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Exact C=1 Boundary Conformal Field Theories

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Abstract

We present a solution of the problem of a free massless scalar field on the half line interacting through a periodic potential on the boundary. For a critical value of the period, this system is a conformal field theory with a non-trivial and explicitly calculable S-matrix for scattering from the boundary. Unlike most other exactly solvable conformal field theories, it is non-rational (*i.e.* has infinitely many primary fields). It describes the critical behavior of a number of condensed matter systems, including dissipative quantum mechanics and of barriers in "quantum wires".

11/93

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1. Boundary Conformal Field Theory

Conformal field theory is usually defined on a two-dimensional manifold without boundaries (the simplest case being the plane). It can also be defined on manifolds with boundaries (like the disk or strip), provided that appropriate boundary conditions are imposed [1]. The Dirichlet and Neumann boundary conditions on scalar worldsheet fields are familiar, if trivial, examples. Non-trivial conformal boundary conditions arise from the interaction of boundary degrees of freedom with worldsheet fields. A wide range of systems, including open string theory [2,3,4], monopole catalysis [5], the Kondo problem [6], dissipative quantum mechanics [7,8] and junctions in quantum wires [9] can be described this way.

The technology for dealing with boundary conformal field theory is easily stated [10]: Consider a bulk conformal field theory C confined to a strip of width L with boundary conditions A and B on the two ends. This theory has a partition function $Z^{AB}_{open} = tr(e^{-TL_0^{AB}})$ where T is the time interval and L_0^{AB} is the open string Hamiltonian. If the boundary conditions are conformal, the partition function will be a sum $Z^{AB}_{open} = \sum n_h \chi_h(e^{-2\pi T/l})$ over Virasoro characters of the open string primary fields (the h are the highest weights and the n_h are the integer multiplicities of the characters). The partition function can also be computed as the amplitude for *closed* string propagation between states $|A\rangle$ and $|B\rangle$ of the bulk *closed* string created by the boundary conditions A and B: $Z^{AB}_{closed} = \langle A|e^{-l(L_0+\widetilde{L}_0)}|B\rangle$, where L_0 and $\widetilde{L}_0$ are the left- and right-moving closed string Hamiltonians. For the theory as a whole to be conformal, the boundary states must satisfy a reparametrization invariance condition $(L_n - \widetilde{L}_{-n})|A\rangle = 0$ [2] which implies that each primary field contributes to $|A\rangle$ a piece $C_h^A \sum_n |n\rangle|\widetilde{n}\rangle$, where the sum is over all the states of the Virasoro module and C_h^A is a coefficient to be determined [11]. This gives a different expansion of the partition function in terms of Virasoro characters: $Z^{AB}_{closed} = \sum_h C_h^A C_h^B \chi_h(e^{-2\pi l/T})$. The dynamical problem is to find the specific primary fields ϕ_h appearing in both the open and closed string expansions along with their multiplicities and weights.

The two expansions must of course be identical under the "modular transformation" $e^{-2\pi T/l} \rightarrow e^{-2\pi l/T}$ between open and closed string variables. This consistency condition is often enough to explicitly determine the (finitely many) boundary states of a rational conformal field theory such as the WZW theory (which underlies the Kondo model [12]). Non-rational conformal field theories are much harder to deal with since they have an infinite number of primary fields and, presumably, boundary states. In this Letter we will present an approach to the solution of what must be the simplest such theory: a single free scalar field interacting only via a boundary periodic potential. Our solution is part conjecture, but its combination of richness and simplicity convinces us that it is exact. As mentioned above, there are several interesting condensed matter systems to which our results apply.

2. The Free Scalar With Periodic Boundary Potential

We will study free massless scalar field theory on the interval $0 < \sigma < l$. A dynamical boundary condition at $\sigma = 0$ is imposed by including a potential term in the otherwise free Lagrangian:

$$L = \frac{1}{8\pi}\int_0^l d\sigma(\partial_\mu X)^2 - \frac{1}{2\epsilon}(g e^{iX(0)/\sqrt{2}} + \bar{g} e^{-iX(0)/\sqrt{2}}) \tag{2.1}$$

where ϵ is the short-distance cutoff and g is a complex renormalized potential strength. To control infrared problems we impose a Dirchlet boundary condition, $X(l)$=0, at $\sigma = l$, but we eventually want to focus on the physics at the $\sigma = 0$ boundary. The potential induces a perturbation away from the (manifestly conformal) free scalar field subject to the Neumann (Dirichlet) boundary condition on the left (right) end of the interval. The specific potential of Eqn. (2.1) was chosen because it has boundary scaling dimension one and induces a marginal perturbation away from the conformal fixed point [13]. We will show that it is in fact *exactly* marginal and induces a conformal boundary condition for *all* values of g.

