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ON THE DERIVATION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATION FOR HAMILTONIAN PARTICLE SYSTEMS.

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Abstract. We consider a Hamiltonian particles system, interacting by means of a pair potential. We look at the behavior of the system on a space scale of order ε^{-1}, times of order ε^{-2}, and mean velocities of order ε, with ε a scale parameter. Assuming that the phase space density of the particles is given by a series in ε (the analogous of the Chapmann-Enskog expansion) the behavior of the system under this rescaling is described, to the lowest order in ε, by the incompressible Navier-Stokes equations. The viscosity is given in terms of microscopic correlations, and its expression agrees with the Green-Kubo formula.

1. Introduction.

A system of many interacting particles, moving according to the Newton equations of motion, can be described on a space scale much larger than the typical microscopic scale (say the range of the interaction) in terms of density, velocity and temperature fields, satisfying hydrodynamic equations, like Euler or Navier-Stokes equations. The scale separation and the local conservation laws are responsible of this reduced description. In fact, on the macroscopic scale the quantities which are locally conserved (slow modes) play a major role in the motion of the fluid. The derivation of the Euler equations is based on the assumption of local equilibrium. On times of order ε^{-1}, the system is expected to be described approximately by a local Gibbs measure, with parameters varying on regions of order ε^{-1}, if ε is a scale parameter. The local equilibrium assumption implies that the parameters of the local Gibbs measures satisfy the Euler equations [1], [2]. The microscopic structure (the potential) appears only in the state equation which links the pressure to the other macroscopic parameters. The microscopic locally conserved

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quantities converge, as $\varepsilon \to 0$, by a law of large numbers, to macroscopic fields. To make this correct, the many particles Hamiltonian system must have good dynamical mixing properties to approach and stay in a state close to the local equilibrium. At the moment it is not understood how to provide such properties. Therefore the only rigorous results are obtained by adding some noise to the Hamiltonian evolution [3] (see [4] for a review on the rigorous results for stochastic systems).

The situation is very different for the derivation of the Navier-Stokes equations. These equations, which describe the behavior of a fluid in the presence of dissipative effects, do not have an immediate interpretation in terms of scale separation. This is not surprising because the NS equations do not have a natural space-time scale invariance like the Euler equations. In fact to see the effect of the viscosity and the thermal conduction one has to look at times such that neighboring regions in local equilibrium exchange a sensible amount of momentum and energy. Simple considerations show that the right scale of time is ε^{-2}. On the other hand we cannot hope to find the Navier-Stokes behavior under the rescaling $x \to \varepsilon^{-1}x$ and $t \to \varepsilon^{-2}t$ since the NS equations are not invariant under this scaling, due to the presence of the transport terms. Therefore we consider the incompressible limit simultaneously, because the incompressible Navier-Stokes equations (INS) have the required scaling invariance.

To explain this point let us recall the derivation of the hydrodynamical equations from the Boltzmann equation, which describes the large scale dynamics of a gas in the low density or kinetic regime. In this regime the typical scales are the mean free path and the mean free time, and every particle undergoes only once in a while to a collision. To recover the hydrodynamical behavior one has to look at the system on space and time scales which are very long with respect to the mean free path and the mean free time, in such a way that every particle can have so many collisions that in the macroscopic time it has thermalized. To be precise, it is has been proved [5] that, if we rescale both space and time by ε^{-1}, the solutions of the rescaled Boltzmann equation looks like a Maxwellian with parameters solving the Euler equations, for small ε. Now ε is the scale separation parameter between the mean free path and the typical macroscopic scale. To get sensible viscous effects, the time has to be of order ε^{-1} compared to the Euler times, hence one has to consider the parabolic space-time scaling ($x = \varepsilon^{-1}x', t = \varepsilon^{-2}t'$). On this time scale one makes the transport term finite by taking the Mach number $Ma = U/c$ (where U is a typical velocity and c is the sound velocity) of order ε. This corresponds to the incompressible regime. In [7] (see also [8] for the non smooth case), it is proved that, if $u(x,t)$ is a sufficiently smooth solution of the incompressible Navier-Stokes equation on a torus for $t \in [0, t_0]$, one can construct a solution f^ε to the rescaled (parabolically)

Boltzmann equation such that, for $t \in [0, t_0]$,

$$||f^\epsilon - M(\rho, \epsilon u, T)||_\infty < c\epsilon^2 \tag{1.1}$$

where ρ and T are given positive constants.

In the incompressible regime the macroscopic state is described by a divergenceless velocity field $u(x,t)$, constant density ρ and constant temperature T. The pressure $p(x,t)$ appearing in the equation is no longer related to the termodynamic parameters by means of a state equation, but has simply the meaning of a lagrangian multiplier for the constraint div$u = 0$. The INS equations are

$$\text{div} u = 0 \tag{1.2}$$

$$\rho\frac{\partial u}{\partial t} + \rho u \cdot \nabla u - \eta \Delta u = -\nabla p \tag{1.3}$$

η is the viscosity and the microscopic interactions enter only in the determination of it.

Above dimensional analysis of course can be carried out as well for a particle system. Along this path, in this paper we give a formal derivation of the INS from a Hamiltonian particles system under the parabolic rescaling, in the low Mach number regime.

The main ingredient is the assumption that the non-equilibrium density can be expressed as a truncated series in the parameter ϵ. We follow a procedure inspired to the Hilbert-Chapmann-Enskog expansion, used to construct the solution of the rescaled Boltzmann equation. From the physical point of view we think of the system as beying in local equilibrium with parameters which are themselves given by a series in ϵ. However there is a non-hydrodynamic correction to the local equilibrium which depends from the non conserved quantities in the system (fast modes) and we assume that this correction does not effect the first order in the expansion, that is at the first order the system is still described by a local Gibbs measure with parameters which differ from constants by terms of order ϵ. This is strictly related to the incompressibility assumption and would be false in the case of finite Mach number. This assumption is the translation of (1.1) to the particle system case. On the other hand, the non hydrodynamical corrections in the second order are important on the scale ϵ^{-2} and give rise to the N.S. terms. It is worth to mention here that, far from being able to give any proof, very strong and rather uncontrollable assumptions are necessary even to give sense to the formal calculations below:

i) the space of the invariant observables for the microscopic dynamics reduces to the locally conserved quantities, mass, momentum and energy;

ii) some equilibrium time correlation functions decay sufficiently fast.

Under this assumptions it is possible to determine the form of the lowest non hydrodynamic correction and thus conclude that the conservation equations are well approximated by the INS for ϵ small.

We only examine the expected values of the empirical fields and do not look for a law of large number. The reason for this choice is that the candidate for approximating the mean velocity field is, as explained below,

$$\epsilon^{d-1} \sum_i v_i \delta(x_i - x) \tag{1.4}$$

In dimension 1 and 2 the fluctuations of momentum are of the same order or even bigger than the the quantity in (1.4). Hence it cannot converge to a deterministic field and a law of large numbers is not true. The situation is different in higher dimension, but we confine ourselves to the analysis of the averages.

