}z$, $\phi \to \sqrt{N}$. This leads to a factor N^{-2} in front of the kinetic term $\int \phi \Pi_z^2$. Inserting the continuum fields we find that the shift does not contribute and

$$H = -a^{1/2} \int d\zeta \left[\frac{1}{N^2 a^5} \Phi \partial_\zeta \Pi + \Phi(\zeta)^2 - \Phi_0^2 \right] \partial \Pi(\zeta) \qquad (56)$$

In the first term we have the correct scaling dimension for the coupling constant $[g] = a^{-5}$. We also find an overall $a^{1/2}$ which is removed by a redefinition of the (stochastic) time. Consequently the time has dimension $a^{-1/2}$. The final continuum Hamiltonian written for the loop fields reads

$$\begin{aligned} H = &-\int_0^\infty dl_1 \int_0^\infty dl_2 \left\{ g\Phi \left(l_1 + l_2 \right) l_1 \Pi(l_1) l_2 \Pi(l_2) + (l_1 + l_2) \phi(l_1) \Phi(l_2) \Pi(l_1 + l_2) \right\} \\ &- \int_0^\infty dl \, \rho_0(l) \Pi(l) \end{aligned} \qquad (57)$$

This indeed is identical to the form guessed in [6]. As we have demonstrated, the terms with the correct scaling survive in the continuum. It is a fact that the cubic interaction remained unchanged. This will be a general feature in all theories.

Let us now turn to some simple generalizations. Since in the above we had a Virasoro generator $O(z) = L(z) = \phi \partial \pi + \phi^2$ coupled to the conjugate $\pi(z)$ with consistency assured by closure of the Virasoro algebra we can contemplate simple extensions by using other algebras.

(i) Supersymmetric $N - S(R)$ algebra. Consider

$$L_n = \frac{1}{2}\Sigma' \alpha_{n+m}\alpha_{-m} + \frac{1}{2}\sum_r \left(r + \frac{1}{2}n\right)b_{n+m}b_{-n}$$

(58)

$$G_r = \sum_r \alpha_{-n}b_{r+n}$$

denote the conjugate pairs as

$$\alpha_n = \phi_n, \alpha_{-n} = n\Pi_n \; n > 0$$

$$b_r, b_{-r} = P_r r > 0$$

(59)

then a natural form is

$$H = -\left(\sum_{n>0} L_{n-2}m\Pi_n - \frac{1}{2}\sum_r G_{r-1}rP_r\right)$$

(60)

(ii) $SL(2, R)$ Kac-Moody algebra. One has the representation for the currents:

$$J^+(z) = -\gamma^2\beta + i\alpha_+\gamma\varphi + \kappa\partial\gamma$$

$$J^0(z) = \beta\gamma - \frac{i}{2}\alpha_+\partial\varphi$$

$$J(z) = \beta$$

(61)

Denoting

$$\partial\varphi(z) + \phi(z) + \partial_z\Pi(z)$$

$$\beta(z) = \sum_n z^{-n-1}\beta_n \; ; \; \beta_n = \frac{\partial}{\partial\gamma_{-n}}$$

(62)

a collective type Hamiltonian reads:

$$H = \int dz \left\{J^+(z)\beta(z) - J^0(z)\partial_z\Pi(z)\right\}$$

(63)

6. Two-matrix Hamiltonian

It is known [11] that the whole sequence of non-critical string models with increasing dimensions $d_k = 1 - \frac{6}{(k+1)(k+2)}, k = 1 \cdots n$,can be generated from critical points of the two matrix model. Denoting the two matrices as $M_1 = X$ and $M_2 = Y$, one has the action:

$$S = Tr(-cXY + v_1(X) + v_2(Y))$$

(64)

The first non-trivial $d = 1/2$ non critical string theory corresponds to a cubic potential:

$$v(X) = X^2/2 - \frac{\lambda}{3}X^3$$

(65)

while the higher models are generated by potentials of higher power in X and Y. We shall concentrate our discussion on the first non trivial cubic case (i.e. $d = 1/2$). Generalisation to arbitrary $d_k \leq 1$ or $k = 3, 4 \cdots$ will be straightforward.

