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BU TP-93/26
November 93

BU TP 93.26
S 203406

The structure of the invariant charge in massive theories with one coupling

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ABSTRACT

Invariance under finite renormalization group (RG) transformations is used to structure the invariant charge in models with one coupling in the 4 lowest orders of perturbation theory. In every order there starts a RG-invariant, which is uniquely continued to higher orders. Whereas in massless models the RG-invariants are power series in logarithms, there is no such requirement in a massive model. Only, when one applies the Callan-Symanzik (CS) equation of the respective theories, the high-energy behavior of the RG-invariants is restricted. In models, where the CS-equation has the same form as the RG-equation, the massless limit is reached smoothly, i.e. the β -functions are constants in the asymptotic limit and the RG-functions starting the new invariant tend to logarithms. On the other hand in the spontaneously broken models with fermions the CS-equation contains a β -function of a physical mass. As a consequence the β -functions depend on the normalization point also in the asymptotic region and a mass independent limit does not exist anymore.

*Supported by Schweizerischer Nationalfonds and Deutsche Forschungsgemeinschaft

1. Introduction

By now in all relevant theories of particle physics the calculations of the 1-loop order are nearly completed and show – especially in the standard model – a very good agreement with the experiment in this approximation. In order to get further restrictions and information about the reliability of the standard model one has to take into account higher order corrections. Whereas one starts to calculate the 2-loop order systematically, the question arises if one could draw some general conclusions from the lowest order to contributions appearing necessarily in higher orders. Of special interest thereby is the dependence on the scales of the theory, which are the physical masses and the normalization point needed in order to fix the couplings at their experimental value. As a natural tool suggests itself renormalization group invariance. Except for the trivial case, where the interactions do not depend on the momenta and are constant, as it happens in the classical approximation, renormalization group invariance is only realised to all orders of perturbation theory. Consequently the lowest order induces the next one necessarily up to the addition of new renormalization group invariants.

In quantum field theory renormalization group invariance is mostly applied in its infinitesimal version, where it is given as a partial differential equation, the renormalization group equation. In this form it has specifically been applied to deduce the high-energy behavior of the interactions. In a 1-coupling theory the renormalization group equation approximated with the 1-loop β -function can be solved analytically and therefore its solution, the 1-loop invariant, is known to all orders of perturbation theory together with its analytical continuation to a non-perturbative regime. The applications of the 1-loop invariant are divided into two different classes: Originally it was used in QED [1], where the coupling has a 1-loop β -function with a positive sign. One can roughly estimate by considering the perturbative power series, that in such theories the 1-loop invariant dominates all higher order contributions in a certain range of asymptotic momenta, where perturbation theory, i.e. the power series expansion of the 1-loop invariant, is meaningful. If one has fixed the electromagnetic coupling at low momenta, with the help of the 1-loop renormalization group invariant it can be calculated at a much higher scale, as it is for example the mass of the Z-boson [2]. In this form it is successfully applied also in the standard model for the coupling of the electromagnetic interaction [3]. It is this aspect of the

renormalization group which is meant by the so-called concept of improvement. On the other hand in theories with a negative 1-loop β -function, as it is e.g. in QCD, the analytical form of the 1-loop invariant is used to prove asymptotic freedom in the ultraviolet, i.e. the coupling approaches zero for large momenta [4]. It is again the 1-loop invariant which determines the behavior of the interaction in infinity, as it can be estimated from the renormalization group equation. For massless theories with a positive β -function similar estimates are valid in the infrared region. Deviations from the leading behavior are calculable in the next-to-leading logarithms summation (cf. [5]). These approximations are not renormalization group invariants by themselves, because the renormalization group equation with the 2-loop β -function added, is not analytically solvable anymore.

In this paper we investigate a different approach to renormalization group invariance, namely finite renormalization group transformations [6, 7]. As a first application we analyze in 1-coupling theories the invariant charge, which is constructed as an invariant under renormalization group transformations. We formulate the requirement of renormalization group invariance order by order in perturbation theory. In contrast to the solutions of the partial differential equation one does not get the all order summation in one stroke, but one is able to structure the Green functions according to their invariance under renormalization group transformations. We show that in every order a new renormalization group invariant starts, whose form is given by the solution of a functional equation. Strings of lower order induced functions run through all orders of perturbation theory and start to be interwoven with each other from three loop order onwards. In this form renormalization group invariance can be applied, if one wants to know, which terms arise necessarily in the second order once one has calculated the finite Green functions in 1-loop order. We perform the explicit calculations for the four lowest orders of perturbation theory.

Not only the renormalization group invariance gives insight into the momentum dependence of the Green functions, but also the dilatations do so. For the theories we consider in this paper they are too expressible as a partial differential equation, the Callan-Symanzik equation, which is – concerning its form – similar to the one of the renormalization group equation [8]. Therefore the invariant charge which we have constructed with the help of finite renormalization group invariance has to be a solution of the Callan-Symanzik equation as well.

In a first stage we consider theories, where the Callan-Symanzik equation has exactly the same form as the renormalization group equation, i.e. they contain both the same differential operators. Examples for such models are the massive ϕ^4 theory, the $O(N)$ -models and the purely scalar $U(1)$ -axial model with spontaneous breaking of the symmetry. Applying the Callan-Symanzik operator to the general renormalization group invariant solution one can assign to any renormalization group function a well defined high-energy behavior, especially for all these theories the massless limit is reached smoothly in the asymptotic region. Apart from these restrictions on the asymptotic behavior the renormalization group solution is shown to be in complete agreement with the Callan-Symanzik equation.

In the last part we carry out the same analysis for the $U(1)$ -axial model with one fermion, which gets its mass via the spontaneous symmetry breaking. In contrast to the models above, the Callan-Symanzik equation differs from the renormalization group equation because a β -function of a physical mass appears [9]. As a consequence of this difference the strings of renormalization group invariants are not separated into 1-loop, 2-loop and so on induced contributions, but 1-loop induced contributions appear in all renormalization group invariant functions. Moreover in the asymptotic limit, which we consider for simplification, the massless theory is not reached anymore. For instance the β -functions of the Callan-Symanzik equation and the renormalization group equation depend on logarithms of the normalization point.

In section 2 of this paper we introduce the 1-coupling models and give the 1-loop invariant solution of the renormalization group equation with its full mass dependence in the massive ϕ^4 -model. To finite renormalization group transformations we turn in section 3. There we solve the four lowest orders, determine the β -functions in terms of the renormalization group functions and consider, how the structure is realised diagrammatically in the 2-loop order of the ϕ^4 -theory. In section 4 and 6 we apply the Callan-Symanzik equation on the invariant charge, first in the pure scalar models encountered above and then in the spontaneously broken $U(1)$ -axial model with fermions. Section 5 contains a few comments on reparametrizations of the coupling. In the last section we give a short summary of the results and an outlook to further applications and consequences.

2. Renormalization group solution in the massive I-coupling theory

For a first application of finite renormalization group transformations we consider simple scalar models with one coupling λ and one mass parameter. Examples for such models are the $O(N)$ -models with the classical action

$$\Gamma_{cl} = \int \frac{1}{2} \partial \vec{\varphi} \cdot \partial \vec{\varphi} - \frac{1}{2} m^2 \vec{\varphi} \cdot \vec{\varphi} - \frac{\lambda}{4!} (\vec{\varphi} \cdot \vec{\varphi})^2, \quad (2.1)$$

where $\vec{\varphi} = (\phi_1, \phi_2, \dots, \phi_N)$. But the analysis comprises also the purely scalar $U(1)$ -axial model in its spontaneously broken phase (linear σ -model with a massless Goldstone boson):

$$\Gamma_{cl} = \int \left(\frac{1}{2} \partial \phi_1 \partial \phi_1 + \partial \phi_2 \partial \phi_2 \right) - \frac{1}{2} m^2 \phi_1^2 - \frac{1}{2} m \sqrt{\frac{\lambda}{3}} \phi_1 (\phi_1^2 + \phi_2^2) - \frac{\lambda}{4!} (\phi_1^2 + \phi_2^2)^2 \quad (2.2)$$

These models are distinguished also by the fact, that one is able to derive a Callan-Symanzik (CS) equation rigorously to all orders of perturbation theory, which contains exactly the same differential operators as the renormalization group (RG) equation. A counterexample to these models we analyze in section 6.

In perturbation theory the Green functions are defined according to the Gell-Mann Low formula, by a suitable subtraction scheme and the respective Ward identities. The free parameters have to be fixed by appropriate normalization conditions, which we choose for the models (2.1) and (2.2) in the following way:

$$\begin{aligned} \partial_p \Gamma_2(p^2) \Big|_{p^2=\kappa^2} &= 1 & \Gamma_2(p^2) \Big|_{p^2=m^2} &= 0 \\ \Gamma_4(p_1, p_2, p_3, p_4) \Big|_{p_i^2=\kappa^2} &= -\lambda, & & \end{aligned} \quad (2.3)$$

where we have defined $\Gamma_2 = \Gamma_{\phi_1 \phi_1}$, $\Gamma_4 = \Gamma_{\phi_1 \phi_1 \phi_1 \phi_1}$. For the spontaneously broken model (2.3) has to be enlarged by the requirement, that the vacuum expectation value is vanishing:

$$\Gamma_{\phi_1} = 0 \quad (2.3a)$$

Throughout the paper we restrict ourselves to on-shell normalization for the mass in order to be able to exploit order by order finite renormalization group invariance. The normalization point for the coupling and the wave function is taken to be in the Euclidean region ($\kappa^2 < 0$).

The behavior of the Green functions under an infinitesimal change of the normalization point κ is expressed by the RG-equation

$$(\kappa \partial_\kappa + \tilde{\beta}_\lambda \partial_\lambda - \tilde{\gamma}_i \mathcal{N}) \Gamma(\phi_i) = 0 \quad \text{with} \quad \mathcal{N} = \sum_{i=1}^N \int \phi_i \frac{\delta}{\delta \phi_i}. \quad (2.4)$$

The CS-equation describes the breaking of the dilatations by the mass term and the dilatational anomalies represented by the function β_λ and the anomalous dimension γ . For the models above it has the general form:

$$(m \partial_m + \kappa \partial_\kappa + \beta_\lambda \partial_\lambda - \gamma \mathcal{N}) \Gamma(\phi_i) = [\Delta m]_2^2 \cdot \Gamma(\phi_i) \quad (2.5)$$

The right-hand side is constructed to behave as a truly soft insertion, i.e. it vanishes for large non-exceptional momenta. In the $O(N)$ -models it is just given by the soft mass insertion

$$\Delta m = \alpha \int (-m^2 \vec{\varphi} \cdot \vec{\varphi}) \quad (2.5a)$$

whereas in the spontaneously broken models the construction is much more subtle (cf. section 6 and [9]):

$$\Delta m = \int (-m^2 \phi_1^2 - \frac{1}{2} m \sqrt{\frac{\lambda}{3}} \phi_1 (\phi_1^2 + \phi_2^2) + O(\hbar)) \quad (2.5b)$$