Our derivation gives the viscosity coefficient η in terms of a global equilibrium time correlation function

$$\eta = \frac{1}{2T}\int_0^\infty d\tau \int d\xi \left\langle \bar{w}^{12}(\xi,\tau)\bar{w}^{12}(0,0)\right\rangle \tag{1.5}$$

where $\bar{w}$ is the "modified"velocity current tensor, defined in Sect.3. The fluctuation-dissipation theory relates the transport coefficients to time-integrated correlation functions [9], [10]. The expression we find agrees with the Green-Kubo formula for the viscosity. We obtain also the Green-Kubo formula for the bulk viscosity ζ, but as it is well known, it does not appear in the INS. In our case the viscosity is not a function of space and time since the correlation is evaluated at the global equilibrium, as a consequence of the fact that we are studying the incompressible regime, where density and temperature are constant. As a by-product of our analysis, we find the analog of the Boussinesq condition for the first correction P_1 of the termodinamic pressure P_ϵ. In fact we have

$$\nabla P_1 = 0 \tag{1.6}$$

Moreover the conservation law for the energy gives, in the incompressible approximation, an equation for the first correction to the temperature T_1

$$c(\partial_t T_1 + u \cdot T_1) = \kappa \Delta T_1 \tag{1.7}$$

where κ is the conductivity which is given again in terms of an integrated equilibrium time correlations and c is the specific heat at constant pressure. Our approach to derive the Green-Kubo formulas sounds similar to the one followed by Green [11]. It is an alternative way to obtain them with respect the linear response theory of Green and

Kubo. A different derivation is due to Zubarev [12], which proposes a specific form of the non-equilibrium distribution which allows to derive the expressions of the transport coefficients as well as the Navier-Stokes corrections [13].

Finally we remark again that, to try to prove the hydrodynamic limit under the above rescaling, we need ergodic properties of the Hamiltonian system even stronger than the ones needed for the Euler regime. Of course they are beyond the available mathematical techniques. It is therefore natural to try to explore this setting in some simple models with stochastic dynamics which have all the necessary ergodic properties. This program has been succesfully accomplished in [14] for the asymmetric simple exclusion process (ASEP) (see [4,15] for references) in dimension bigger than 2. In fact the ASEP is probably the simplest non trivial model for which an analog of the incompressible limit makes sense. In this model there is only one conserved quantity, the density and the analog of the Euler limit is well known [16]. The limiting equation for the density is the non-viscous Burgers equation. The diffusive scaling limits presents mostly the same difficulties (absence of scale invariance) discussed above, for non weak asymmetries. Moreover, from the technical point of view, the study of the model is made difficult from the fact that it is a non-gradient system (in the sense that the current is not the "gradient" of some function, see [4]) and the dynamics does not satisfy the detailed balance with respect to the invariant product measure. In [14] it is considered a product initial state with a density profile with spatial fluctuation of order ε (the analog of the assumption of mean velocities of order ε in Hamiltonian systems). Using the Varadhan method [17] of dealing with non-gradient systems, the entropy method [18, 3] and a kind of multiscale analysis, it is possible to prove that, with probability 1, the fluctuation of the empirical density around the constant profile is of order ε and the rescaled fluctuation of density, after a suitable space shift of order ε^{-1}, satisfies the viscous Burgers equation in the limit $\varepsilon \to 0$. An important feature, to our purposes, is that the diffusion matrix computed in [14] is strictly bigger than the one of the corresponding symmetric process. This means that the diffusivity of the model is not just the one due to the assumed stochasticity of the model (the symmetric part), but there is a contribution from the "deterministic" motion of the particles (the asymmetric part), which can be interpreted as a "Navier-Stokes" contribution in analogy with the Hamiltonian case. This is in agreement with the heuristic green-Kubo formula for this model, as computed in [19]

2. The conservation laws.

We consider a system of N identical particles of mass 1 in a cube of size ε^{-1} in $\mathbb{R}^d$,

with periodic b.c., interacting through a pair central potential V of finite range. After rescaling space as ε^{-1} and time as ε^{-2} the Newton equations become

$$\frac{dx_i}{dt}(t) = \varepsilon^{-1} v_i(t)$$
$$\frac{dv_i}{dt}(t) = -\varepsilon^{-2} \sum_{i\neq j} \nabla V(\varepsilon^{-1}(x_i - x_j)) \tag{2.1}$$

The number of particles N is assumed to be of order ε^{-d} to keep the density finite. The total number of particles, the d components of the total momentum and the total energy are the conserved quantities. We construct the corresponding empirical fields:

empirical density

$$z^0(x) = \varepsilon^d \sum_i \delta(x_i - x) \tag{2.2}$$

empirical velocity field density

$$z^\alpha(x) = \varepsilon^d \sum_i v_i^\alpha \delta(x_i - x), \quad \alpha = 1, \dots, d \tag{2.3}$$

empirical energy density

$$z^{d+1}(x) = \varepsilon^d \sum_i \frac{1}{2}[v_i^2 + \sum_{j\neq i} V(\varepsilon^{-1}|x_i - x_j|)]\delta(x_i - x) \tag{2.4}$$

Their meaning is as follows: The average of the integral of z^α over a small region is equal to the average number of particles, momentum, energy associated to the region. We will write also

$$z^\mu(x) = \varepsilon^d \sum_i \delta(x_i - x) z_i^\mu \tag{2.5}$$

with $z_i^0 = 1$; $z_i^\alpha = v_i^\alpha, \alpha = 1, \dots, d$; $z_i^{d+1} = \frac{1}{2}[v_i^2 + \sum_{i\neq j} V(\varepsilon^{-1}|x_i - x_j|)]$. The generalized functions z^α on the phase space are expected to be approximated, to the lowest order in ε, by the macroscopic hydrodynamic fields, in the sense that, with probability 1, for any smooth function f, we have

$$\int dx z(x) f(x) = \int dx b(x) f(x) + o(1) \tag{2.6}$$

where $o(1)$ denotes a quantity going to 0 as $\varepsilon \to 0$, $z = \{z^\alpha\}$ and $b = \{\rho, U, e\}$, the macroscopic density, velocity field and energy respectively. The empirical fields satisfy the following local conservation laws, which are obtained differentiating $z^\alpha(x,t)$ with respect to the time and using the Newton equations,

$$\frac{d}{dt}\varepsilon^d \sum_i f(x_i) = \varepsilon^{-1}\varepsilon^d \sum_i \frac{\partial f}{\partial x_i^\alpha}(x_i) v_i^\alpha \tag{2.7}$$

$$\frac{d}{dt}\varepsilon^d \sum_i f(x_i)v_i^\beta = \varepsilon^{-1}\varepsilon^d \sum_i \{\frac{\partial f}{\partial x_i^\alpha}(x_i)v_i^\alpha v_i^\beta - \varepsilon^{-1}\sum_{j\neq i}\nabla_\beta V(\varepsilon^{-1}(x_i - x_j))f(x_i)\} \quad (2.8)$$

$$\frac{d}{dt}\varepsilon^d \sum_i f(x_i)z_i^{d+1} = \varepsilon^{-1}\varepsilon^d \sum_i \{\frac{\partial f}{\partial x_i^\alpha}(x_i)v_i^\alpha z_i^{d+1} - \frac{1}{2}\varepsilon^{-1}\sum_{i\neq j}\nabla_\alpha V(\varepsilon^{-2}(x_i - x_j))v_i^\alpha f(x_i)\} \quad (2.9)$$