For the subsequent analysis it will be helpful to recall some facts about the primary fields of conventional $c = 1$ CFT: Modulo some subtleties which do not affect our application, there is a continuum of holomorphic primary fields $e^{ikX(z)}$ of weights $h = k^2/2$ (and corresponding antiholomorphic fields). The associated Virasoro characters are $\chi_k(q) = q^{k^2/2}/f(q)$, where $f(q) = \prod_{n=1}^{\infty}(1-q^n)$. For special values of the "momentum" k, some descendant states have vanishing norm and new primaries, the famous discrete states appear [14]. They are organized in $SU(2)$ multiplets of spin $J = 0, \frac{1}{2}, 1, \ldots$. The (J,m) primary, $\psi_{(J,m)}$, has weight $h = J^2$ and a Virasoro character which turns out to be $\chi_{(J,m)}(q) = (q^{J^2} - q^{(J+1)^2})/f(q)$. There is an explicit representation for $\psi_{(J,m)}$ [15],

$$\psi_{(J,m)}(0) \sim \left[\oint \frac{dz}{2\pi i} e^{-i\sqrt{2}X(z)}\right]^{J-m} e^{iJ\sqrt{2}X(0)}, \tag{2.2}$$

which shows that it is a polynomial in ∂X, $\partial^2 X$, etc. times a zero mode piece $e^{im\sqrt{2}X}$. The lowest-lying fields, $\psi_{1/2,\pm 1/2} \sim e^{\pm iX/\sqrt{2}}$, are precisely the terms appearing in the boundary potential in (2.1). Since products of $\psi_{(J,m)}$ fuse to other $\psi_{(J,m)}$ by an $SU(2)$ fusion algebra, the operators appearing in a perturbation expansion in the boundary potential should be spanned only by the discrete states. This strongly suggests that the exact boundary states are sums over the Virasoro modules of the discrete state primary fields $\psi_{(J,m)}$.

Let us first check that the conjecture is true when the potential vanishes (free field with one Neumann and one Dirichlet boundary condition). The partition function is easily found to be

$$Z_0 = w^{1/48} \prod_{n=1}^{\infty} \frac{1}{1 - w^{n-\frac{1}{2}}} = \frac{w^{-1/24}}{f(w)} \sum_{j=0}^{\infty} w^{(j+1/2)^2/4} ,$$

where $w = e^{-2\pi T/l}$. This shows that the energy levels of this open string organize themselves into a set of Virasoro modules with highest weights $h_j = \pi(j + \frac{1}{2})^2/(2l)$. Using standard technology, we can re-express Z_0 in terms of the closed string variable $q^2 = e^{-2\pi l/T} = e^{-2\pi\tau}$, with the result

$$Z_0 = \frac{(q^2)^{-1/24}}{f(q^2)} \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} = \frac{(q^2)^{-1/24}}{f(q^2)} \theta_4(0|2i\tau) , \tag{2.3}$$

where the theta functions are defined using the conventions of [16]. Because of the discrete state subtlety, it is not so obvious how to read off the Virasoro modules which propagate in the closed string channel. However, a little experimentation shows that the contributions of the discrete modules to the boundary states are

$$\begin{aligned} |B_N\rangle &= \sum_{J=0}^{\infty} (-1)^J |J, 0 \gg , \\ |B_D\rangle &= \sum_{J=0}^{\infty} \sum_{m=-J}^{J} |J, m \gg , \end{aligned} \tag{2.4}$$

where $|J, m \gg$ is the module associated with $\psi_{(J,m)}$ ($|B_D\rangle$ also receives contributions from the continuum states e^{ikX}, although they have no effect on our considerations). Eqs. (2.4) reproduce (2.3) and all partition functions arising from other combinations of Neumann and Dirichlet boundary conditions. To see this one just expands the basic boundary state formula $Z_0 = \langle B_N | e^{-l(L_0 + \widetilde{L}_0)} | B_D \rangle$ with the help of the discrete state character formula

$$\ll J, m | e^{-l(L_0 + \widetilde{L}_0)} | J', m' \gg = \delta_{JJ'} \delta_{mm'} \frac{(q^2)^{-1/24}}{f(q^2)} \left[q^{2J^2} - q^{2(J+1)^2} \right] .$$