In this paper we restrict the considerations concerning the RG-transformations to the invariant charge defined as a combination of perturbatively constructed Green functions

$$\begin{aligned} Q(p^2, m^2, \kappa^2, \lambda) &= \\ &= -\Gamma_4(p_1, p_2, p_3, p_4, m^2, \kappa^2, \lambda) \prod_{k=1}^4 \left(\partial_{p_k} \Gamma_2(p_k^2, m^2, \kappa^2, \lambda) \right)^{-\frac{1}{2}} \Big|_{p_i^2=p^2} \\ & \quad r_i p_i = -\frac{p^2}{2} \end{aligned} \quad (2.6)$$

where again $p^2 < 0$ for definiteness. According to (2.3) it has well-defined normalization properties

$$Q(p^2, m^2, \kappa^2, \lambda) \Big|_{p^2=\kappa^2} = \lambda. \quad (2.7)$$

Furthermore it is dimensionless

$$(p^2 \partial_{p^2} + m^2 \partial_{m^2} + \kappa^2 \partial_{\kappa^2}) Q(p^2, m^2, \kappa^2, \lambda) = 0 \quad (2.8)$$

and depends therefore only on the dimensionless ratios $\frac{p^2}{\kappa^2}$ and $\frac{m^2}{p^2}$.

$$Q(p^2, m^2, \kappa^2, \lambda) = Q\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) \quad (2.9)$$

The most important property of the invariant charge concerning the RG is the fact, that it is an invariant under the RG-transformations. For this reason it satisfies the homogeneous RG-equation

$$(\kappa\partial_\kappa + \tilde{\beta}_\lambda\partial_\lambda)Q\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) = 0 \quad (2.10)$$

and the CS-equation without anomalous dimensions:

$$(m\partial_m + \kappa\partial_\kappa + \beta_\lambda\partial_\lambda)Q\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) = Q_m\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) \quad (2.11)$$

with

$$Q_m\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) = \left([\Delta_m]_2^2 \cdot \Gamma\right)_4(\partial_p^2 \Gamma_2)^{-2} - 2\partial_p^2([\Delta_m]_2^2 \cdot \Gamma)_2 \Gamma_4 \quad (2.11a)$$

all functions understood to be perturbatively expanded.

Before we turn to the finite RG-transformations we want to give the 1-loop induced RG-invariant as it is calculated from the RG-equation in the $O(N)$ -models. From the 1-loop invariant charge

$$\begin{aligned} Q\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) &= \lambda + \frac{1}{16\pi^2} \frac{N+8}{6} \lambda^2 \left(\sqrt{1 - \frac{3m^2}{p^2}} (\ln(\sqrt{1 - \frac{3m^2}{p^2}} + 1) - \ln(\sqrt{1 - \frac{3m^2}{p^2}} - 1)) \right. \\ &\quad \left. - \sqrt{1 - \frac{3m^2}{\kappa^2}} (\ln(\sqrt{1 - \frac{3m^2}{\kappa^2}} + 1) - \ln(\sqrt{1 - \frac{3m^2}{\kappa^2}} - 1)) \right) + O(\lambda^3) \\ &\equiv \lambda - \lambda^2(Q^{(1)}(\frac{m^2}{p^2}) - Q^{(1)}(\frac{m^2}{\kappa^2})) + O(\lambda^3), \end{aligned} \quad (2.12)$$

the RG-function $\tilde{\beta}_\lambda^{(1)}$ is calculated to be

$$\tilde{\beta}_\lambda^{(1)}\left(\frac{m^2}{\kappa^2}\right) = \frac{1}{16\pi^2} \frac{N+8}{6} \lambda^2 \left(\frac{3m^2}{\kappa^2} \frac{1}{\sqrt{1 - \frac{3m^2}{\kappa^2}}} \ln \frac{\sqrt{1 - \frac{3m^2}{\kappa^2}} + 1}{\sqrt{1 - \frac{3m^2}{\kappa^2}} - 1} + 2 \right) \quad (2.13)$$

In the limit $\kappa^2 \rightarrow -\infty$ the RG-function $\tilde{\beta}_\lambda(\frac{m^2}{\kappa^2})$ becomes κ -independent and coincides with the CS-function $\beta_\lambda^{(1)}$ and the one of the corresponding massless models:

$$\beta_\lambda^{(1)} = \lim_{\kappa^2 \rightarrow -\infty} \tilde{\beta}_\lambda^{(1)}\left(\frac{m^2}{\kappa^2}\right) = \frac{1}{16\pi^2} \frac{N+8}{3} \lambda^2, \quad (2.14)$$

In the limit $\kappa^2 \rightarrow 0$ the RG-function vanishes:

$$\lim_{\kappa^2 \rightarrow 0} \tilde{\beta}_\lambda^{(1)}\left(\frac{m^2}{\kappa^2}\right) = 0 \quad (2.15)$$

With the mass-dependent 1-loop RG-function $\tilde{\beta}_\lambda^{(1)}$ (2.13) one is able to solve the RG-equation (2.10) analytically in this approximation:

$$\begin{aligned} \tilde{Q}_1\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) &= \frac{\lambda}{1 + \lambda(Q^{(1)}(\frac{m^2}{p^2}) - Q^{(1)}(\frac{m^2}{\kappa^2}))} \\ &= \sum_{i=0}^{\infty} \lambda^{i+1} (Q^{(1)}(\frac{m^2}{\kappa^2}) - Q^{(1)}(\frac{m^2}{p^2}))^i \end{aligned} \quad (2.16)$$

$\tilde{Q}_1(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda)$ is the 1-loop induced invariant charge; it continues in terms of 1-loop Green function $Q^{(1)}(\frac{m^2}{p^2}) - Q^{(1)}(\frac{m^2}{\kappa^2})$ (2.12) the renormalization group invariance to all orders. Because we restrict ourselves to the region where perturbation theory can be applied we understand and this non-perturbative solution always expanded as a power series in λ , i.e. treat it perturbatively.

The same form (2.16) can be deduced for the spontaneously broken model (2.2), where $Q^{(1)}(\frac{m^2}{p^2})$ is a different, more complicated function due to the appearance of further finite diagrams. In the asymptotic limit, $|\kappa^2| \gg m^2$ and $|p^2| \gg m^2$, its invariant charge coincides with the one of the $O(2)$ -model and its β -function $\tilde{\beta}_\lambda^{(1)}$ is the same as the one of the CS-equation and the purely massless model (2.14). This is a general feature of the asymptotic limit in these models, and – as we will point out – it can be derived as a consequence of the fact, that a CS-equation of the same form as the RG-equation exists.

Concluding the discussion of the 1-loop invariant charge we calculate the action of the CS-operator on $\tilde{Q}_1(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda)$ (2.16):

$$(m\partial_m + \kappa\partial_\kappa + \beta_\lambda^{(1)}\partial_\lambda)\tilde{Q}_1\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) = \tilde{Q}_1^2\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right)(\beta_\lambda^{(1)} - \tilde{\beta}_\lambda^{(1)}\left(\frac{m^2}{p^2}\right)), \quad (2.17)$$

where $\beta_\lambda^{(1)}$ is the Callan-Symanzik function (2.14) and $\tilde{\beta}_\lambda^{(1)}$ is the RG-function (2.13). As one easily verifies, the right-hand side is well-defined: It is soft, i.e. it vanishes for asymptotic momenta, and it is a RG-invariant in the same sense as $\tilde{Q}_1(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda)$.

3. The structure of the invariant charge

3.1. RENORMALIZATION GROUP INVARIANTS

In the last section we have calculated the 1-loop induced invariant charge (2.16) as the solution of the RG-equation with the 1-loop β -function. In order to get deeper insight into the meaning of it we will structure the higher order contributions according to their invariance with respect to RG-transformations. For this purpose we consider again the invariant charge as it is defined in (2.6) normalized according to (2.7). (As in section 2 p^2 and κ^2 are always taken in the Euclidean region for definiteness.)

$$Q\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right)\Big|_{p^2=\kappa^2} = \lambda \quad (3.1)$$

Finite RG-transformations of the Green functions and especially of Q can be derived by a formal integration of the RG-equation (2.4), which expresses the effect of infinitesimal RG-transformations in the differential form [10]. But on a much more fundamental level invariance of the Green functions under finite RG-transformations up to field redefinitions, the anomalous dimensions, can be postulated directly as such [6, 7]. Due to its construction the invariant charge is an invariant under the RG-transformations: If one has fixed Q at a different point κ_1^2 calculating with a different coupling λ_1 ,

$$Q\left(\frac{p^2}{\kappa_1^2}, \frac{m^2}{p^2}, \lambda_1\right)\Big|_{p^2=\kappa_1^2} = \lambda_1, \quad (3.2)$$

the RG-invariance requires that for all momenta the result has to be the same:

$$Q\left(\frac{p^2}{\kappa_1^2}, \frac{m^2}{p^2}, \lambda\right) = Q\left(\frac{p^2}{\kappa_1^2}, \frac{m^2}{p^2}, \lambda_1\right) \quad (3.3)$$

By means of the normalization conditions we find

$$Q\left(\frac{\kappa_1^2}{\kappa_1^2}, \frac{m^2}{\kappa_1^2}, \lambda\right) = \lambda_1 \quad (3.4)$$

from which one can immediately derive the multiplication law of the RG:

$$Q(\tau\tau_1, u, \lambda) = Q(\tau, u, Q(\tau_1, u\tau, \lambda)) \quad (3.5)$$

where

$$\tau = \frac{p^2}{\kappa_1^2}, \quad \tau_1 = \frac{\kappa_1^2}{\kappa_1^2}, \quad u = \frac{m^2}{p^2}.$$

From now on we restrict ourselves again to perturbation theory, where the invariant charge is calculated in powers of the coupling λ

$$Q(\tau, u, \lambda) = \sum_{i=0}^{\infty} \lambda^{i+1} f_i(\tau, u) \quad (3.6)$$

where according to (3.1)

$$f_0(\tau, u) = 1 \quad \text{and} \quad f_i(1, u\tau) = 0, \quad i \geq 1 \quad (3.7)$$

Inserting the perturbatively calculated $Q(\tau, u, \lambda)$ into the equation (3.5) one gets

$$\sum_{i=0}^{\infty} \lambda^{i+1} f_i(\tau\tau_1, u) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \lambda^{j+1} f_j(\tau_1, u\tau) \right)^{i+1} f_i(\tau, u) \quad (3.8)$$

Perturbatively we are able to solve the equation recursively by comparing the same powers in the coupling λ :

$$\begin{aligned} \lambda^1: & \quad 1 = 1 \\ \lambda^2: & \quad f_1(\tau\tau_1, u) = f_1(\tau_1, u\tau) + f_1(\tau, u) \\ \lambda^3: & \quad f_2(\tau\tau_1, u) = f_2(\tau_1, u\tau) + f_2(\tau, u) + 2f_1(\tau_1, u\tau)f_1(\tau, u) \\ \lambda^4: & \quad f_3(\tau\tau_1, u) = f_3(\tau_1, u\tau) + f_3(\tau, u) + 3f_2(\tau_1, u)f_1(\tau, u\tau) \\ & \quad + f_1(\tau, u)(f_1^2(\tau_1, u\tau) + 2f_2(\tau_1, u\tau)) \end{aligned} \quad (3.9)$$

The general expression in order k takes the form

$$\begin{aligned} f_k(\tau\tau_1, u) = & f_k(\tau_1, u\tau) + f_k(\tau, u) \\ & + \sum_{m=1}^{k-1} f_m(\tau, u) \sum_{[a_i]_{m+1}} (m+1; a_1, \dots, a_m, 0) f_0^{a_1}(\tau_1, u\tau) f_1^{a_2}(\tau_1, u\tau) \dots f_{k-1}^{a_k}(\tau_1, u\tau) \end{aligned} \quad (3.10)$$

where

$$(m+1; a_1, \dots, a_l) = \frac{(m+1)!}{a_1! \dots a_l!}$$

and $\sum_{[a_i]_{m+1}}$ is a sum over all integer $a_i \geq 0$ with $a_1 + \dots + a_k = m+1$ and $a_1 + 2a_2 + \dots + ka_k = k+1$.