Here $\nabla_\beta V(\xi) = \partial V(\xi)/\partial\xi_\beta$. Because of the simmetry properties of the potential we can write, as usual, the second term in the r.h.s. of (2.8) as

$$-\frac{1}{2}\varepsilon^{d-2}\sum_{i\neq j}\nabla_\beta V(\varepsilon^{-1}(x_i - x_j))[f(x_i) - f(x_j)] \quad (2.10)$$

Since f is slowly varying on the microscopic scale we can write, with $\xi_i = \varepsilon^{-1}x_i$,

$$f(\varepsilon\xi_i) - f(\varepsilon\xi_j) = \sum_\gamma \frac{\partial f}{\partial x_i^\gamma}(x_i)\varepsilon[\xi_i^\gamma - \xi_j^\gamma]$$
$$+ \sum_{\gamma,\nu}\frac{\partial^2 f}{\partial x_i^\gamma \partial x_i^\nu}(x_i)\varepsilon^2[\xi_i^\gamma - \xi_j^\gamma][\xi_i^\nu - \xi_j^\nu] + \varepsilon^3 D(x_i - x_j) + O(\varepsilon^4) \quad (2.11)$$

where

$$D(x_i - x_j) = \sum_{\gamma,\nu,\alpha}\frac{\partial^3 f}{\partial x_i^\gamma \partial x_i^\nu \partial x_i^\alpha}(x_i)[\xi_i^\gamma - \xi_j^\gamma][\xi_i^\nu - \xi_j^\nu][\xi_i^\alpha - \xi_j^\alpha]$$

Due to the simmetry of the potential the second term of the Taylor expansion of f does not contribute and the last term of eq.(2.8) becomes

$$\frac{1}{2}\varepsilon^{-1}\varepsilon^d \sum_{i,j}\sum_\gamma \frac{\partial f}{\partial x_i^\gamma}(x_i)\Psi^{\beta\gamma}(\varepsilon^{-1}(x_i - x_j)) + O(\varepsilon) \quad (2.12)$$

with

$$\Psi^{\beta\gamma}(\xi) = -\nabla_\beta V(\xi)\xi^\gamma \quad (2.13)$$

An analogous computation can be done for the energy equation. The general form of the rescaled local conservation laws is

$$\frac{\partial}{\partial t}\int dx f(x)z^\beta(x) = \varepsilon^{-1}\int dx \sum_{k=1}^{3}\frac{\partial f}{\partial x^k}w^{\beta k}(x) + O(\varepsilon) \quad (2.14)$$

where $w^{\beta k}$, $\beta = 0,\dots,d+1$; $k = 1,\dots,d$ are the currents associated to the fields z^β and are explicitely given by

$$w^{0k}(x) = \varepsilon^d \sum_i \delta(x_i - x)v_i^k \quad (2.15)$$

$$w^{\beta k}(x) = \varepsilon^d \sum_i \delta(x_i - x)\{v_i^\beta v_i^k + \frac{1}{2}\sum_j \Psi^{\beta k}(\varepsilon^{-1}(x_i - x_j))\}, \quad \beta = 1,\dots,d \quad (2.16)$$

$$w^{d+1\,k}(x) = \varepsilon^d \sum_i \{v_i^k z_i^{d+1} + \frac{1}{2}\sum_{j,\gamma}\Psi^{\gamma k}(\varepsilon^{-1}(x_i - x_j))\frac{1}{2}[v_i^\gamma + v_j^\gamma]\} \quad (2.17)$$

We put also $w^{\beta k}(x) = \varepsilon^d \sum_i \delta(x_i - x)w_i^{\beta k}$.

The empirical fields $z^\alpha(x)$ are approximate integrals of the motion in the sense that, defining the Liouville operator in terms of microscopic variables $\xi_i = \varepsilon^{-1}x_i$ as

$$\mathcal{L}f(\xi, v) = \sum_i \{v_i^\alpha \frac{\partial f}{\partial \xi_i^\alpha} - \sum_{i\neq j}\frac{\partial V}{\partial \xi_i^\alpha}(|\xi_i - \xi_j|)\frac{\partial f}{\partial v_i^\alpha}\} \quad (2.18)$$

and denoting by ζ_i^α the quantities z_i^α as functions of the microscopic variables ξ_i, it follows from the previous calculation that

$$\mathcal{L}[\varepsilon^d \sum_i f(\varepsilon\xi_i)\zeta_i^\alpha] = O(\varepsilon) \quad (2.19)$$

We call the observables with this property *local integrals of motion.* This is consistent with the following definition of *local equilibrium distribution* on the phase space (in microscopic variables):

$$G = Z^{-1}\exp\sum_i \sum_{\alpha=0}^{d+1}\lambda^\alpha(\varepsilon\xi_i)\zeta_i^\alpha \quad (2.20)$$

with Z the normalization factor. In fact the distribution G is locally stationary for $\mathcal{L}$ in the sense that

$$\mathcal{L}G = O(\varepsilon) \quad (2.21)$$

In other words if we look at a region around the point x microscopically very large but macroscopically small, the system appears to be in equilibrium in this region and its distribution is the Gibbs measure G restricted to the variables localized there. In regions of this type we can follow the evolution of the system for very large microscopic times τ such that it makes sense to consider ergodic properties of the unitary group S_τ generated by $\mathcal{L}$:

For any local observable ϕ of mean zero with respect to G we put

$$\hat{\phi} = \lim_{\tau\to\infty}\tau^{-1}\int_0^\tau d\tau' S_{\tau'}\phi \quad (2.22)$$

$\hat{\phi}$ represents the part of ϕ which is invariant under S_τ.

We assume that the set of all the invariant local observables contains only combinations of the empirical fields associated to the particle number, momentum and energy, and any function of them.