The Hamiltonian operator for two matrices reads

$$H = -Tr((\frac{\partial}{\partial X} - \frac{\partial S}{\partial X})\frac{\partial}{\partial X} + (\frac{\partial}{\partial Y} - \frac{\partial S}{\partial Y})\frac{\partial}{\partial Y})$$

(66)

As we have explained in the general discussion the information on the specific potential comes in through the source term

$$\hat{S} = -Tr(\frac{\partial S}{\partial X}\frac{\partial}{\partial X} + \frac{\partial S}{\partial Y}\frac{\partial}{\partial Y})$$

(67)

which is added to the two-matrix Laplacian operator $\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial X^2}$. In the transition to the collective field theory it is the Laplacian which produces the cubic loop space interaction while $\hat{S}$ in general only adds a loop space source term. The loop space

variables are now given by the traces

$$\Phi_C = Tr(X^{n_1} Y^{n_2} X^{n'_1} Y^{n'_2} \cdots) \tag{68}$$

with $C = \{n_1, n_2, n'_1, n'_2 \cdots\}$ denoting the set of looops around two fixed points. This represents an extremely large space. However' it is known from study of Schwinger-Dyson equations of the two matrix problem [12] that for specific (polynomial) potentials the space reduces and the S-D equations close within a subset of loops. For the first non-trivial case of a cubic potential we have two sets of loops

$$\Phi_n^0 = Tr X^n, n = 0, 1... \tag{69}$$

$$\Phi_n^1 = Tr X^n Y, n = 0, 1... \tag{70}$$

We shall now show that on this subset one can define a consistent Hamiltonian. In general as the order of the potential $v_2(Y)$ increases to Y^{k+1} one will define a reduced theory with k independent fields.

We now evaluate the joining and splitting interactions specified by $\Omega(C, C')$ and $\omega(C)$. First:

$$\Omega(\Phi_n^0, \Phi_m^0) = Tr(\frac{\partial \Phi_n^0}{\partial X} \frac{\partial \Phi_m^0}{\partial X}) = nm\Phi_{n+m}^0 \tag{71}$$

$$\Omega(\Phi_n^0, \Phi_m^1) = Tr(\frac{\partial \Phi_n^0}{\partial X} \frac{\partial \Phi_m^1}{\partial X}) = nm\Phi_{n+m}^1 \tag{72}$$

Here loops of type 0 join amongst themselves while a loop of type 1 joins with a loop of type 0 into a loop of type 1. Similarly one can evaluate $\Omega(\Phi_n^1, \Phi_m^1)$. Here one encounters new loops of the type Y^2. We will be truncating the system and will specify $\Omega(\Phi_n^1, \Phi_m^1)$ in such a way as to achieve consistency (in the reduced S-D equations this quantity does not participate).

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For the splitting interaction one simply has

$$\omega(\Phi_n^0) = n \sum_{r=0}^{n-2} \Phi_r^0 \Phi_{n-r-2}^0 \tag{73}$$

$$\omega(\Phi_n^1) = n \sum_{r=0}^{n-2} \Phi_r^0 \Phi_{n-r-2}^1 \tag{74}$$

Here loops of type 0 split as in the one-matrix case while a loop of type 1 splits into 0 and 1. Next we evaluate the term induced by the potential

$$\hat{S} = -\Omega(S, \Phi_n^0) \frac{\partial}{\partial \Phi_n^0} - \Omega(S, \Phi_n^1) \frac{\partial}{\partial \Phi_n^1} \tag{75}$$

with

$$-\Omega(S, \Phi_n^0) = n(c\Phi_{n-1}^1 - \Phi_n^0 + \lambda\Phi_{n+1}^0) \tag{76}$$

$$-\Omega(S, \Phi_n^1) = n(\frac{c}{\lambda}(\Phi_{n-1}^1 - c\Phi_n^0) - \Phi_n^1 + \lambda\Phi_{n+1}^1) \tag{77}$$