In this paper we will only evaluate the four lowest orders of perturbation theory. But with the help of (3.10) it is a straightforward calculation to show that the RG-solution (2.16) fulfills the equations of finite RG-transformations, too, where each order induces the next one necessarily.

The crucial equation appearing in the successive solution of the system (3.9) is the functional equation

$$f_k(\tau\tau_1, u) = f_k(\tau_1, u\tau) + f_k(\tau, u) \quad (3.11)$$

It is the starting equation of the RG-invariant functions. A solution of it can be added in every order being determined by true n -loop contributions. In the massive case ($u \neq 0$) the general solution of (3.11) is given by

$$f_k(\tau, u) = g_k(\tau u) - g_k(u) \quad (3.12)$$

with $g_k(y)$ an arbitrary function of one argument. In the massless case ($u = 0$) (3.11) is much more restrictive and the unique solution is the logarithm:

$$f_k(\tau\tau_1) = f_k(\tau) + f_k(\tau_1) \iff f_k(\tau) = \alpha_k \ln \tau \quad (3.13)$$

In order to prove that (3.12) is the general solution of the functional equation (3.11) we rewrite it by a change of variables into ($y = \tau\tau_1, y' = \tau$)

$$f(y, u) - f(y', u) = f\left(\frac{y}{y'}, uy'\right) \quad (3.11')$$

We are now able to use the techniques of solving functional equations: Differentiating (3.11') with respect to y, y' and u one gets the following equations ($w = \frac{y}{y'}, v = uy'$):

$$\begin{aligned} y\partial_y f(y, u) &= \frac{y}{y'}\partial_w f(w, v) \\ -y'\partial_{y'} f(y', u) &= -\frac{y}{y'}\partial_w f(w, v) + uy'\partial_v f(w, v) \end{aligned} \quad (3.14)$$

$$u\partial_u f(y, u) - u\partial_u f(y', u) = uy'\partial_v f(w, v)$$

Combining the equations in a way that the explicit dependence on w and v cancels we remain with

$$y\partial_y f(y, u) - u\partial_u f(y, u) = y'\partial_{y'} f(y', u) - u\partial_u f(y', u) \quad (3.15)$$

This is a partial differential equation with the left hand side independent of y' and the right hand side independent of y and consequently:

$$y\partial_y f(y, u) - u\partial_u f(y, u) = F(u) \quad (3.16)$$

with the solution

$$f(y, u) = g(yu) + \hat{g}(u), \quad \text{where } u\partial_u \hat{g}(u) = F(u) \quad (3.17)$$

$\hat{g}(u)$ is fixed by inserting (3.17) into (3.11')

$$\hat{g}(uy) = -\hat{g}(uy) \quad (3.18)$$

and the general solution of the functional equation (3.11) is given by (3.12).

In agreement with the explicit calculation of section 1 for the $O(N)$ -models we have in 1-loop (cf. (2.12) with $Q^{(1)} \equiv g_1$)

$$Q_\infty^{(1)}\left(\frac{p^2}{k^2}, \frac{m^2}{p^2}, \lambda\right) = \lambda^2 \left(g_1\left(\frac{m^2}{k^2}\right) - g_1\left(\frac{m^2}{p^2}\right) \right) \quad (3.19)$$

and for the massless case

$$Q_\infty^{(1)}\left(\frac{p^2}{k^2}, \lambda\right) = \lambda^2 \alpha_1 \ln \frac{p^2}{k^2} \quad \text{with } \alpha_1 = \frac{1}{16\pi^2} \frac{N+6}{6} \quad (3.20)$$

The solution of the next order works straightforwardly with the result

$$f_2(\tau, u) = g_2(\tau u) - g_2(u) + (g_1(\tau u) - g_1(u))^2 \quad (3.21)$$

$g_2(\tau u) - g_2(u)$ is a new true 2-loop function, starting a new RG-invariant, whereas $(g_1(\tau u) - g_1(u))^2$ continues the 1-loop function to the next order to preserve the all order RG-invariance. It is uniquely determined up to the addition of the RG-invariant $g_1^2(\tau u) - g_1^2(u)$. The ordering we use here is guided to achieve in the RG-equation minimal β -functions, in a sense we will specify later (cf. (3.26))

In a similar way one can solve the recursion formula (3.10) to calculate the 3- and 4-loop order contribution:

$$\begin{aligned} f_3(\tau, u) &= g_3(\tau u) - g_3(u) \\ &\quad + \frac{5}{2}(g_2(\tau u) - g_2(u))(g_1(\tau u) - g_1(u)) + \frac{1}{2}(g_2(\tau u)g_1(u) - g_2(u)g_1(\tau u)) \\ &\quad + (g_1(\tau u) - g_1(u))^3 \\ f_4(\tau, u) &= g_4(\tau u) - g_4(u) \\ &\quad + 3(g_3(\tau u) - g_3(u))(g_1(\tau u) - g_1(u)) + (g_3(\tau u)g_1(u) - g_3(u)g_1(\tau u)) \\ &\quad + \frac{3}{2}(g_2(\tau u) - g_2(u))^2 + \frac{13}{3}(g_2(\tau u) - g_2(u))(g_1(\tau u) - g_1(u))^2 \\ &\quad + \left(\frac{5}{3}g_1(\tau u) - \frac{4}{3}g_1(u)\right)(g_2(\tau u)g_1(u) - g_2(u)g_1(\tau u)) \\ &\quad + (g_1(\tau u) - g_1(u))^4 \end{aligned} \quad (3.22)$$

Before we discuss the structure of the RG-solution in more detail we want to compare the result with the purely massless case in the same model. One has

$$\begin{aligned}
f_1(\tau) &= \alpha_1 \ln \tau \\
f_2(\tau) &= \alpha_2 \ln \tau + \alpha_1^2 \ln^2 \tau \\
f_3(\tau) &= \alpha_3 \ln \tau + \frac{5}{2} \alpha_2 \alpha_1 \ln^2 \tau + \alpha_1^3 \ln^3 \tau \\
f_4(\tau) &= \alpha_4 \ln \tau + (3\alpha_3 \alpha_1 + \frac{3}{2} \alpha_2^2) \ln^2 \tau + \frac{13}{3} \alpha_2 \alpha_1^2 \ln^3 \tau + \alpha_1^4 \ln^4 \tau
\end{aligned} \tag{3.23}$$

From here the structure is obvious: In every order there starts a new RG-invariant, i.e. a solution of (3.11). In the massless case it is $\ln \tau$ times an arbitrary coefficient which has to be determined by an explicit evaluation of diagrams, in the massive case it is a new function of the form $g_n(\tau u) - g_n(u)$. Besides this new invariant there appear strings of lower order induced functions, which are series in $\ln \tau$ in the massless case, whereas in the massive case those are combinations of the functions $g_i(u)$ and $g_i(u)$, which start to be interwoven with each other from 3-loop onwards. Especially there appear also antisymmetric combinations of the lower order functions, e.g. $g_2(\tau u)g_1(u) - g_2(u)g_1(\tau u)$ in 3-loop order. The 1-loop induced solution of the RG-equation we have calculated in (2.16) is the RG-invariant induced string of the lowest order. It has to appear in every order in addition to the higher order contributions as it is required by RG-invariance. Especially one can verify that it solves itself the functional equation (3.10), if one puts all higher order invariants equal to zero, i.e. $g_i(u) = 0$ for $i \geq 2$.

We want to mention once again that the ordering of the lower loop induced contributions is unique up to the addition of new invariants of the form $g_1(\tau u) \dots g_n(\tau u) - g_1(u) \dots g_n(u)$. As for $f_2(\tau, u)$ (3.21) the structure we have imposed is again due to the minimality of the β -functions: Because $Q(\tau, u, \lambda)$ as given in (3.19, 21, 22) is constructed as an invariant under finite RG-transformation, it is obvious, that it fulfills the RG-equation, the infinitesimal form of (3.5):

$$\mathcal{R}Q = (\kappa \partial_\kappa + \tilde{\beta}_\lambda \partial_\lambda) Q = 0 \tag{3.24}$$

Therefore we are able to calculate the β -functions $\tilde{\beta}_\lambda$ up to four loops in terms of the RG-function $g_i(y)$. Applying (3.24) on the 1-loop expression (3.19), one gets:

$$-2\tau u g_1'(\tau u) + \tilde{\beta}_\lambda^{(1)} = 0 \implies \tilde{\beta}_\lambda^{(1)}(\tau u) = 2\tau u g_1'(\tau u) \tag{3.25}$$

In the same way all the other β -functions are calculated:

$$\begin{aligned}
\tilde{\beta}_\lambda^{(2)}(u\tau) &= \lambda^3 2u\tau g_2'(u\tau) \\
\tilde{\beta}_\lambda^{(3)}(u\tau) &= \lambda^4 (2u\tau g_3'(u\tau) + \frac{1}{2} g_1(u\tau) 2u\tau g_2'(u\tau) - \frac{1}{2} g_2(u\tau) 2u\tau g_1'(u\tau)) \\
\tilde{\beta}_\lambda^{(4)}(\tau u) &= \lambda^5 (2\tau u g_4'(\tau u) + 2\tau u g_3'(\tau u) g_1(\tau u) - 2\tau u g_1'(\tau u) g_3(\tau u) \\
&\quad + \frac{1}{3} g_1(\tau u) (2\tau u g_2'(\tau u) g_1(\tau u) - 2\tau u g_1'(\tau u) g_2(\tau u)))
\end{aligned} \tag{3.26c}$$

The ordering of the RG-invariants is due to the principle, that one has avoided to introduce lower order RG-functions in the β -functions, except for the anti-symmetric combinations which are unavoidable and vanish if $g_i(y) = c_i g_1(y)$. To make this statement of the minimality of β -functions clear, we add to the 2-loop solution (3.21) an invariant of the form $g_1^2(u\tau) - g_1^2(u)$ with an arbitrary coefficient r

$$f_2(\tau, u) = g_2(\tau u) - g_2(u) + r(g_1^2(u\tau) - g_1^2(u)) + (g_1(\tau u) - g_1(u))^2 \tag{3.21'}$$

The 2-loop β -function is then calculated to be

$$\tilde{\beta}_\lambda^{(2)}(u\tau) = \lambda^3 (2u\tau g_2'(u\tau) + 2r \tilde{\beta}_\lambda^{(1)}(u\tau) g_1(u\tau)) \tag{3.26d'}$$

i.e. it contains a contribution appearing with the 1-loop β -function. If one solves the RG-equation in the approximation of the 1-loop β -function, as we did in section 2, it is just assumed that $\tilde{\beta}_\lambda^{(2)}$ is independent of $g_1(u\tau)$ and can be therefore neglected in some approximation. But, in section 6 we will show, that through the CS-equation there arise exactly such terms in the spontaneously broken model in the presence of fermions.