To make this concept more precise we refer to [20], [4] where it is introduced the Hilbert space of the local observables equipped with the scalar product

$$(\phi,\psi) = \int dx[\langle\phi\tau_x\psi\rangle - \langle\phi\rangle\langle\psi\rangle] \tag{2.23}$$

Here $\langle\cdot\rangle$ is the average on the local Gibbs measure. Our assumption means that, introducing the projector on the invariant space defined as

$$\mathcal{P}\phi = \sum_{\mu=0}^{d+1}(\phi,z^\mu)(z,z)^{-1}_{\mu\nu}z^\nu \tag{2.24}$$

where $(z,z)^{-1}$ denotes the inverse of the matrix with elements (z_μ,z_ν), then

$$\hat{\phi} = \mathcal{P}\phi \tag{2.25}$$

To show how the conservation laws give the hydrodynamic equations (INS) we follow a procedure similar to the one Chapmann-Enskog proposed to approximate the solutions of the Boltzmann equation. Let us start by the phase space distribution function F_ϵ for the rescaled system, which satisfy the Liouville equation

$$\frac{\partial F_\epsilon}{\partial t} = \epsilon^{-2}\mathcal{L}^* F_\epsilon \tag{2.26}$$

where $\mathcal{L}^*$ is the adjoint, w.r.t. the Liouville measure, of the Liouville operator on the phase space, formally given by $\mathcal{L}^* = -\mathcal{L}$.

We write F_ϵ as a part which is Gibbsian with parameters slowly dependending on the microscopic variables and depending on ϵ by means of a series in ϵ, and a remainder. More explicitely, we put

$$F_\epsilon = G_\epsilon + \epsilon^2 G_0 R_\epsilon \tag{2.27}$$

with

$$G_\epsilon = G_\epsilon^{-1}\exp\{\sum_{i,\mu}\lambda^\mu_\epsilon(x_i,t)z^\mu_i\};\quad \lambda^\mu_\epsilon(x,t) = \sum_{n=0}^{\infty}\epsilon^n\lambda^\mu_n(x_i,t);\quad \lambda^\mu_0 = \text{const.} \tag{2.28}$$

The structure of F_ϵ given by (2.27), (2.28) can be justified looking at the diverging terms in the Liouville equation. In fact the term $\epsilon^{-2}\mathcal{L}^*G_0$ is of order ϵ^{-1} by (2.21), so we are forced to take $\mathcal{L}^*G_0 = 0$ and hence $\lambda^\mu_0 = \text{const.}$ for $\mu = 0,\ldots,d+1$. The fact that there is not a non-Gibbsian correction of order ϵ is due to the same considerations. In fact, suppose that in (2.27) there is also a term of order ϵ, say $\epsilon G_0\tilde{R}_1$. The contribution to the Liouville equation due to such a term would be $\epsilon^{-1}\mathcal{L}^*G_0\tilde{R}_1$, so we would be forced

to assume $\mathcal{L}^*G_0\tilde{R}_1 = O(\epsilon)$. By the assuptions on the kernel of $\mathcal{L}^*$ this means that $G_0\tilde{R}_1$ is Gibbsian and then can be included in G_ϵ.

By the same argument we can assume that in R_ϵ there are no terms which are combinations of the invariant quantities z^α with coefficients depending on the macroscopic variables, since these terms are already present in G_ϵ. Hence we put

$$\hat{R}_\epsilon = 0 \tag{2.29}$$

We also assume that for any t

$$R_\epsilon(t) = R(t) + O(\epsilon) \tag{2.30}$$

Inserting the expression for F_ϵ in the Liouville equation and integrating on time, we have

$$[G_\epsilon(t) - G_\epsilon(0)] + \epsilon^2[R_\epsilon(t) - R_\epsilon(0)] = \int_0^t [\epsilon^{-2}\mathcal{L}^*G_\epsilon + \mathcal{L}^*G_0 R_\epsilon] \tag{2.31}$$

The l.h.s. of (2.31) goes to 0 in the limit $\epsilon \to 0$, since the only term of order 1 is constant in time due to the assumptions on λ_0. Hence we have

$$\int_0^t \{\epsilon^{-1}\mathcal{L}^*g_1 + \mathcal{L}^*G_0 h\} = -\int_0^t \mathcal{L}^*R + O(\epsilon) \tag{2.32}$$

where $g_1 = \sum_{j,\mu}\lambda^\mu_1(x_j,t)z^\mu_j$ and we have used

$$G_\epsilon = G_0\{1 + \epsilon[g_1 - < g_1 >] + \epsilon^2 h\} + O(\epsilon^3) \tag{2.33}$$

Here h is a function of the invariant fields and $< \cdot >$ is the average w.r.t. G_0.

We observe that, since g_1 is a linear combination of the invariant quantities z with coefficients depending on the macroscopic variables, the action of $\mathcal{L}^*$ on it gives a linear combination of the currents w with a factor ϵ. Therefore the first term in the l.h.s.of (2.32) is of order 1. Moreover, for the same reasons the second term go to zero. In conclusion R satisfies the equation:

$$\int_0^t [\mathcal{L}^*G_0 R - \epsilon^d \sum_i \frac{\partial\lambda^\mu_1}{\partial x^\gamma_i}(x_i,s)w^{\mu\gamma}_i G_0] = 0 \tag{2.34}$$

3. Incompressible Navier-Stokes equations.

The incompressible limit corresponds to the assumption that the velocity field is small compared with the sound speed. In other words we assume that $U^\mu(x,t) \equiv \langle z^\mu(x)\rangle_{F_\epsilon(t)}$, $\mu = 1,\dots,d$, starts with a term of order ϵ. Under the assumptions on F_ϵ, this corresponds to choose $\lambda_0^\mu = 0$ for $\mu = 1,\dots,d$. On the other hand, it results $U^\mu(x,t) = \frac{1}{2}\epsilon\rho T\lambda_1^\mu(x,t) + O(\epsilon^2)$ with T, the constant temperature of the Gibbs state G_0, given by $(\lambda_0^{d+1})^{-1}$ and ρ the constant density of the Gibbs state G_0 corresponding to the constant chemical potential λ_0^0. We denote by $u^\mu(x,t)$ the rescaled velocity field given by $u^\mu(x,t) = \frac{1}{2}T\lambda_1^\mu(x,t)$. In the incompressible case the continuity equation reduces to the incompressibility condition div $u = 0$. To obtain it, we start from the conservation law for the empirical density (2.7) and we take the expectation with respect the non-equilibrium measure $F_\epsilon(t)$

$$\left\langle \epsilon^d \sum_i f(x_i)\right\rangle_{F_\epsilon(t)} - \left\langle \epsilon^d \sum_i f(x_i)\right\rangle_{F_\epsilon(0)} = \epsilon^{-1}\int_0^t ds \left\langle \epsilon^d \sum_{i,k} \frac{\partial f}{\partial x_i^k}(x_i)v_i^k\right\rangle_{F_\epsilon(s)} \tag{3.1}$$