In this last expression the first term comes from eliminating a higher loop variable Φ_m^2. The substitution is justified by the S-D equation :

$$\frac{\partial S}{\partial Y} \frac{\partial \Phi_m^1}{\partial Y} - \frac{\partial}{\partial Y} \frac{\partial \Phi_m^1}{\partial Y} = 0 \tag{78}$$

which gives

$$\Phi_m^2 = 1/\lambda(\Phi_{m-1}^1 - c\Phi_m^0) \tag{79}$$

We now have all the ingredients to define the reduced Hamiltonian. The fields are now coupled to the two operators :

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$$O_n^0 = \Phi_{n+m-2}^0 m \frac{d}{d\Phi_m^0} + \Phi_m^0 \Phi_{n+m-2}^0 + \Phi_{n+m-2}^1 m \frac{d}{d\Phi_m^1} + \Omega(S, \Phi_n^0) \qquad (80)$$

$$O_n^1 = \Phi_{n+m-2}^1 m \frac{d}{d\Phi_m^1} + \Phi_m^1 \Phi_{n+m-2}^0 + \Phi_{n+m-2}^1 m \frac{d}{d\Phi_m^0} + \Omega(S, \Phi_n^1) \qquad (81)$$

As predicted by the general theory one can see that:

$$< O_n^{(0),+} \Pi_i \Phi_{n_i}^0) >= 0 \qquad (82)$$

$$< O_n^{(1),+} \Pi_i \Phi_{n_i}^0) >= 0 \qquad (83)$$

give the correct S-D equations. Indeed these equations read :

$$< (c\Phi_{n-1}^1 - \Phi_n^0 + \lambda\Phi_{n+1}^0 + \sum \Phi_r^0 \Phi_{n+r-2}^0)\Pi_i \Phi_{n_i}^0 > + \sum_i n_i < \Phi_{n+n_i-2}^0 \Pi_{j \neq i} \Phi_{n_j}^0 >= 0 \qquad (84)$$

$$< (c/\lambda(\Phi_{n-1}^1 - c\Phi_n^0) - \Phi_n^1 + \lambda\Phi_{n+1}^1 + \sum \Phi_r^1 \Phi_{n+r-2}^0 \Pi_i \Phi_{n_i}^0 >$$
$$+ \sum_i n_i < \Phi_{n+n_i-2}^1 \Pi_{j \neq i} \Phi_{n_j}^0 >= 0 \qquad (85)$$

which can be compared with [12].

Continuing with the Hamiltonian, it is given as :

$$H = - \int dz(O^0(z)\Pi_0(z) + O^1(z)\Pi_1(z)) \qquad (86)$$

where we have the fields

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$$\Phi^i(z) = \sum_{n>0} z^{-n-1}\Phi_n^i, i = 0, 1 \qquad (87)$$

$$\Pi^i(z) = \sum_{n>0} z^{n-1} n \frac{\partial}{\partial\Phi_n^i}, i = 0, 1 \qquad (88)$$

Explicitely:

$$H = - \int [\Phi^0(z)\Pi^0(z) + (\Phi^0(z))^2 + \Phi^1(z)\Pi^1(z)]\Pi^0(z)$$
$$- \int [\Phi^0(z)\Pi^1(z) + \Phi^1(z)\Pi^0(z) + \Phi^1(z)\Phi^0(z)]\Pi^1(z) + \hat{S}(z) \qquad (89)$$

Let us now elaborate on the consistency of the truncation followed to reach this Hamiltonian. As we have explained, a consistent collective field theory requires a closure of the operator algebra $\{O_i\}$. Here we can easily see that this holds. Shifting $\Pi_1 \rightarrow \Pi_1 + \Phi_1$ and defining conformal fields :