Furthermore, as it can be noticed from the 3-loop function, the form required by the minimality of the RG-functions is at most compatible with the massless theory: If all functions $g_i(y)$ tend to logarithms in the asymptotic region:

$$g_i(y) \sim \ln(-y) \quad \text{if } y \rightarrow 0 \tag{3.27}$$

the antisymmetric combinations vanish and the massless limit (3.23) results to all orders. For the models introduced in section 2 the RG-functions behave in fact as given in (3.27). But this result is not a consequence of RG-invariance, but can be only derived by using the CS-equation.

3.2. THE INVARIANT CHARGE AT ARBITRARY MOMENTA

In the defining equation of the invariant charge (2.6) we have taken the momenta symmetrically, and the Lorentz invariant combinations of external momenta are given therefore in expressions of one momentum parameter p^2 . In order to derive the string structure of the invariant charge, this a unnecessary restriction. In general the invariant charge of the scalar models depends of six independent Lorentz invariants

$$Q = Q\left(\frac{p_1^2}{\kappa^2}, \frac{m^2}{p_1^2}, P_{12}, P_{13}, P_{23}, P_{22}, P_{33}\right) \quad \text{with} \quad P_{ij}(ip_k) = \frac{p_i \cdot p_j}{p_k^2} \quad (3.28)$$

At the symmetric normalization point κ_{sym} (cf. (2.3)) the Lorentz invariants are numbers

$$P_{ij}(\kappa_{sym}) \equiv K_{ij} = -\frac{1}{3} \quad \text{if } i \neq j \quad \text{and} \quad P_{ii}(\kappa_{sym}) \equiv K_{ii} = 1 \quad (3.29)$$

Moreover also the restriction to one mass parameter has been a simplification in order to make the results more transparent, but is completely unnecessary. A further mass term can be consistently introduced for example in the spontaneously broken model (2.2), if one breaks the symmetry explicitly by a soft mass term for the Goldstone boson. Fixing all the masses at the pole, RG-invariance is defined as in (3.5):

$$Q(\tau\tau_1, u, P_{ij}, \alpha_l, \lambda) = Q(\tau, u, P_{ij}, \alpha_l, Q(\tau_1, u\tau, K_{ij}, \alpha_l, \lambda)) \quad (3.30)$$

where now $\tau = \frac{p_1^2}{\kappa^2}$, $u = \frac{m^2}{p_1^2}$ and α_l denotes mass ratios $\frac{m^2}{m_l^2}$. In formula (3.30) we have implicitly restricted the momenta to be in the same sheet as the normalization point, i.e. they are Euclidean ones, because we have assumed that there appears only one function Q . Taking again a perturbative power series expansion for the invariant charge the lowest order equation, the general starting equation of RG-invariants, has the same form as above (3.11) except for the dependence on further parameters.

$$f_k(\tau\tau_1, u, P_{ij}, \alpha_l) = f_k(\tau_1, u\tau, K_{ij}, \alpha_l) + f_k(\tau, u, P_{ij}, \alpha_l) \quad (3.31)$$

If all momenta are taken in the Euclidean region, as it is for the symmetric point, and the invariant charge is real, it can be solved as above with the result:

$$Q^{(1)}(\tau, u, P_{ij}, \alpha_l, \lambda) = \lambda^2 (g_1(\tau u, K_{ij}, \alpha_l) - g_1(u, P_{ij}, \alpha_l)) \quad (3.32)$$

Having solved (3.31) the solution of the respective 2-loop equation (3.21) can be calculated straightforward

$$Q^{(2)}(\tau, u, P_{ij}, \alpha_l, \lambda) = \lambda^3 (g_2(\tau u, K_{ij}, \alpha_l) - g_2(u, P_{ij}, \alpha_l) + (g_1(\tau u, K_{ij}, \alpha_l) - g_1(u, P_{ij}, \alpha_l))^2) \quad (3.33)$$

Further information on momentum and mass dependence could be drawn from symmetry properties of the Feynman diagrams, but such considerations are beyond the purpose of this paper. Already in such simple theories with spontaneous breaking of the symmetry, there appear in 2-loop order a lot of 1-loop induced counterterms. Thereby the knowledge of the general structure (3.33) should be helpful at least for a check of the results and for the correct adjustment of finite counterterms having e.g. calculated in a scheme with asymptotic normalization conditions as it is the MS-scheme [11].

3.3. THE DISTRIBUTION OF RG-INVARIANTS TO DIAGRAMS

After having ordered the n-loop terms according to their properties under RG-transformations the question arises, if the lower loop induced contributions can be assigned to certain Feynman diagrams. As we will point out, this structure can not be associated to individual diagrams. Especially the 1-loop induced invariant charge Q_1 (2.16) is not just related to the sum of bubble diagrams as it is often argued. The situation seems to be rather analogous to gauge invariance which is also not realized diagram by diagram but as a rule only for Green functions.

In order to settle this issue we consider the 2-loop diagrams of the ϕ^4 -theory. According to our construction of the invariant charge (2.6) we have to add the self energy contribution $\partial_{p^2}\Gamma_2(p^2)$ to the 4-point function Γ_4 :

$$Q^{(2)}\left(\frac{m^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) = -\Gamma_4^{(2)}\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) - 2\lambda\partial_{p^2}\Gamma_2^{(2)}\left(p^2, \frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) \quad (3.34)$$

RG-invariance of the invariant charge states that $Q^{(2)}\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right)$ has to be of the form (3.21)

$$Q^{(2)}\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) = Q^{(2)}\left(\frac{m^2}{\kappa^2}\right) - Q^{(2)}\left(\frac{m^2}{p^2}\right) + \left(Q^{(1)}\left(\frac{m^2}{\kappa^2}\right) - Q^{(1)}\left(\frac{m^2}{p^2}\right)\right)^2 \quad (3.21')$$

$Q^{(1)}\left(\frac{m^2}{\kappa^2}\right) - Q^{(1)}\left(\frac{m^2}{p^2}\right)$ is the 1-loop Green function calculated in (2.12), and $Q^{(2)}(y) \equiv g_2(y)$, the genuine 2-loop function.

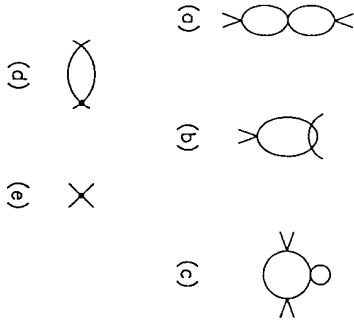


Fig. 1 : The 2-loop diagrams of the 4-point vertex in the ϕ^4 -theory.

Calculating the bubble diagram fig. 1a in the BPHZL-scheme one gets:

$$\Gamma_4^{(1a)} = -\frac{1}{3}\lambda^3 \left(Q^{(1)}\left(\frac{m^2}{p^2}\right) - Q^{(1)}\left(\frac{m^2}{p^2}\right)|_{p^2=0} \right)^2 \quad (3.35)$$

The subtraction at $p^2 = 0$ is due to the scheme we use. From the counterterm inserted the 1-loop diagram (fig. 1d) one gets

$$\Gamma_4^{(1d)} = 2\lambda^3 \left(Q^{(1)}\left(\frac{m^2}{p^2}\right) - Q^{(1)}\left(\frac{m^2}{p^2}\right)|_{p^2=0} \right) \left(Q^{(1)}\left(\frac{m^2}{\kappa^2}\right) - Q^{(1)}\left(\frac{m^2}{\kappa^2}\right)|_{\kappa^2=0} \right) \quad (3.36)$$

The counterterm diagram fig. 1e is independent of p^2 and in the BPHZL-scheme the diagram of fig. 1c is zero. In order to obtain in 2-loop order the structure predicted by the RG-invariance (3.21') the diagram of fig. 1b has to contain a momentum dependent square of $Q^{(1)}\left(\frac{m^2}{p^2}\right)$,

$$\Gamma_4^{(1b)} = -\frac{2}{3}\lambda^3 \left(Q^{(1)}\left(\frac{m^2}{p^2}\right) \right)^2 + \tilde{Q}^{(2)}\left(\frac{m^2}{p^2}\right) + \text{constant} \quad (3.37)$$

with $\tilde{Q}^{(2)}(y) \sim \ln y$ if $y \rightarrow 0$. In [12] the diagram fig. 1b is calculated with the momenta taken on mass-shell and the result confirms (3.37).

In the massless theory, where all momentum dependence is logarithmic at the symmetric point and the structure is determined by (3.23), one has in the BPHZL-

scheme (M^2 is the auxiliary mass) (cf. for example [13]):

$$\begin{aligned} \Gamma_4^{(1a)} &= -\frac{3}{4} \left(\frac{1}{16\pi^2} \right)^2 \lambda^3 \left(\ln \left| \frac{4p^2}{3M^2} \right| - 2 \right)^2 \\ \Gamma_4^{(1b)} &= \left(\frac{1}{16\pi^2} \right)^2 \left(-\frac{3}{2} \ln^2 \left| \frac{4p^2}{3M^2} \right| + 9 \ln \left| \frac{4p^2}{3M^2} \right| + C \right) \\ \Gamma_4^{(1c)} &= 0 \\ \Gamma_4^{(1d)} &= \frac{9}{2} \left(\frac{1}{16\pi^2} \right)^2 \left(\ln \left| \frac{4p^2}{3M^2} \right| - 2 \right) \left(\ln \left| \frac{4\kappa^2}{3M^2} \right| - 2 \right) \\ \Gamma_4^{(1e)} &= \left(\frac{1}{16\pi^2} \right)^2 \left(-\frac{9}{4} \ln^2 \left| \frac{4\kappa^2}{3M^2} \right| + 6 \ln \left| \frac{4\kappa^2}{3M^2} \right| - C \right) \end{aligned} \quad (3.38)$$

The 2-point function is given by

$$\Gamma_2(p^2, \kappa^2, \lambda) = -\frac{1}{12} \left(\frac{1}{16\pi^2} \right)^2 \left(p^2 \ln \frac{p^2}{\kappa^2} - p^2 \right) \quad (3.39)$$

Only the sum of all the diagrams has the desired form:

$$Q^{(2)}\left(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda\right) = \left(\frac{1}{16\pi^2} \right)^2 \left(9 \ln^2 \frac{p^2}{\kappa^2} - \frac{17}{6} \ln \frac{p^2}{\kappa^2} \right) \quad (3.40)$$

It is independent of the auxiliary mass M^2 and the square of the logarithm appears in RG-invariant form, i.e. its coefficient is the 1-loop coefficient squared (cf. (3.20)). The correct coefficient arises only when the diagrams fig. 1a and 1b are summed up. The square of the logarithmic terms is in all schemes uniquely associated with fig. 1a and 1b, whereas the single logarithmic term originates from different diagrams in different schemes. The appearance of the 1-loop induced contributions seems to us to be strongly related to the assignment of sectors to diagrams as they have been introduced e.g. in the framework of dimensional regularization [14]. If such a technique is helpful also for singling out finite contributions remains to be clarified.