3. Some Perturbative Results

We now want to turn on the potential and expand the partition functions in powers of g and $\bar{g}$. Since only one boundary is dynamical, the only new element we need is the massless scalar propagator between two points on the same boundary of a cylinder of length $\tau = l/T$ and circumference 1. This standard object is expressed in terms of theta functions as

$$\langle X(t_1)X(t_2)\rangle_{\sigma=0} = -2\log\frac{\theta_1^2(t_1-t_2|2i\tau)}{\theta_4^2(t_1-t_2|2i\tau)} \tag{3.1}$$

Expanding the partition function to first order gives

$$\begin{aligned} Z_1 =& - Z_0\frac{(g+\bar{g})}{2\epsilon}\int_0^1 \langle e^{iX(t)/\sqrt{2}}\rangle \\ =& - Z_0\,\frac{(g+\bar{g})}{2\epsilon}e^{-\langle X(0)X(\epsilon)\rangle/4} = -Z_0\,\frac{(g+\bar{g})}{2}\,\frac{\theta_1'(0|2i\tau)}{\theta_4(0|2i\tau)}\,. \end{aligned} \tag{3.2}$$

Note that we have eliminated the divergence of this amplitude by regulating the integrand (by point-splitting of the coincident-point propagators) and then by renormalizing (by taking the potential strength proportional to $1/\epsilon$). The net first-order result, expressed in closed string channel variables, is

$$Z_1 = \pi(g+\bar{g})\frac{(q^2)^{-1/24}}{f(q^2)}\sum_{n=0}^{\infty}(-1)^{n+1}(2n+1)q^{(2n+1)^2/2}. \tag{3.3}$$

The powers of q^2 which appear in the sum correspond to the weights of the discrete states $\psi_{(J,\pm1/2)}$ for all possible half-integer J and we can find a corrected $|B_N\rangle$, containing such states, which reproduces (3.3). The dual transformation to the open string channel gives

$$Z_1 = -\frac{\pi T}{4l}(g+\bar{g})\frac{w^{-1/24}}{f(w)}\sum_{j=0}^{\infty}(-1)^j(2j+1)w^{(j+1/2)^2/4}\ ,$$

This can be interpreted as a shift of the highest weights of the Virasoro modules appearing in Z_0, with the j-th module being shifted by $(-1)^j(2j+1)\pi(g+\bar{g})/(4l)$. The main point here is that the perturbation causes all the energy levels of any given Virasoro module to have a common energy shift, which they must if the perturbation in (2.1) is truly conformal.

4. Conjectured Exact Solution

Now consider the expansion of $Z(g,\bar{g})$ to higher orders. A new power law divergence, contributing to a shift of the open string vacuum energy and arising from the collision of an $e^{iX/\sqrt{2}}$ with an $e^{-iX/\sqrt{2}}$ insertion first appears in second order. It turns out to be possible to subtract the divergence in this, and all higher, orders by a simple principal value prescription. The second-order terms (*i.e.* the g^2, $\bar{g}^2$ and $g\bar{g}$ terms) organize nicely into

$$Z_2 = \pi^2(g+\bar{g})^2 \frac{(q^2)^{-1/24}}{f(q^2)} \sum_{n=1}^{\infty} (-1)^{n+1} q^{2n^2} n^2.$$

To this order, the partition function $Z_0 + Z_1 + Z_2$, when reexpressed in open string channel variables, can once again be interpreted as a sum over open string Virasoro modules with shifted highest weights. This is a new piece of evidence that the theory specified by (2.1) is exactly conformal.