Using (2.27), (2.28) and (2.30) we see that the l.h.s. of (3.1) goes to zero and (3.1) becomes

$$\int_0^t ds\left\langle g_1 \sum_{k=1}^d \frac{\partial f}{\partial x_i^k}(x_i)v_i^k\right\rangle = \int_0^t ds\left\langle \sum_j \sum_{\mu=0}^{d+1} \lambda_1^\mu(x_j,t)z_j^\mu \epsilon^d \sum_{i,k}\frac{\partial f}{\partial x_i^k}(x_i)v_i^k\right\rangle = O(\epsilon) \tag{3.2}$$

where $\langle\,\cdot\,\rangle$ is the average with respect the Gibbs measure G_0. Since G_0 is gaussian in the velocities, the terms with $\mu = 0, d+1$ do not contribute (they are averages of odd polinomials in v) and the l.h.s. of (3.2) becomes

$$\left\langle \epsilon^d \sum_i \sum_{\mu,k=1}^d \lambda_1^\mu(x_i,t)\frac{\partial f}{\partial x_i^\mu}(x_i)v_i^\mu v_i^k\right\rangle \tag{3.3}$$

Since the average on G_0 of $v_i^\mu v_i^k$ contributes only for $k=\mu$, in the limit $\epsilon \to 0$ we have,

$$\int_0^t ds \int dx \sum_{\mu=1}^d \frac{\partial(\rho u^\mu)}{\partial x^\mu}(x,t) f(x) = 0 \tag{3.4}$$

for any test function f and for any t. Hence

$$\operatorname{div} u = 0 \tag{3.5}$$

We examine now the second conservation law (2.8). By averaging as before, for $\beta = 1,\dots,d$, we get

$$\left\langle \epsilon^d \sum_i f(x_i)v_i^\beta\right\rangle_{F_\epsilon(t)} - \left\langle \epsilon^d \sum_i f(x_i)v_i^\beta\right\rangle_{F_\epsilon(0)} =$$

$$\epsilon^{-1}\int_0^t ds\left\langle \epsilon^d \sum_i\sum_{k=1}^d \frac{\partial f}{\partial x_i^k}(x_i)\left\{v_i^k v_i^\beta + \frac{1}{2}\sum_{j\neq i}\Psi^{\beta k}(\epsilon^{-1}(x_i-x_j))\right\}\right\rangle_{F_\epsilon(s)} + O(\epsilon) \tag{3.6}$$

Using the assumptions on $F_\epsilon(t)$ we see that the l.h.s. is of order ϵ, since the term of order 1 vanishes because λ_0^μ are constant. It is convenient to rewrite the integrand in the r.h.s. of (3.6) as

$$\int dx \frac{\partial f}{\partial x^k}(x)\left\langle \epsilon^{-1} w^{\beta k}(x)\right\rangle_{F_\epsilon} \tag{3.7}$$

Using the assumptions (2.27)-(2.30) we have:

$$\epsilon^{-1}\left\langle w^{\beta k}\right\rangle_{F_\epsilon} = \epsilon^{-1}\left\langle w^{\beta k}\right\rangle_{G_\epsilon} + \epsilon\left\langle w^{\alpha\beta}R_\epsilon\right\rangle_{G_0} \tag{3.8}$$

We introduce the currents $\bar w^{\beta k}$ as given by the expression (2.16) with the velocities v_i replaced by $\bar v_i = v_i - \epsilon u(x_i)$. Then

$$w_i^{\beta k} = \bar w_i^{\beta k} + \epsilon^2 u^k(x_i)u^\beta(x_i) + \epsilon u^k(x_i)\bar v_i^\beta + \epsilon u^\beta(x_i)\bar v_i^k \tag{3.9}$$

For the simmetry of the measure G_ϵ we have $\left\langle \bar w^{\beta k}(x)\right\rangle_{G_\epsilon} = O(\epsilon^4)$, if $k \neq \beta$. The average of $\bar w^{\beta\beta}$, $\beta = 1,\dots,d$, with respect the local Gibbs state G_ϵ is, by the virial theorem, the termodinamic pressure P_ϵ in the state G_ϵ [21]. Therefore eq. (3.6) and assumption (2.30) say that

$$\epsilon^{-1}\int dx \nabla f(x) P^\epsilon(x,s) = O(\epsilon) \tag{3.10}$$

Since P^ϵ is a function of the termodynamic parameters λ_ϵ, we can expand it in series of ϵ as $\sum_k \epsilon^k P_k$, where $P_k = \frac{d^k P^\epsilon}{d\epsilon^k}\big|_{\epsilon=0}$ We have that P_0 is constant since it is a function of the constants λ_0^0 and λ_0^{d+1}, while $P_1 = \sum_{\mu=0}^{d+1}\frac{\partial P^\epsilon}{\partial\lambda_\epsilon^\mu}\big|_{\epsilon=0}\lambda_1^\mu$. In order to fulfill (3.10) for any test function f, P_1 has to be constant.

To determine the equation for $u^\mu(x,t)$ we have to rescale the empirical velocity field. This means that we have to look at the empirical field

$$\bar z^\alpha(x) = \epsilon^{-1}\epsilon^d \sum_i v_i^\alpha \delta(x_i - x), \quad \alpha = 1,\dots,d$$

We proceed as we did before to obtain (3.6), but we have to look to the explicit form of the term $O(\varepsilon)$ because it has to be divided by ε. We have:

$$\Big\langle \varepsilon^{d-1}\sum_i f(x_i)v_i^\beta\Big\rangle_{F_\varepsilon(t)} - \Big\langle \varepsilon^{d-1}\sum_i f(x_i)v_i^\beta\Big\rangle_{F_\varepsilon(0)} =$$

$$\varepsilon^{-2}\int_0^t ds\Big\langle \varepsilon^d \sum_i\sum_{k=1}^d \frac{\partial f}{\partial x_i^k}(x_i)\{v_i^k v_i^\beta + \frac{1}{2}\sum_{j\neq i}\Psi^{\beta k}(\varepsilon^{-1}(x_i - x_j))\}\Big\rangle_{F_\varepsilon(s)} +$$

$$\int_0^t ds\Big\langle \frac{1}{2}\varepsilon^d\sum_{i\neq j}\nabla_\beta V(\varepsilon^{-1}(x_i-x_j))D(x_i-x_j)\Big\rangle_{F_\varepsilon(s)} + O(\varepsilon) \tag{3.11}$$

This equation, to the lowest order in ε, reduces to the Navier-Stokes equation for the velocity field u. The argument is the following. The l.h.s. of (3.11) is given by

$$\int dx f(x)\rho[u(x,t) - u(x,0)] \tag{3.12}$$

up to terms of order ε.