4. The CS-equation

In the last section we have shown, how the requirement of RG-invariance structures the invariant charge. To get further information on the RG-functions $g_i(y)$ we have to use the CS-equation. In this section we consider such theories, as they have been introduced in section 2, where the CS-equation has the same form as the RG-equation:

$$CQ = (m\partial_m + \kappa\partial_\kappa + \beta_\lambda\partial_\lambda)Q = Q_m \quad (2.11')$$

We apply the CS-operator order by order on the invariant charge as it is given in (3.19, 21, 22) and calculate β_λ in expressions of the RG-functions $g_i(y)$. Thereby we will derive the high-energy behavior we have mentioned in the last section (3.27) and mass independence of the β_λ and β_λ for asymptotic normalization conditions. Besides this we will not find any further restrictions on the RG-functions, $g_i(y)$, $i = 1, 4$ and the CS-equation is completely consistent with RG-invariance by itself.

Starting point is the 1-loop solution (3.19)

$$Q^{(1)}(\tau, u, \lambda) = \lambda^2 (g_1(u\tau) - g_1(u)) \quad (4.1)$$

with $u = \frac{m^2}{p^2}$ and $\tau = \frac{p^2}{\kappa^2}$ and $\kappa^2 < 0$ as above. Inserting (4.1) into the CS-equation gives

$$-2ug_1'(u)\lambda^2 + \beta_\lambda^{(1)} = Q_m^{(1)}(u) \quad (4.2)$$

The CS-equation has to be valid for all momenta p^2 , therefore especially for large ones, where the right hand side will vanish according to its construction, if one is not at an exceptional momentum. The symmetric point is non-exceptional and one has:

$$\lambda^2 \lim_{u \rightarrow 0} 2ug_1'(u) = \beta_\lambda^{(1)} \equiv b_0\lambda^2 \quad (4.3)$$

From here it follows that $\beta_\lambda^{(1)}$ is independent of κ^2 , i.e. a constant, as it is well-known. Furthermore by integration one derives that $g_1(u)$ has logarithmic behavior for asymptotic momenta:

$$g_1(u) = \frac{1}{2}b_0 \ln(-u) + C_1 \quad \text{for } u \rightarrow 0 \quad (4.4)$$

Therefore the asymptotic behavior of the 1-loop function $g_1(y)$ is in fact a consequence of the existence of the CS-equation. Because RG-invariance has only determined the difference $g_1(\tau u) - g_1(u)$ we are free to normalize $g_i(y)$ in such a way that $C_1 = 0$ for $y \rightarrow 0$. This normalization leads to some simplification later on, especially it simplifies the transition to the massless theory. From (4.3) and (3.25) it follows also the result we have mentioned in section 2, that the β -function of the RG-equation (3.25) is the same as the CS- β -function in the limit of an asymptotic normalization point:

$$\lim_{\kappa^2 \rightarrow -\infty} \tilde{\beta}_\lambda^{(1)}\left(\frac{m^2}{\kappa^2}\right) = \beta_\lambda^{(1)} \quad (4.5)$$

In agreement with our explicit calculation (cf. (2.17)) one finds for the right-hand side, the soft mass insertion:

$$Q_m^{(1)}(u) = \beta_\lambda^{(1)} - \tilde{\beta}_\lambda^{(1)}(u) \quad (4.6)$$

It is κ -independent as required by the consistency equation of the RG- and the CS-equation:

$$[\mathcal{R}, C]Q = \mathcal{R}Q_m \quad (4.7)$$

which means in 1-loop

$$\kappa\partial_\kappa\beta_\lambda^{(1)} = \kappa\partial_\kappa Q_m^{(1)} = 0 \quad (4.8)$$

The 2-loop order works in the same way as the 1-loop order. Taking the $Q^{(2)}(\tau, u, \lambda)$ of (3.21) one finds for the CS- β -function

$$\beta_\lambda^{(2)} = \lambda^3 \lim_{u \rightarrow 0} 2ug_2'(u) \quad (4.9)$$

Therefrom we deduce the same results as above: The β -function of the CS-equation in 2-loop order is κ -independent again, consequently in the limit $u \rightarrow 0$ the function $g_2(u)$ has logarithmic behavior. For an asymptotic normalization point the β -function of the CS-equation and of the RG-equation (3.26a) are equal.

$$\beta_\lambda^{(2)} = \lim_{\kappa^2 \rightarrow -\infty} \tilde{\beta}_\lambda^{(2)}\left(\frac{m^2}{\kappa^2}\right) \equiv b_1\lambda^3 \quad \text{and} \quad \lim_{u \rightarrow 0} g_2'(u) = \frac{1}{2}b_1 \ln(-u) \quad (4.10)$$

where we have chosen the arbitrary integration constant to be zero, using the same arguments as above (cf. (4.4)). The right-hand side of the CS-equation can be calculated in terms of the functions $g_i(y)$ and their derivatives again:

$$Q_m^{(2)}(\tau, u) = \beta_\lambda^{(2)} - \tilde{\beta}_\lambda^{(2)}(u) + 2(\beta_\lambda^{(1)} - \tilde{\beta}_\lambda^{(1)}(u))(g_1(\tau u) - g_1(u)) \quad (4.11)$$

According to (4.10, 4, 5) in the limit of asymptotic p^2 , i.e. $u \rightarrow 0$, $u\tau = \frac{m^2}{\kappa^2} = \text{const.}$, $Q_m^{(2)}$ is vanishing and therefore soft, as required. All these findings are in complete agreement with the consistency equation (4.7) of order 2-loop:

$$0 = \kappa \partial_\kappa \beta_\lambda^{(2)} = \kappa \partial_\kappa Q_m^{(2)}(\tau, u) + \tilde{\beta}_\lambda^{(1)}(\tau u) \partial_\lambda Q_m^{(1)}(\tau, u) \quad (4.12)$$

No further restrictions on $g_2(u)$ and $g_1(u)$ are required.

Due to the anti-symmetric combinations we have found in the RG-solutions from 3-loop onwards (3.22) the structure starts to become more complicated, especially the β -function of the CS-equation starts to become κ -dependent as expected. Using the arguments as above the CS-function is found to be

$$\beta_\lambda^{(3)}(u\tau) = \lambda^4 \left(\lim_{u \rightarrow 0} (2u g_3'(u) - \frac{1}{2} b_1 g_1(u) + \frac{1}{2} b_0 g_2(u)) + b_1 g_1(\tau u) - b_0 g_2(\tau u) \right) \quad (4.13)$$

Therefore $\beta_\lambda^{(3)}$ is κ -dependent, but the momentum dependent terms have to tend to a constant in the asymptotic limit,

$$\lim_{u \rightarrow 0} (2u g_3'(u) - \frac{1}{2} b_1 g_1(u) + \frac{1}{2} b_0 g_2(u)) \equiv b_2 \quad (4.14)$$

With the normalization of the functions $g_1(u)$ and $g_2(u)$ we have chosen in (4.4, 10) ($C_1 = 0$) one has:

$$\lim_{u \rightarrow 0} (b_1 g_1(u) - b_0 g_2(u)) = 0$$

and we find the logarithmic behavior of $g_3(u)$

$$\lim_{u \rightarrow 0} g_3(u) = \frac{1}{2} b_2 \ln(-u) \quad (4.15)$$

For asymptotic κ^2 the CS-function is independent of κ and agrees with the RG-function (3.26b)

$$\lim_{\kappa^2 \rightarrow \infty} \beta_\lambda^{(3)}\left(\frac{m^2}{\kappa^2}\right) = b_2 \lambda^4 = \lim_{\kappa^2 \rightarrow \infty} \tilde{\beta}_\lambda^{(3)}\left(\frac{m^2}{\kappa^2}\right) \quad (4.16)$$

For finite κ the κ -dependence of $\beta_\lambda^{(3)}$ gives a measure, how far the 2-loop function $g_2(u)$ and 1-loop function $g_1(u)$ differ in the mass-dependent region. Especially it is just the antisymmetric term in (3.22), which causes the κ -dependence of $\beta_\lambda^{(3)}$. E.g. in the case, if $g_2(u) = b_2 g_1(u)$ for all u , the antisymmetric contribution as well as the κ -dependence in the β -function of the CS-equation would cancel.

Finally looking into the consistency equation (4.7) of 3-loop order one verifies the following identities, which are valid without any restrictions on $g_i(x)$:

$$\begin{aligned} \kappa \partial_\kappa \beta_\lambda^{(3)}\left(\frac{m^2}{\kappa^2}\right) &= \tilde{\beta}_\lambda^{(2)}\left(\frac{m^2}{\kappa^2}\right) \beta_\lambda^{(1)} - \tilde{\beta}_\lambda^{(1)}\left(\frac{m^2}{\kappa^2}\right) \beta_\lambda^{(2)} \\ \text{and} \\ (\kappa \partial_\kappa + \tilde{\beta}_\lambda\left(\frac{m^2}{\kappa^2}\right) \partial_\lambda) Q_m^{(3)} &= 0 \end{aligned} \quad (4.17)$$

For the 4 loop order (3.22) the calculations work in the same way as for the 3-loop order and we only state the result:

$$\begin{aligned} \beta_\lambda^{(4)}(\tau u) &= \lambda^5 \left(\lim_{u \rightarrow 0} (2u g_4'(u)) \right. \\ &\quad \left. + 2(b_2 g_1(\tau u) - b_0 g_3(\tau u)) + g_1(\tau u)(b_1 g_1(\tau u) - b_0 g_2(\tau u)) \right) \end{aligned} \quad (4.18)$$

Therefore, the RG-function $g_4(u)$ starting the new RG-invariant of order 4-loop has logarithmic behavior for $u \rightarrow 0$, i.e. in the massless limit,

$$\lim_{u \rightarrow 0} u g_4'(u) \equiv b_3 \implies g_4(u) = \frac{1}{2} b_3 \ln(-u) \quad \text{for } u \rightarrow 0 \quad (4.19)$$

taking the normalization for the integration constant as above (4.10). In the limit of an asymptotic normalization point the β -functions (3.26c) and (4.18) agree and are given by the constant b_3 . All the results are again in perfect agreement with the consistency equation (4.7) of 4-loop order:

$$\begin{aligned} \kappa \partial_\kappa \beta_\lambda^{(4)}\left(\frac{m^2}{\kappa^2}\right) &= 2(\tilde{\beta}_\lambda^{(3)}\left(\frac{m^2}{\kappa^2}\right) \beta_\lambda^{(1)} - \tilde{\beta}_\lambda^{(1)}\left(\frac{m^2}{\kappa^2}\right) \beta_\lambda^{(3)}\left(\frac{m^2}{\kappa^2}\right)) \\ \text{and} \\ (\kappa \partial_\kappa + \tilde{\beta}_\lambda\left(\frac{m^2}{\kappa^2}\right) \partial_\lambda) Q_m^{(4)} &= 0 \end{aligned} \quad (4.20)$$

In a purely massive theory, e.g. massive ϕ^4 -theory, where all functions have to exist at $p^2 = 0$, too, one can simply derive that the RG- β -functions vanishes at $\kappa^2 = 0$. Therefore one has to test the CS-equation at $p^2 = 0$ with the explicit expressions for the β -functions and the soft insertion on the right-hand side using that the differential operator $m \partial_m + \kappa \partial_\kappa$ commutes with the test at $p^2 = 0$

$$\lim_{\kappa^2 \rightarrow 0} \tilde{\beta}_\lambda^{(i)}\left(\frac{m^2}{\kappa^2}\right) = 0 \quad i = 1, \dots, 4 \quad \text{if all fields are massive} \quad (4.21)$$

Although we did not succeed to find an all order recursion formula we are convinced that the same results will be achieved in any order of perturbation theory, especially

the structure of RG-strings and their associated logarithmic high-energy behavior. At the same time we do not expect any further restrictions on the functions $g_i(x)$ following from consistency of the RG- and CS-equation. Therefore a RG-solution with an appropriate high-energy behavior seems to be automatically consistent with the CS-equation in the one coupling theory.