We have carried out the expansion of Z to fourth order and continue to find results consistent with exact conformal invariance. We have even found a general expression for the partition functions which we believe to summarize the behavior of the theory to all orders: Everything we know is consistent with the net effect of the boundary interaction being a shift of the highest weights of the open string Virasoro modules by a universal, coupling constant dependent, shift function. We claim that the exact weight of the j-th open string module has the form

$$h_j = \frac{\pi \left(j + \frac{1}{2} + (-1)^j \Delta(g,\bar{g})/\pi\right)^2}{2l}$$

so that the open string channel partition function has the form

$$Z(g,\bar{g}) = \frac{w^{-1/24}}{f(w)} \sum_{j=0}^{\infty} w^{[j+1/2+(-1)^j \Delta(g,\bar{g})/\pi]^2/4}. \tag{4.1}$$

Calculations out to fourth order are all consitent with (4.1) with

$$\Delta(g,\bar{g}) = \frac{\pi}{2}(g+\bar{g}) + \frac{\pi^3}{48}(g^3 + \bar{g}^3 - 3g^2\bar{g} - 3g\bar{g}^2) + \ldots . \tag{4.2}$$

When (4.1) is transformed to closed string channel variables, we obtain

$$Z(g,\bar{g}) = \frac{(q^2)^{-1/24}}{f(q^2)} \left(1 + 2\sum_{n=1}^{\infty} q^{n^2/2} \cos\left[\frac{n\pi}{2} + n\Delta(g,\bar{g})\right]\right) . \tag{4.3}$$

The remarkable thing about this expression is that, for any value of Δ, it involves *only* the weights of the discrete states. Actually, some further conditions have to be met in order for it to be possible to construct (4.3) by sandwiching the closed string propagator between discrete state boundary states. It is easy to see that on expanding (4.3) in powers of g and $\bar{g}$, the term of order $g^k\bar{g}^l$ must come from (J,m) modules with $m = \frac{k-l}{2}$. Since such modules have $J \geq |\frac{k-l}{2}|$, and since the discrete state weights are J^2, the q^2 expansion of the $g^k\bar{g}^l$ term must begin at $q^{(k-l)^2/2}$. If these conditions are met one can find an explicit expansion of the dynamical boundary state of the form

$$|B\ (g,\bar{g})\rangle = \sum_{J=0}^{\infty} \sum_{m=-J}^{J} C_{Jm}(g,\bar{g})|J,m\gg \tag{4.4}$$

where the C_{Jm} are expressed in terms of operations carried out on Δ. Low order perturbation theory calculations imply that theory,

$$C_{Jm} = (-1)^J\delta_{m,0} + \frac{\pi}{2}(-1)^{(2J+1)/2}(2J+1)\left(g\delta_{m,1/2} + \bar{g}\delta_{m,-1/2}\right) + \dots\ .$$

(Some notational license has been taken in that the first(second) term is present only for integer (half-integer) J.)

Although we don't have complete knowledge of Δ, some interesting things can be said about it. The consistency conditions fix all g^k or $\bar{g}^k$ terms in the expansion of $\Delta(g,\bar{g})$ in terms of the leading $O(g)$ term. This turns out to imply that $\Delta(g,0) = \arcsin(\pi g/2)$. Other information on $\Delta(g,\bar{g})$ comes from considering the strong potential limit, *e.g.* $g = \bar{g} \to \infty$. It is physically clear that, in this limit the boundary state should reduce to a sum over the Dirichlet boundary states localized at the minima of $\cos(X/\sqrt{2})$. Similarly, for $g \to -\infty$ the end of the string is localized at the maxima of $\cos(X/\sqrt{2})$. Indeed, the correct partition function in these limits results from Eqn. (4.1) if $\Delta(g\to\infty) = -\Delta(g\to-\infty) = \pi/2$.

5. The Exact S-Matrix

The universal function $\Delta(g,\bar{g})$ should implicitly contain all the physical information about the theory. A particularly interesting set of questions arises in calculation of the reflection S-matrix from the dynamical boundary at $\sigma = 0$, which is well-defined for a semi-infinite string. The S-matrix is determined by correlation functions of operators ∂X and $\bar{\partial} X$ on a Euclidean half-plane with the interaction of Eq. (2.1) integrated along the boundary. For example, doing straightforward perturbation theory in g and $\bar{g}$, we find that the 2-point function is

$$\langle \partial X(z) \bar{\partial} X(\bar{w}) \rangle = \frac{S(g,\bar{g})}{(z-\bar{w})^2} \ ,$$

where the $1 \to 1$ amplitude turns out to have the expansion $S(g,\bar{g}) = -1 + 2\pi^2 g\bar{g} + \ldots$. In the operator formalism,