To get the INS we have to compute again the non-equilibrium average of the velocity current tensor $w^{\beta k}$ but now there is a factor ε^{-2} in front of it. Therefore we see that in this case also the terms of order ε^2 in (2.27) have to be taken into account. First of all we observe that the term containing D goes to 0 as $\varepsilon \to 0$ because the lowest order is given by an average with respect to G_0 (Gibbs measure with constant parameters), hence it vanishes because it contains derivatives of f. The other terms are at least of order ε, hence do not contribute to the lowest order.

Moreover we have

$$\varepsilon^{-2}\Big\langle w^{\beta k}(x)\Big\rangle_{G_\varepsilon} + \Big\langle w^{\beta k}(x)R_\varepsilon\Big\rangle_{G_0} = \varepsilon^{-2}\Big\langle \bar{w}^{\beta k}\Big\rangle_{G_\varepsilon} + \rho u^\beta u^k + \Big\langle w^{\beta k}R_\varepsilon\Big\rangle_{G_0} + O(\varepsilon) =$$

$$\varepsilon^{-2}P_0 + \varepsilon^{-1}P_1 + P_2 + \rho u^\beta u^k + \Big\langle R w^{\beta k}(x)\Big\rangle + O(\varepsilon) \tag{3.13}$$

The first two terms of (3.13) do not contribute to INS because P_0 and P_1 are constant. The fourth term in (3.13) gives the non linear transport term, while P_2 represents the second order correction to the termodynamic pressure P_ε and gives rise to the unknown pressure p appearing in the INS.

The last term in the r.h.s of (3.13) is determined by R, the non-equilibrium part of the distribution F_ε, which takes into account the fast modes in the system, namely the non-conserved quantities. They appear at the hydrodynamic level only through dissipative effects and determine the expression of the transport coefficients.

To compute it, let us first introduce $\bar{w}^{\beta k} = \bar{w}^{\beta k} - \mathcal{P}\bar{w}^{\beta k}$ and notice that (2.29) implies

$$\Big\langle R\mathcal{P}w^{\alpha\beta}(x)\Big\rangle = 0 \tag{3.14}$$

Again by (2.29) we can use the "identity" $(\mathcal{L}^*)^{-1}\mathcal{L}^*R = R$ and (2.34) to get (recall the notation introduced after (2.17)):

$$\int dx \sum_k \frac{\partial f}{\partial x^k}\Big\langle \bar{w}^{\beta k}(x)(\mathcal{L}^*)^{-1}\mathcal{L}^*R\Big\rangle = \Big\langle \mathcal{L}^*R\mathcal{L}^{-1}\varepsilon^d\sum_i\sum_{k=1}^d \bar{w}_i^{\beta k}\frac{\partial f}{\partial x_i^k}(x_i)\Big\rangle$$

$$= \Big\langle \sum_j\sum_{\mu=0}^{d+1}\sum_{l,k=1}^{d}\frac{\partial\lambda_1^\mu}{\partial x_j^l}(x_j,s)w_j^{\mu l}\mathcal{L}^{-1}\varepsilon^d\sum_i \bar{w}_i^{\beta k}\frac{\partial f}{\partial x_i^k}(x_i)\Big\rangle \tag{3.15}$$

We remark that $\mathcal{L}^{-1}$ is "well defined" on $\bar{w}$ by the assumptions discussed in section 2.

To find the expression of the transport coefficient we consider

$$\Big\langle \sum_j \frac{\partial\lambda_1^\mu}{\partial x_j^l}(x_j,s)w_j^{\mu l}\mathcal{L}^{-1}\varepsilon^d\sum_{i,k}\bar{w}_i^{\beta k}\frac{\partial f}{\partial x_i^k}(x_i)\Big\rangle =$$

$$\int dy \frac{\partial\lambda_1^\mu}{\partial y^l}(y,s)\int \varepsilon^{-d}dz\frac{\partial f}{\partial z^k}(z)\Big\langle \bar{w}^{\mu l}(x)\mathcal{L}^{-1}\bar{w}^{\beta k}(z)\Big\rangle \tag{3.16}$$

The substitution of the current w with $\bar{w}$ is correct because the range of $\mathcal{L}^{-1}$ is orthogonal to $\mathcal{P}w$. Since the Gibbsian state G_0 is invariant under translations on $\mathbb{R}^d$ we have

$$\text{r.h.s. of (3.16)} = \int dy\frac{\partial\lambda_1^\mu}{\partial y^l}(y,s)\int d\xi\frac{\partial f}{\partial y^k}(y+\varepsilon\xi)\Big\langle \bar{w}^{\mu l}(0)\mathcal{L}^{-1}\bar{w}^{\beta k}(\xi)\Big\rangle \tag{3.17}$$

where we have changed the variable z in $\xi = \varepsilon^{-1}(z-y)$ absorbing the factor ε^{-d}. Hence, provided that $\langle\bar{w}^{\mu l}(0)\mathcal{L}^{-1}\bar{w}^{\beta k}(\xi)\rangle$ decay fast enough for large ξ, up to $O(\varepsilon)$ we have

$$\text{r.h.s. of (3.16))} = \int dy\frac{\partial\lambda_1^\mu}{\partial y^l}(y,s)\frac{\partial f}{\partial y^k}(y)\int d\xi\Big\langle \bar{w}^{\mu l}(0)\mathcal{L}^{-1}\bar{w}^{\beta k}(\xi)\Big\rangle \tag{3.18}$$

To conclude the argument we transform $\mathcal{L}^{-1}$ in a time-integral of $\exp[t\mathcal{L}]$:

$$-\Big\langle \bar{w}^{\mu l}(0)\mathcal{L}^{-1}\bar{w}^{\beta k}(\xi)\Big\rangle = \int_0^\infty d\tau\Big\langle \bar{w}^{\mu l}(\xi,\tau)\bar{w}^{\beta k}(0,0)\Big\rangle \tag{3.19}$$

The symmetries of the microscopic current-current correlations imply (see [4]) that the correlations for $\mu = 0, d+1$ vanish and

$$\int d\xi\Big\langle \bar{w}^{\mu l}(\xi,\tau)\bar{w}^{\beta k}(0,0)\Big\rangle = c(\tau)[\delta_{kl}\delta_{\beta k} + \delta_{k\mu}\delta_{\beta l}] + c'(\tau)\delta_{\beta k}\delta_{l\mu} \tag{3.20}$$

Therefore the time integral of (3.20) has only two independent coefficients

$$\int_0^\infty d\tau c(\tau) = 2\eta T; \quad \int_0^\infty d\tau c'(\tau) = 2T(\zeta - \frac{2}{d}\eta) \tag{3.21}$$

where η and ζ are the shear viscosity and the bulk viscosity respectively. They are finite if the correlations decay sufficiently fast to make the time integrals in (3.21) convergent. Notice that the subtraction of $\mathcal{P}\bar{w}^{\alpha\beta}$ has been crucial, because the self-correlation of the slow part of the current does not decay in time.