With these results it is obvious that the asymptotic limit goes smoothly into the logarithmic structure of the massless theory (3.23) with $\alpha_i = \frac{1}{2}b_{i-1}$. Having derived the string structure of the RG-invariant solution and the logarithmic behavior of the $g_i(u)$ the use of improvement for a theory with a positive β -function as mentioned in the introduction can be clarified: If one has fixed the coupling at a specific normalization point $\kappa_0^2 = c_0 m^2$ for example in the low energy region and takes the limit to large p^2 , all the momentum dependent terms tend to logarithms. In each order the highest power in the logarithms is given by the power of $g_1(\frac{m^2}{p^2})$, all the other momentum dependence has lower powers in the logarithms. Therefore as long as perturbation theory is valid, the 1-loop induced invariant charge will dominate all the other contributions for large p^2 .

5. Reparametrizations of the coupling

The structure of the invariant charge we have found in (3.19, 21, 22) is related to the choice of normalization conditions (3.1) identifying the invariant charge with the coupling itself at a normalization point $p^2 = \kappa^2$ to all orders. Calculations carried out in schemes without specific normalization conditions can change this structure due to the fact that one is not able to find such a point κ^2 to all orders. But in any allowed scheme there has to be a point $p^2 = \kappa^2$ – mostly in the asymptotic region – where the invariant charge is related to a power series in the coupling¹⁾:

$$Q(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda') \Big|_{p^2=\kappa^2} = \lambda' + \rho_1 \lambda'^2 + \rho_2 \lambda'^3 + \dots \quad (5.1)$$

The change in the structure of the invariant charge arising from such normalization conditions can be taken into account by considering reparametrizations of the coupling λ :

$$\lambda(\lambda') = \lambda' + \rho_1 \lambda'^2 + \rho_2 \lambda'^3 + \dots \quad (5.2)$$

¹⁾ ρ_1 can be always made to vanish by choosing $p^2 = c\kappa^2$, with c a number.

For the invariant charge calculated with the redefined coupling

$$Q(\frac{p^2}{\kappa^2}, \frac{m^2}{p^2}, \lambda') = \sum_{k=0}^{\infty} \lambda^{k+1} f_k(\tau, u, \rho_i)$$

one gets immediately the following expression. $f_k(\tau, u)$ denotes the functions of the properly normalized invariant charge as given in (3.19, 21, 22):

$$\begin{aligned} \lambda^1 : \quad & f_0(\tau, u, \rho_i) = 1 \\ \lambda^2 : \quad & f_1(\tau, u, \rho_i) = \rho_1 + f_1(\tau, u) \\ \lambda^3 : \quad & f_2(\tau, u, \rho_i) = \rho_2 + 2\rho_1 f_1(\tau, u) + f_2(\tau, u) \\ \lambda^4 : \quad & f_3(\tau, u, \rho_i) = \rho_3 + (\rho_1^2 + 2\rho_2) f_1(\tau, u) + 3\rho_1 f_2(\tau, u) + f_3(\tau, u) \\ \lambda^5 : \quad & f_4(\tau, u, \rho_i) = \rho_4 + 2(\rho_2 \rho_1 + 2\rho_3) f_1(\tau, u) + 3(\rho_2 + \rho_1^2) f_2(\tau, u) \\ & \quad \quad \quad + 4\rho_1 f_3(\tau, u) + f_4(\tau, u) \end{aligned} \quad (5.3)$$

The leading contribution $(g_1(u\tau) - g_1(u))^n$ in the n -loop order is not affected by such a reparametrizations, whereas asymptotically the coefficients of all the lower logarithms are shifted by constants. The general structure remains: the starting of a new renormalization group invariant and the necessary appearance of lower order introduced functions. The β -functions of the RG-equation and of the CS-equation are changed according to the well-known formula:

$$\beta_{\lambda'}(\frac{m^2}{\kappa^2}, \rho_i) = \left(\frac{d\lambda'}{d\lambda}\right)^{-1} \beta_{\lambda(\lambda')}(\frac{m^2}{\kappa^2}) \quad \text{and} \quad \tilde{\beta}_{\lambda'}(\frac{m^2}{\kappa^2}, \rho_i) = \left(\frac{d\lambda'}{d\lambda}\right)^{-1} \tilde{\beta}_{\lambda(\lambda')}(\frac{m^2}{\kappa^2}) \quad (5.4)$$

As it is well-known the two lowest orders of the β -functions are not changed by such a reparametrization but are given with their κ -dependence quite generally by (3.25, 26a) and (4.3, 9). The 3-loop order is changed to

$$\tilde{\beta}_{\lambda'}^{(3)}(\frac{m^2}{\kappa^2}, \rho_i) = \tilde{\beta}_{\lambda'}^{(3)}(\frac{m^2}{\kappa^2}) + \rho_1 \tilde{\beta}_{\lambda'}^{(2)}(\frac{m^2}{\kappa^2}) - \rho_2 \tilde{\beta}_{\lambda'}^{(1)}(\frac{m^2}{\kappa^2}) + \rho_1^2 \tilde{\beta}_{\lambda'}^{(1)}(\frac{m^2}{\kappa^2}) \quad (5.5)$$

with the equivalent formula for the CS- β -function, too. One should notice that in a massive theory with general non-asymptotic normalization conditions one cannot make the β -functions of the CS- or RG-equation vanish from 3-loop order onwards by a reparametrization as it is possible for the massless theory, because the explicit κ -dependence will just be canceled if $g_1(y) \sim g_2(y)$. Such a cancellation would be intrinsic to a theory and could not be forced from outside.

6. The asymptotic invariant charge of the scalar particle in the spontaneously broken $U(1)$ -axial model

In this section we will apply the methods developed above to the spontaneously broken $U(1)$ -axial model with a scalar/pseudoscalar doublet A/B and one fermion ψ [9]: the classical action is given by

$$\Gamma_{cl} = \int \left(\frac{1}{2} (\partial_\mu A \partial_\mu A + \partial_\mu B \partial_\mu B) + i \bar{\psi} \not{\partial} \psi - m_f \bar{\psi} \psi - \frac{m_f}{m_H} \sqrt{\frac{1}{2}} \bar{\psi} (A + i \tau_2 B) \psi - \frac{1}{2} m_H^2 A^2 - \frac{1}{2} m_H \sqrt{\frac{1}{2}} A (A^2 + B^2) - \frac{\lambda}{4!} (A^2 + B^2)^2 \right) \quad (6.1)$$

In contrast to the models considered in section 4 the GS-equation and RG-equation do not have the same structure, i.e. do not contain the same hard differential operators in the physical parametrization, where the physical masses are fixed at the pole of the respective propagators. But from the point of view of the RG-equation it is a one-coupling theory as the models considered above. Therefore the whole analysis of sect. 3 is valid for the invariant charge of the scalar coupling especially the ordering according to RG-invariant functions can be applied as it is. The crucial point is the action of the GS-equation on the RG-invariant functions $g_i(y)$. In order to simplify the analysis and to get a first impression of the consequences of the physical mass β -function in the GS-equation we consider the invariant charge normalized at an asymptotic normalization point and consequently at asymptotic momenta. As a result we derive that the RG-functions $g_i(y)$ contain well-defined powers of logarithms in higher orders inducing at the same time a certain κ -dependence into the β -functions, which does not vanish in the asymptotic region.

Before turning to the calculations a few words about the normalization conditions we impose are in order: Throughout the paper we have normalized the masses on-shell. Especially in the spontaneously broken models considerations concerning the RG-equation are often presented in the symmetric way specifying the couplings and taking the shift parameter v parametric in the Ward identity. If such considerations are carried out with mass-independent β -functions one calculates the symmetric limit. Moreover there is no way to reach the massive region by a renormalization group integration for such parametrizations even in lowest order or in any approximation, because typically terms as $\frac{\lambda v}{\kappa^2} \ln \frac{\lambda v}{\kappa^2}$ appear in the β -functions. This is not just a annoying technical problem, but it is deeply connected with the fact, that the

order by order finite RG-invariance in the sense we have formulated in sect. 2 is lost. In order to detect finite RG-invariance in perturbation theory one has to expand the RG-functions $g_i(y)$, i.e. the Green functions, as a Taylor series in the coupling:

$$g_i \left(\frac{m^2}{p^2} + \rho \lambda \frac{m^2}{p^2} \right) = g_i \left(\frac{m^2}{p^2} \right) + \rho \lambda \frac{m^2}{p^2} g_i' \left(\frac{m^2}{p^2} \right) + \dots \quad (6.2)$$

Such an expansion is only valid for a restricted range of momenta. Therefore normalization conditions, which do not fix the pole of the propagator, seem to us to make the perturbative expansion even worse than it is expected to be anyway.

Apart from these theoretical aspects the $U(1)$ -axial model can be considered as a toy model of the matter sector of the standard model: One reason for carrying out such a RG-analysis is to get estimates of the higher orders, as e.g. of the normalization point dependence, if one knows the 1-loop order completely. With this aim in mind one has to parametrize in a way as it is done in realistic models. In the standard model the masses of the particles are known to a high accuracy and are therefore taken as an experimental input for all further calculations. Only with such normalization conditions one can expect to get useful results for the 2- or 3 loop order.