$$\partial\phi_0 = \sqrt{2}\Phi_0 + 1/\sqrt{2}\Pi_0 \qquad (90)$$

$$\partial\phi_1 = \sqrt{2}\Phi_1 + 1/\sqrt{2}\Pi_1 \qquad (91)$$

we have

$$O_0(z) = 1/2((\partial\phi_0)^2 + (\partial\phi_1)^2)_- \qquad (92)$$

$$O_1(z) = (\partial\phi_0\partial\phi_1)_- \qquad (93)$$

These operators close the algebra

$$[O_n^0, O_m^0] = (n - m)O_{n+m}^0 \qquad (94)$$

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$$[O_n^0, O_m^1] = (n - m)O_{n+m}^1 \tag{95}$$

$$[O_n^1, O_m^1] = (n - m)O_{n+m}^0 \tag{96}$$

Clearly this can be decoupled by setting $O^\pm \equiv O^0 \pm O^1$ into two decoupled Virasoro algebras.

Introducing the linear combinations

$$\Phi_\pm = \Phi^0 \pm \Phi^1 \tag{97}$$

$$\Pi_\pm = \frac{1}{2}(\Pi^0 \pm \Pi^1) \tag{98}$$

the Hamiltonian is written in a more symmetric form as:

$$H = -\int [2\Phi_+\Pi_+ + \frac{1}{2}\Phi_+\Phi_+ + \frac{1}{2}\Phi_+\Phi_-]\Pi_+ \\ + \int [2\Phi_-\Pi_- + \frac{1}{2}\Phi_-\Phi_- + \frac{1}{2}\Phi_+\Phi_-]\Pi_- + \hat{S}(z)\} \tag{99}$$

In the continuum limit one expects that the non-scaling term disappears The cubic interaction terms certainly survive; the fields now have the dimension $[\Phi(\zeta)] = a^{-4/3}, [\Pi] = a^{10/3}$. As always the coupling constant g can be introduced into the Hamiltonian by a constant rescaling of the fields.

If one expands, taking the following:

$$\Phi_+(z) = \sum_{\alpha>0} z^{-\alpha-1}\Phi_\alpha^+ \tag{100}$$

$$\Phi_-(z) = \sum_{\beta>0} z^{\beta-1}\Phi_\beta^- \tag{101}$$

the resulting Hamiltonian $H = H_+ + H_-$ gets the mode expansion:

$$H_+ = -\sum(2g\Phi_{\alpha+\alpha'-2}^+\Pi_{\alpha'}^+ + \frac{1}{2}\Phi_{\alpha-2}^+\Phi_{-\alpha'}^+ + \frac{1}{2}\Phi_{\alpha+\alpha'-2}^+\Phi_{\alpha'}^-)\Pi_\alpha^+ \tag{102}$$

where $[\Pi_\alpha^\pm, \Phi_\alpha^\pm] = \alpha\delta_{\alpha,\alpha'}$.

Let us end with several comments. It is simple to extend the above derivations to higher critical models. For a potential of power Y^{k+1} one truncates the system to k loop space fields $\{\Phi_0, \Phi_1, \cdots\}$. Their splitting and rearrangement processes are given by Ω and ω. Concerning these series of Hamiltonians it may still be relevant to study the transition from the discrete to the critical continuum theory. For example, already in the above construction the expected W_3 structure or more specifically the W_3 generators are not manifestly visible. Since they are certainly present in the S-D equations this deserves further study.

Conclusions

In the present work we have considered matrix models in the loop space representation and used them to construct field theories of non-critical strings, interpreting the origin of time in terms of stochastic quantization. Hamiltonians of Fokker-Planck type naturally arise. They are represented in loop space through collective field formalism. The Hamiltonians in general take the form: $H = -\sum_i O_i \frac{d}{d\phi_i}$ exhibiting a coupling between generators of an algebra $\{O_i\}$ and the conjugate fields. Hermiticity of such Hamiltonians in general requires closure of the operator algebra and it follows that the partition function obeys the corresponding constraints. The approach is exhibited giving simple string field theories for $d < 1$. The procedure of a continuum limit is adressed. It is certainly not uninteresting to study this correspondence between matrix models and string field theory. It might teach us about similar correspondences in higher dimensions.

Note During the writing of this paper we have received the work referenced as [13]. There appears to be a considerable agreement between the two constructions.

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