$$\langle \partial X(z) \bar{\partial} X(\bar{w}) \rangle = \langle B(g,\bar{g}) | \partial X(z) \bar{\partial} X(\bar{w}) | 0 \rangle \ , \tag{5.1}$$

where $\langle B(g,\bar{g})|$ is the exact state for the dynamical boundary, which is determined implicitly by Eqn. (4.3). Let us note that the only contribution to Eqn. (5.1) comes from the $(1,0)$ module. The coefficient of this module in the expansion of $\langle B(g,\bar{g})|$ can be read off from Eqn. (4.3)with the result $S(g,\bar{g}) = 1 - 2[\cos 2\Delta(g,\bar{g})]_{g\bar{g}}$, The subscript means that we are to keep only the powers of $g\bar{g}$ in the perturbative expnsion of the right-hand side.

The most remarkable feature of our reflection S-matrix is that its $n \to m$ connected pieces, while non-trivial, are entirely determined by the $1 \to 1$ amplitude. To show how this works, let us consider the $2 \to 2$ amplitude,

$$\langle \partial X(z) \partial X(u) \bar{\partial} X(\bar{w}) \bar{\partial} X(\bar{v}) \rangle = G(z,u,\bar{w},\bar{v}) =$$
$$\frac{1}{(z-u)^2(\bar{w}-\bar{v})^2} + \frac{S^2}{(z-\bar{v})^2(u-\bar{w})^2} + \frac{S^2}{(z-\bar{w})^2(u-\bar{v})^2} + F(z,u,\bar{w},\bar{v}) \ ,$$

where F is the connected part. The existence of a null state among the descendants of ∂X at level 3 gives rise to a 3-rd order differential equation [17],

$$\left[\frac{\partial^3}{\partial z^3} - 4 \frac{\partial}{\partial z} \sum_{i=1}^{3} \frac{\partial}{\partial w_i} \frac{1}{z-w_i} - 6 \sum_{i=1}^{3} \frac{\partial}{\partial w_i} \frac{1}{(z-w_i)^2} \right] G(z,w_1,w_2,w_3) = 0 \ . \tag{5.2}$$

This equation determines the connected part in terms of the disconnected parts, and we find

$$F(z,u,\bar{w},\bar{v}) = \frac{2(1-S^2)}{(z-\bar{v})(z-\bar{w})(u-\bar{v})(u-\bar{w})}$$

The Fourier transform of the Minkowskian continuation of this is

$$\tilde{F}(E_1,E_2;E_3,E_4) = 2(1-S^2)\delta(E_1+E_2-E_3-E_4)(E_1+E_2-|E_1-E_3|-|E_2-E_3|) \ , \tag{5.3}$$

where $E_i > 0$. Curiously, this formula bears a strong resemblance to the $2 \to 2$ amplitude found in the $c=1$ matrix model [18].

It appears that recursive application of the differential equations to higher-point functions determines them entirely. We find that the connected part of any $1 \to n$ amplitude vanishes, while for the $2 \to 2n$ amplitudes

$$\langle \partial X(z_1) \partial X(z_2) \bar{\partial} X(\bar{w}_1) \ldots \bar{\partial} X(\bar{w}_{2n}) \rangle_{\text{conn}} = -(-2)^n (1 - S^2) \frac{(z_2 - z_1)^{2n-2}}{\prod_{i=1}^{2n} (z_1 - \bar{w}_i)(z_2 - \bar{w}_i)} .$$

In Fourier space, these become piecewise linear functions of the energies, similar to Eqn. (5.3). This should be compared with some results in dissipative quantum mechanics [19].

6. Conclusions

In this Letter we have presented a solution of the $c = 1$ conformal field theory with a boundary sine-Gordon interaction. Conformal invariance imposes tight constraints on the partition function when viewed both from the open string and from the closed string point of view. Examination of the closed string channel allows us to deduce the exact boundary state in terms of a universal function of the complex potential strength. Remarkably, the boundary state is built out of the Virasoro modules of the well-known discrete states of the $c = 1$ CFT. From this information we determine a new non-trivial S-matrix for the scattering of the massless scalar quanta from the dynamical boundary. The operator ∂X has a null state at level 3, and this gives rise to BPZ differential equations which appear to fix all of the correlations functions recursively. Every correlator is a simple rational function of coordinate differences, which is certainly extraordinarily simple behavior for a nontrivial field theory. In future work we plan to expand on our results, and to discuss their applications to specific physical systems.

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