The Green functions of the model are constructed according to the Gell-Mann Low formula and with a suitable renormalization prescription, which we do not specify, because we just consider the finite Green functions. The model is defined by the requirement that the Green functions fulfill the Ward-identity of the spontaneously broken $U(1)$ -axial symmetry

$$W \Gamma = 0 \quad \text{with} \quad W = -i \int \left((A+v) \frac{\delta}{\delta B} - B \frac{\delta}{\delta A} - \frac{i}{2} \frac{\delta}{\delta \psi} \gamma_5 \psi - \frac{i}{2} \bar{\psi} \gamma_5 \frac{\delta}{\delta \bar{\psi}} \right), \quad (6.3)$$

and by normalization conditions to fix the free parameters of the theory:

$$\begin{aligned} \partial_{p^2} \Gamma_{AA} |_{p^2=\kappa^2} &= 1 & \gamma^\mu \partial_{p^\mu} \Gamma_{\psi\psi} |_{p=\kappa} &= 1 & \Gamma_A &= 0 \\ \Gamma_{AA} |_{p^2=m_H^2} &= 0 & \Gamma_{\psi\psi} |_{p=m_f} &= 0 & & \\ \Gamma_{AAA} |_{(p_i)=\kappa_{sym}} &= -\lambda & & & & \end{aligned} \quad (6.4)$$

With these normalization conditions the Green functions are calculated perturbatively in powers of the coupling $\sqrt{\lambda}$, the shift parameter is determined by the Ward-identity:

$$\dot{v} \equiv \dot{v}(m_H, m_f, \kappa, \lambda) = \sqrt{\frac{3}{\lambda}} m_H + O(\hbar) \quad (6.5)$$

In ref. [9] we have shown that the CS-equation exists to all orders with a soft insertion on the right-hand side:

$$(m\partial_m + \beta_{m_f} m_f \partial_{m_f} + \beta_\lambda \partial_\lambda - \gamma_B \mathcal{N}_B - \gamma_F \mathcal{N}_F) \Gamma = \hat{v} (1 + \rho) \int \left(\frac{\delta}{\delta A} + \hat{\alpha} \frac{\delta}{\delta q} \right) \Gamma \Big|_{q=0} \quad (6.6)$$

where q is an external field coupled to the invariant of infrared dimension 2 and

$$\begin{aligned} m\partial_m &= m_H \partial_{m_H} + m_f \partial_{m_f} + \kappa \partial_\kappa \\ \mathcal{N}_B &= \int \left(A \frac{\delta}{\delta A} + B \frac{\delta}{\delta B} \right) \quad \mathcal{N}_F = \int \left(\psi \frac{\delta}{\delta \psi} + \bar{\psi} \frac{\delta}{\delta \bar{\psi}} \right) \\ \hat{v} \rho &= (\beta_\lambda \partial_\lambda + \beta_{m_f} m_f \partial_{m_f} + \gamma_B) \hat{v} \end{aligned} \quad (6.6a)$$

The renormalization group equation has due to the physical normalization condition the simple form

$$(\kappa \partial_\kappa + \beta_\lambda \partial_\lambda - \tilde{\gamma}_B \mathcal{N}_B - \tilde{\gamma}_F \mathcal{N}_F) \Gamma = 0 \quad (6.7)$$

with the additional constraint:

$$\kappa \partial_\kappa \hat{v} + \beta_\lambda \partial_\lambda \hat{v} + \tilde{\gamma}_B \hat{v} = 0 \quad (6.7a)$$

The invariant charge of the scalar field A defined according to (2.6) with $\Gamma_2 \equiv \Gamma_{AA}$ and $\Gamma_4 \equiv \Gamma_{AAAA}$ has the same properties as the invariant charge of the scalar models: It is dimensionless

$$Q(p^2, m_f, m_H, \kappa, \lambda) = Q\left(\frac{p^2}{\kappa^2}, \frac{m_f}{p^2}, \frac{m_H}{m_H}, \lambda\right), \quad (6.8)$$

it has well-defined normalization properties

$$Q\left(\frac{p^2}{\kappa^2}, \frac{m_f}{p^2}, \frac{m_H}{m_H}, \lambda\right) \Big|_{p^2=\kappa^2} = \lambda \quad (6.9)$$

and furthermore it is a RG-invariant, i.e. it satisfies the homogeneous RG-equation and the CS-equation without anomalous dimension.

A complete analysis of the 1-loop induced higher order contributions requires the complete knowledge of the finite Green functions in 1-loop order. All these Green functions are calculated in the literature but for a first approach we want to restrict ourselves to asymptotic normalization conditions taking κ^2 in the asymptotic region. Therefore one will only obtain information about the invariant charge in

the asymptotic region where the momenta are large compared to the masses. Because we have fixed all the physical masses the asymptotic invariant charge of the spontaneously broken model is not equivalent to the one of the massless symmetric model especially as far as dependence on the masses and the normalization point is concerned.

The 1-loop β -functions of the model in the asymptotic normalization are given by

$$\begin{aligned} \beta_{\lambda,as}^{(1)} = \beta_\lambda^{(1)} &= \frac{1}{8\pi^2} \frac{1}{3} \left(5 - 8 \frac{m_f^4}{m_H^4} + 4 \frac{m_f^2}{m_H^2} \right) \lambda^2 \equiv b_\lambda^{(1)} \left(\frac{m_f}{m_H} \right) \lambda^2 \\ \beta_{m_f}^{(1)} &= -\frac{1}{16\pi^2} \frac{1}{3} \left(5 - 8 \frac{m_f^4}{m_H^4} \right) \lambda \equiv b_{m_f}^{(1)} \left(\frac{m_f}{m_H} \right) \lambda \end{aligned} \quad (6.10)$$

By subtracting the RG-equation and the CS-equation one can show that the β -function of the CS-equation β_λ is identical to the β -function of the RG-equation $\tilde{\beta}_{\lambda,as}$ for asymptotic normalization conditions. But in this case, where the CS- and the RG-equation have not the same structure, they are allowed to depend on κ (cf. [11]).

$$\tilde{\beta}_{\lambda,as} \left(\frac{m_H^2}{\kappa^2}, \frac{m_f}{m_H} \right) = \beta_{\lambda,as} \left(\frac{m_H^2}{\kappa^2}, \frac{m_f}{m_H} \right) \quad (6.11)$$

As a simple check on the difference to the models considered in section 4, we integrate the RG-equation with the 1-loop β -function (6.10)

$$(\kappa \partial_\kappa + b_\lambda^{(1)}(\alpha) \partial_\lambda) Q(\tau, u, \alpha, \lambda) = 0 \quad (6.12)$$

with the asymptotic result

$$\bar{Q}_{1,as} = \frac{\lambda}{1 - \frac{1}{2} \lambda b_\lambda^{(1)}(\alpha) \ln \tau} = \sum_{k=0}^{\infty} \lambda^{k+1} \left(\frac{1}{2} b_\lambda^{(1)}(\alpha) \ln \tau \right)^k \quad (6.13)$$

Thereby we have denoted

$$\tau = \frac{p^2}{\kappa^2}, \quad u = \frac{m_H^2}{p^2}, \quad \alpha = \frac{m_f}{m_H} \quad (6.13a)$$

Whereas in the models of section 4 RG-invariants are respected by the CS-equation, it is obvious that the RG-invariant $\bar{Q}_{1,as}$ is not a solution of the CS-equation from 2-loop order onwards but is broken by hard terms:

$$\begin{aligned} C\bar{Q}_{1,as} &= (\kappa \partial_\kappa + m_H \partial_{m_H} + \beta_{m_f}^{(1)} \alpha \partial_\alpha + \beta_\lambda^{(1)}) \bar{Q}_{1,as} = \beta_{m_f}^{(1)} \alpha \partial_\alpha \bar{Q}_{1,as} \\ &= \lambda^3 b_{m_f}^{(1)}(\alpha) \alpha \partial_\alpha b_\lambda^{(1)}(\alpha) + O(\lambda^4) \neq 0 \end{aligned} \quad (6.14)$$

In order to get insight into the problems arising thereby we turn to the finite RG-transformations as formulated in sect. 3. As we have pointed out in section 3.2, a further dependence on the mass ratio α does not affect the analysis of section 3.1, as long as all masses are normalized on-shell. Therefore the structure of the 1 lowest orders is given by (3.19, 21) and (22), where now the RG-functions depend on α , too:

$$g_i(y) \longrightarrow g_i(y, \alpha) \quad (6.15)$$

Therefore we have

$$\begin{aligned} Q^{(1)}(\tau, u, \alpha, \lambda) &= \lambda^2 (g_1(u\tau, \alpha) - g_1(u, \alpha)) \\ Q^{(2)}(\tau, u, \alpha, \lambda) &= \lambda^3 (g_2(\tau u, \alpha) - g_2(u, \alpha) + (g_1(\tau u, \alpha) - g_1(u, \alpha))^2) \end{aligned} \quad (6.16)$$

and respectively for the 3 and 4-loop order. The β -functions of the RG-equation are therefore likewise determined by the expressions (3.25, 3.26), where the ordinary derivative is replaced by a partial one with respect to $u\tau$, e.g.:

$$\begin{aligned} \tilde{\beta}_\lambda^{(1)}(y, \alpha) &= 2y \partial_y g_1(y, \alpha) \lambda^2 \\ \tilde{\beta}_\lambda^{(2)}(y, \alpha) &= 2y \partial_y g_2(y, \alpha) \lambda^3 \end{aligned} \quad (6.17)$$

The string structure and the β -functions belonging to it are the only information contained in finite RG-invariance. In order to get restrictions on the RG-functions $g_i(y, \alpha)$ we have – as above – to use the CS-equation. From now on we restrict our considerations to an asymptotic normalization point and consequently asymptotic momenta, which means for the 1-loop expression

$$Q_{\text{as}}^{(1)} = \frac{1}{2} \lambda^2 b_\lambda^{(1)}(\alpha) (\ln |\frac{m^2}{\kappa^2}| - \ln |\frac{m^2}{\mu^2}|) \quad (6.18a)$$

and therefore

$$g_{1, \text{as}}(y) = \frac{1}{2} b_\lambda^{(1)}(\alpha) \ln(-y) \quad (6.18b)$$

It fulfils the asymptotic CS-equation in 1-loop with the same β -function as in the RG-equation (cf. (6.10)). We apply the CS-equation on the 2-loop RG-invariant (6.16) taking into account that the right-hand side, the soft insertion, is vanishing and find

$$\begin{aligned} \beta_{\lambda, \text{as}}^{(2)}(u\tau, \alpha) &= \lambda^3 (2u \partial_u g_{2, \text{as}}(u, \alpha) - \frac{1}{2} b_{m_f}(\alpha) \partial_\alpha b_\lambda^{(1)} \ln(-u) \\ &\quad + \ln(-u\tau) \frac{1}{2} b_{m_f}(\alpha) \partial_\alpha b_\lambda^{(1)}(\alpha)) \end{aligned} \quad (6.19)$$

Because the CS-equation is seen to exist from the general proof in [9] we know that the momentum dependent terms have to sum up to a constant – these are the same arguments as in section 4 – , i.e.:

$$g_{2, \text{as}}(u, \alpha) = -\frac{1}{8} b_{m_f}^{(1)}(\alpha) \partial_\alpha b_\lambda^{(1)}(\alpha) \ln^2(-u) + \frac{1}{2} b_\lambda^{(2)}(\alpha) \ln(-u) \quad (6.20)$$

where we have fixed the integration constant to zero (cf. comments to (4.4)). Therefore the 2-loop RG-function will start with a quadratic term in the logarithm of the same power as $g_{1, \text{as}}^2(y)$. Summarizing the results we find for the 2-loop invariant charge:

$$\begin{aligned} Q_{\text{as}}^{(2)} \frac{m_f^2}{\kappa^2}, \frac{m_f^2}{\mu^2}, \alpha, \lambda &= \lambda^3 \left(\frac{1}{2} b_\lambda^{(2)}(\alpha) \ln(\frac{E^2}{\kappa^2}) - \frac{1}{8} b_{m_f}^{(1)}(\alpha) \partial_\alpha b_\lambda^{(1)}(\alpha) (\ln^2 |\frac{m_f^2}{\kappa^2}| - \ln^2 |\frac{m_f^2}{\mu^2}|) \right. \\ &\quad \left. + (b_\lambda^{(1)}(\alpha) \ln(\frac{E^2}{\kappa^2}))^2 \right) \end{aligned} \quad (6.21)$$

and the β -function depends on $\ln |\frac{m^2}{\kappa^2}|$

$$\beta_{\lambda, \text{as}}^{(2)} = \tilde{\beta}_\lambda^{(2)} = \lambda^3 \left(-\frac{1}{2} b_{m_f}^{(1)}(\alpha) \partial_\alpha b_\lambda^{(1)}(\alpha) \ln |\frac{m^2}{\kappa^2}| + b_\lambda^{(2)}(\alpha) \right) \quad (6.22)$$

$b_\lambda^{(2)}(\alpha)$ is a true 2-loop function. In contrast to the scalar models the 2-loop function starts with a quadratic term in the logarithm of the same power as the 1-loop induced RG-invariant $g_{1, \text{as}}^2(y)$ appearing in 2-loop order. At the same time the β -function starts to depend logarithmically on the ratio $\frac{m^2}{\kappa^2}$. Interestingly enough, an asymptotic theory in the sense of mass-independence, if the normalization point and all momenta are taken at infinity, does not exist: The asymptotic normalization conditions are defined by the requirement that the terms of order $\frac{m^2}{\kappa^2} \ln |\frac{m^2}{\kappa^2}|$ can be neglected. But the smaller these terms are chosen, the larger the logarithmic term $\ln \frac{m^2}{\kappa^2}$ in the 2-loop invariant charge will grow.