Since we already know that div $u = 0$ the term proportional to the bulk viscosity does not appear in the limiting equation. Putting all the terms toghether we have the following equation to the lowest order in ε:

$$\int dx f(x)\rho[u^\beta(x,t) - u^\beta(x,0)] =$$
$$\int_0^t ds \int dy \sum_{k=1}^d \frac{\partial f}{\partial y^k}(y)\{\rho u^\beta(y,s)u^k(y,s) - \eta\frac{\partial u^\beta}{\partial y^k}(y,s)\} \tag{3.22}$$

for any test function f, and hence the incompressible Navier-Stokes equation. The viscosity is given by

$$\eta = \frac{1}{2T}\int_0^\infty d\tau \int d\xi \left\langle \bar{w}^{12}(\xi,\tau)\bar{w}^{12}(0,0)\right\rangle \tag{3.23}$$

and is independent of the space coordinates because the current-current correlation involved is computed at the global equilibrium.

The computation gives also the Green-Kubo formula for the bulk viscosity ζ

$$\zeta = \frac{1}{2d^2T}\int_0^\infty d\tau \int d\xi [\left\langle \sum_\alpha \bar{w}^{\alpha\alpha}(\xi,\tau)\sum_\gamma \bar{w}^{\gamma\gamma}(0,0)\right\rangle - \left\langle \sum_\alpha \bar{w}^{\alpha\alpha}\right\rangle\left\langle \sum_\gamma \bar{w}^{\gamma\gamma}\right\rangle] \tag{3.24}$$

The usual expression given in [4] is recovered using the explicit form of the projector $\mathcal{P}$.

By means of the arguments developed before it is possible to find an equation for the first correction to the kinetic energy. We will use below the following remark on the conservation of the mass. Since the density current is a locally invariant field, taking into account (2.29), it follows that the non equilibrium average of (2.15) will be determined only by the Gibbsian part G_ε. Thus the equation for the averages is

$$\int dx[\rho_\varepsilon(x,t) - \rho_\varepsilon(x,0)] = -\int_0^t \varepsilon^{-1}\int dx \mathrm{div}(\rho_\varepsilon u_\varepsilon)f(x) \tag{3.25}$$

where $\rho_\varepsilon = \langle z^0\rangle_{G_\varepsilon}$ and $\rho_\varepsilon u_\varepsilon^\mu = \langle z^\mu\rangle_{G_\varepsilon}$, $\mu = 1,\ldots,d$.

Now we consider the conservation law for the energy (2.9). We are interested in the first correction to the energy since at zero order the energy is a constant, which we call e_0. Therefore we look for the equation for the quantity $\varepsilon^{-1}(z_i^{d+1} - e_0)$). By (2.9) we have

$$\frac{d}{dt}\varepsilon^{-1}\varepsilon^d\sum_i f(x_i)(z_i^{d+1} - e_0) = \varepsilon^{-2}\varepsilon^d\sum_i\sum_{k=1}^d \frac{\partial f}{\partial x^k}(x_i)w_i^{d+1\,k} + O(\varepsilon) \tag{3.26}$$

We only sketch the argument to get the limiting equation, since the procedure is the same as in the previous cases. We need to evaluate $\langle w^{d+1\,k}(x)R_\varepsilon\rangle$ and $\varepsilon^{-2}\langle w^{d+1\,k}(x)\rangle_{G_\varepsilon}$

The first of them gives the diffusive correction. We introduce $\bar{w}^{d+1\,k}$ and $\tilde{z}^{d+1}$ defined by the (2.17) and (2.4) with v replaced by $\tilde{v}$. We have

$$w_i^{d+1\,k} = \bar{w}_i^{d+1\,k} + \varepsilon\{u^k(x_i)z_i^{d+1} + \sum_\gamma (u^\gamma\tilde{v}_i^\gamma\tilde{v}_i^k + \Psi^{\gamma k}(\varepsilon^{-1}|x_i - x_j|)\frac{1}{2}[u^\gamma(x_i) + u^\gamma(x_j)])\} \tag{3.27}$$

Then, as before,

$$\left\langle \bar{w}^{d+1\,k}R_\varepsilon\right\rangle = \int_0^t ds \left\langle \sum_j\sum_{l=1}^d \frac{\partial\lambda_1^\mu}{\partial x_j^l}(x_j,s)\bar{w}_j^{\mu l}\mathcal{L}^{-1}\bar{w}^{d+1\,k}(x)\right\rangle + O(\varepsilon) \tag{3.28}$$

where $\bar{w}^{d+1\,k} = \bar{w}^{d+1\,k} - \mathcal{P}(\bar{w}^{d+1\,k})$. Because of time-reversal and rotation invariance of the Gibbs state the only correlations different from zero are (see [4])

$$\int dx \left\langle \bar{w}^{d+1\,k}(x,\tau)\bar{w}^{d+1\,l}(0,0)\right\rangle = \delta_{lk}a(\tau) \tag{3.29}$$

and $\int d\tau a(\tau) = 2\kappa T^2$. Therefore the conductivity κ is given by

$$\kappa = \frac{1}{2dT^2}\int d\tau \left\{ \int d\xi \left\langle \sum_{k,l} w^{d+1\,k}(x,\tau)w^{d+1\,l}(0,0)\right\rangle - d(T(e+P)^2/\rho\right\} \tag{3.30}$$

We observe that, since $\lambda_\varepsilon = -(T_\varepsilon)^{-1}$, λ_1^{d+1} is given by $T_1(T^{-2})$.

Using the previous arguments one can see that the second term in (3.27) gives no contribution to $\langle w^{d+1\,k}R_\varepsilon\rangle$ in the limit $\varepsilon \to 0$.

The mean of the energy current on the Gibbs state G_ε, i.e. $\langle w^{d+1\,k}\rangle_{G_\varepsilon}$, is nothing but $(\rho_\varepsilon e_\varepsilon + P_\varepsilon)u_\varepsilon$ (see [4]), where $\rho_\varepsilon e_\varepsilon = \langle z^{d+1}\rangle_{G_\varepsilon}$.