The 2-loop RG-function we have calculated in (6.22) is again in agreement with the consistency equation (4.7) tested for the invariant charge

$$\kappa \partial_\kappa \beta_{\lambda, \text{as}}^{(2)} = \beta_{m_f}^{(1)} \alpha \partial_\alpha \beta_{\lambda, \text{as}}^{(1)} \quad (6.23)$$

For completeness we want to give also the 3-loop order β -functions and the RG-function $g_{3, \text{as}}(y)$. The calculation works as it did in 2-loop order, whereby now also

the κ -dependent part of the β -function $\beta_{m_f,as}^{(2)}$ is determined:

$$\beta_{m_f,as}^{(2)} = \lambda^2 \left(\frac{1}{2} b_{m_f}^{(1)}(\alpha) b_\lambda^{(1)}(\alpha) \ln \left| \frac{m_f^2}{\kappa^2} \right| + b_{m_f}^{(2)}(\alpha) \right) \quad (6.24a)$$

and

$$\begin{aligned} \beta_{\lambda,as}^{(3)} = \lambda^4 \left(-\frac{1}{8} b_{m_f}^{(1)}(\alpha) b_\lambda^{(1)}(\alpha) \alpha \partial_\alpha b_\lambda^{(1)}(\alpha) \ln^2 \left| \frac{m_f^2}{\kappa^2} \right| \right. \\ \left. + \frac{1}{8} b_{m_f}^{(1)}(\alpha) \alpha \partial_\alpha \left(b_{m_f}^{(1)}(\alpha) \alpha b_\lambda^{(1)}(\alpha) \right) \ln^2 \left| \frac{m_f^2}{\kappa^2} \right| \right. \\ \left. - \frac{1}{2} \left(b_{m_f}^{(1)}(\alpha) \alpha \partial_\alpha b_\lambda^{(3)}(\alpha) + b_{m_f}^{(2)}(\alpha) \alpha \partial_\alpha b_\lambda^{(1)}(\alpha) \right) \ln \left| \frac{m_f^2}{\kappa^2} \right| \right. \\ \left. + b_\lambda^{(3)}(\alpha) \right) \end{aligned} \quad (6.24b)$$

and $g_{3,as}(y, \alpha)$ is calculated to be

$$\begin{aligned} g_{3,as}(y, \alpha) = \frac{1}{24} b_{m_f}^{(1)}(\alpha) \alpha \partial_\alpha \left(b_{m_f}^{(1)}(\alpha) \alpha b_\lambda^{(1)}(\alpha) \right) \ln^3(-y) \\ - \frac{1}{24} b_{m_f}^{(1)}(\alpha) b_\lambda^{(1)}(\alpha) \alpha \partial_\alpha b_\lambda^{(1)}(\alpha) \ln^3(-y) \\ - \frac{1}{8} \left(b_{m_f}^{(1)}(\alpha) \alpha \partial_\alpha b_\lambda^{(2)}(\alpha) + b_{m_f}^{(2)}(\alpha) \alpha \partial_\alpha b_\lambda^{(1)}(\alpha) \right) \ln^2(-y) \\ + \frac{1}{2} b_\lambda^{(3)}(\alpha) \ln(-y) \end{aligned} \quad (6.25)$$

As expected the 3-loop RG-invariant depends on $\ln \frac{m^2}{\kappa^2}$ to the third power and the β -function to the second power. Note that the β -function β_λ of three loop order is not just given by the differentiation of $g_3(y)$ but has according to (6.17) an anti-symmetric contribution of the 1- and 2-loop order. $Q_{as}^{(3)}$ can immediately be calculated by inserting (6.25, 6.20, 6.18b) into the 3-loop expression of (3.22). The β -functions (6.24) are in agreement with the consistency equation:

$$\begin{aligned} \kappa \partial_\kappa \beta_{m_f,as}^{(2)} \partial_\alpha Q_{as}^{(1)} &= -\beta_\lambda^{(1)} \partial_\lambda \beta_{m_f}^{(1)} \partial_\alpha Q_{as}^{(1)} \\ \kappa \partial_\kappa \beta_{\lambda,as}^{(3)} &= \beta_{m_f,as}^{(2)} \alpha \partial_\alpha \beta_\lambda^{(1)} + \beta_{m_f}^{(1)} \alpha \partial_\alpha \beta_{\lambda,as}^{(2)} \end{aligned} \quad (6.26)$$

Although we have used the RG-invariance and the CS-equation only in order to compute the invariant charge of the scalar field in a restricted range of momenta, namely asymptotic ones, the results show that there is a far reaching difference between the spontaneously broken model, which contains only scalar fields, and the one, which contains fermions: In the pure scalar model considered in section 4 strings of lower loop induced contributions are in one to one correspondence with the RG-invariants. This means in particular, that one will get a sensible result, if one puts higher order RG-invariants to zero and calculates with the lower orders

in some approximation a RG-invariant result. The most prominent example is the string of the 1-loop induced RG-invariant as calculated from the differential equation including the 1-loop β -functions and all higher order β -function taken to be zero. In the model with fermions as considered in this section such an separation in RG-invariants and e.g. the 1-loop contribution is not possible anymore: Through the CS-equation 1-loop induced contributions appear in every RG-invariant, therefore the approximate solution of the RG-equation (6.12) neglecting all higher order β -functions does not make any sense. This result is general and not related to the asymptotic normalization condition we have chosen for simplification. Another important difference to the pure scalar models is the fact, that in presence of fermions an asymptotic limit does not exist. Considering the three lowest orders and taking into account the consistency equation of the CS- and the RG-equation we can conclude that in the $(n+1)$ -loop order there appears a logarithmic term of the ratio $\frac{m^2}{\kappa^2}$ to the n^{th} power

$$\beta_{\lambda,as}^{(n+1)} \sim \ln \left| \frac{m^2}{\kappa^2} \right| \quad (6.27)$$

The logarithmic dependence on the ratio of mass and normalization point can be understood from actually having fixed two interactions at different scales: The scalar interaction by the choice of the normalization point and the Yukawa interaction by the Ward-identity connected with the pole of the fermion propagator, the physical fermion mass. To fix couplings at different scales is a very realistic scenario thinking at the wide range of masses, which appear in the standard model. Concluding from the calculations above, this means, that one has to expect under such circumstances a sensitive dependence on the point where one has normalized the couplings, i.e. adjusted at their experimental value.

These considerations of the spontaneously broken case have to be understood as a first look into the usefulness of structuring Green functions with the help of RG-invariance and the CS-equation. For all further applications it is unavoidable to use the complete 1-loop order with all finite diagrams. Then one can try to estimate, which terms have to be expected necessarily in the next order.

7. Conclusions

In this paper our principal aim was to gain information on the structure of the invariant charge in a 1-coupling theory from a combined use of the CS-equation and RG-invariance. Although they both look similar in the infinitesimal version, in which they are derived in perturbation theory, their meaning is completely different. RG-invariance is – up to field redefinitions – a symmetry of the Green functions, which has to be realized in order to make the outcome of calculations independent of the arbitrary normalization point, one has to choose to adjust the coupling to its experimental value. The invariance under RG-transformations thus makes the calculations universal and is one aspect, in which a renormalizable field theory is distinguished from an effective field theory. In contrast to this there does not exist a compelling physical reason to require dilatational invariance. Dilatations are broken already classically by the mass terms and moreover in most 4-dimensional quantum field theories by hard anomalies. But in the models we have considered in this paper the action of dilatations on the Green functions can be expressed in form of a partial differential equation, the CS-equation.

As usual for every symmetry in principle RG-invariance needs to be only realized for physical observables as for example the S-matrix, but if it holds for the Green functions up to field redefinitions, the S-matrix elements are invariant as a consequence. In a 1-coupling theory finite RG-invariance of the invariant charge can be deduced from the formal integration of the RG-equation. Therefore the invariant charge, as it is calculated in perturbation theory, is an invariant under RG-transformations. Because RG-invariance is realized only to all orders of perturbation theory, every order of the perturbative power series induces contributions to the next one necessarily. We have shown that RG-invariance structures the Green functions according to their transformation properties: In particular apart from a new RG-invariant strings of lower order RG-functions run through all orders of perturbation theory. The RG-invariants themselves start with arbitrary functions depending in an unique way on the renormalization point and the mass. Concerning the special form of the RG-functions $g_i(y)$ or their high-energy behavior one does not get any further information from RG-invariance in a massive theory.

As we have shown, it is the existence of the CS-equation which restricts the high-

energy behavior of the RG-functions appearing in the structure of the invariant charge. In such models, where the CS-equation has exactly the same form as the RG-equation, it turns out that all RG-functions tend to single logarithms in the asymptotic limit and as a consequence the β -functions are mass independent in this limit. Therefore the asymptotic limit goes smoothly into the massless theory. Apart from this high-energy restrictions every RG-invariant is consistent with the CS-equation by itself. This is in marked contrast to the $U(1)$ -axial model with fermions, which get their mass via the spontaneous breaking of the symmetry. There the CS-equation contains a β -function belonging to a physical mass differential operator. In this case the individual RG-invariants are not by themselves solutions of the CS-equation, but in order to satisfy the CS-equation they have to contribute all at the same time. In every RG-invariant there appears a 1-loop induced contribution, which means, all RG-invariants are excited by the 1-loop contribution, none can be neglected against the other. Moreover the β -function of the RG-equation and the CS-equation depend in the asymptotic limit logarithmically on the ratio of mass and normalization point and the same happens for the invariant charge from two loop order onwards. Consequently a massless limit in the asymptotic region does not exist anymore.

The structuring according to RG-invariants combined with the respective high-energy behavior as derived from the CS-equation is certainly a helpful tool in any check of calculations beyond 1-loop order. As it can be seen from the results in section 6 there appear also in such simple models as the spontaneously broken model with fermions plenty of differently ordered 1-loop induced terms, which can be controlled knowing the complete 1-loop order of the respective model. For such practical applications it seems to be of utmost interest to extend the considerations to general Green functions including the anomalous dimension.

Acknowledgements The author is grateful to C. Becchi, J. Gasser, H. Leutwyler, P. Minkowski and K. Sibold for useful discussions.

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