Summarizing, if we take the average of (3.26) with respect F_ε we obtain

$$\varepsilon^{-1}\int dx[\rho_\varepsilon e_\varepsilon(x,t) - \rho_\varepsilon e_\varepsilon(x,0)] =$$
$$\int_0^t ds \int dx f(x)\{-\varepsilon^{-2}\mathrm{div}[(\rho_\varepsilon e_\varepsilon + P_\varepsilon)u_\varepsilon](x,s) + k\Delta T_1(x,s)\} + O(\varepsilon) \tag{3.31}$$

for any test function f. Hence, using (3.25),

$$\varepsilon^{-1}\rho_\varepsilon\partial_t e_\varepsilon + \varepsilon^{-2}u_\varepsilon\rho_\varepsilon\nabla e_\varepsilon + \varepsilon^{-2}\mathrm{div}(P_\varepsilon u_\varepsilon] = \kappa\Delta T_1 + O(\varepsilon) \tag{3.32}$$

Writing $e_\varepsilon, P_\varepsilon, u_\varepsilon$ as series in ε with coefficients e_n, P_n and u_n (recall $u_0 = 0$, $u_1 = u$), we have

$$\text{l.h.s. of (3.32)} = \rho\partial_t e_1 + P_0\mathrm{div}u_2 + \rho u\cdot\nabla e_1 + O(\varepsilon) \tag{3.33}$$

As next step we eliminate $\mathrm{div} u_2$ by means of the equation (3.25) which we rewrite in the form

$$D_t\rho_1 + \rho \mathrm{div} u_2 = O(\varepsilon) \tag{3.34}$$

where $D_t f \equiv \partial_t f + u \cdot \nabla f$. We get

$$\rho D_t e_1 - \rho^{-1} P_0 D_t \rho_1 = \kappa \Delta T_1 + O(\varepsilon) \tag{3.35}$$

Let us note that only the internal energy contributes to e_1 because the kinetic energy is of order ε^2. Hence e_1 can be written as a function of ρ_1 and T_1 and we have

$$[\rho \frac{\partial e_1}{\partial \rho_1} - \frac{P_0}{\rho}] D_t\rho_1 + \frac{\partial e_1}{\partial T_1} D_t T_1 = \kappa \Delta T_1 + O(\varepsilon) \tag{3.36}$$

The Boussinesq condition, stating that P_1 is constant, implies

$$\frac{\partial P_\varepsilon}{\partial \rho}\Big|_0 D_t\rho_1 + \frac{\partial P_\varepsilon}{\partial T}\Big|_0 D_t T_1 = 0 \tag{3.37}$$

We can eliminate $D_t\rho_1$ using above relation and, up to $O(\varepsilon)$, we get

$$c[\partial_t T_1 + u \cdot T_1] = \kappa \Delta T_1 \tag{3.38}$$

where c is given by ρ times the specific heat at constant pressure. Equation (3.38) for $u = 0$ is the Fourier law.

4. Asymmetric simple exclusion process.

In this section we shortly review the result in [14] because it supports our previous considerations in the sense that in a situation where the analysis can be made rigorous, the results confirm the arguments presented here.

In fact we consider a lattice gas with hard core exclusion as a simple example of stochastic system of particles for which it is possibile to prove rigorously a sort of INS limit. The model is as follows: we consider a system of particles on a d-dimensional lattice $\mathbb{Z}^d$, with periodic boundary conditions on a cubic region of size ε^{-1} which we denote by $\mathbf{T}_\varepsilon$. The particles jump independently with intensity $p_{x,y} \geq 0$ from the site x to the site y if it is empty. We denote by $\eta(x) = 0, 1$ the occupation number per site; η is a configuration of the system and the configuration space is $\{0,1\}^{\mathbf{T}_\varepsilon}$.

The generator of the stochastic dynamics is

$$\mathcal{L} f = \sum_{xy} c(x,y,\eta)[f(\eta^{xy}) - f(\eta)] \tag{4.1}$$

with

$$c(x,y,\eta) = p_{x,y}\eta(x)(1-\eta(y)) \tag{4.2}$$

and η^{xy} is the configuration in which $\eta(x)$ and $\eta(y)$ are exchanged. We restrict ourselves to the nearest neighborogh case, i.e. we assume that $p_{x,y}$ are non vanishing if and only if $|y - x| = 1$. We denote by e the unit vectors on the lattice with non negative components and put $p_e = p_{x,y}$ if $y = x + e$ and $p_{-e} = p_{y,x}$. It is also convenient to fix $p_e + p_{-e} = 2$ for all e.

This system has only one locally conserved field, the density. An invariant measure for the dynamics is the product measure $Z_\lambda^{-1} \prod_{x \in \mathbf{T}_\varepsilon} \exp \lambda\eta(x)$.

On the Euler time scale, for any smooth function f on $\mathcal{R}^d$, define the locally conserved empirical density field as,

$$r_t(f) = \varepsilon^d \sum_{x \in \mathbf{T}_\varepsilon} f(\varepsilon x)\eta_{\varepsilon^{-1}t}(x) \tag{4.3}$$

where $\eta_t(x)$ denotes the number of particles in x at time t. It has been proven in [16] that $r_t(f)$ converges, as $\varepsilon \to 0$, to a limit given by $\int dz f(z)\, \rho(z,t)$, with $\rho(z,t)$ solution of the d-dimensional non viscous Burgers equation

$$\partial_t \rho + F \cdot \nabla[\rho(1-\rho)] = 0 \tag{4.4}$$

where F is the driving field given by $F_e = p_e - p_{-e}$.

To see diffusive effects, as usual we have to wait for microscopic times of order ε^{-2}. The analysis of the corrections of order ε to (4.4) suggest that the macroscopic equation is the viscous Burgers equation. On the other hand such equation is not invariant under the diffusive scaling. The situation is quite similar to the one we described for the Navier-Stokes equation, but in a simpler case with only one conserved quantity. In fact the viscous Burgers equation is invariant if in addition to the diffusive scaling for space and time we consider perturbations of order ε to the constant density profile. Let $0 < \theta < 1$ be a constant density and assume $\rho = \theta - \varepsilon u$. Then, on times $\varepsilon^{-2}t$ we expect for u an equation of the form

$$\frac{\partial u}{\partial t} + \varepsilon^{-1} v \cdot \nabla u + F \cdot \nabla u^2 = \sum_{i,j=1}^{d} D_{i,j} \frac{\partial^2 u}{\partial z_i \partial z_j} \tag{4.5}$$

with $v = (1 - 2\theta)F$. We remove the diverging term by considering, instead of u, $m(z,t) = u(z + \varepsilon^{-1}vt, t)$, which satisfies

$$\frac{\partial m}{\partial t} + F \cdot \nabla m^2 = \sum_{i,j=1}^{d} \frac{\partial^2 m}{\partial z_i \partial z_j} \tag{4.6}$$

Above considerations suggest to introduce the rescaled empirical field defined for any test function f as

$$z_t^\varepsilon(f) = \varepsilon^{d-1} \sum_{x\in\mathbb{T}_\varepsilon} f(\varepsilon x + \varepsilon^{-1} vt)[\theta - \eta_{\varepsilon^{-2}t}(x)]$$

In [14], using the "non gradient" method [17], the entropy method [18, 3] and a multiscale analysis, the following theorem is proved:

Theorem. *Let $\eta(t)$ be the stochastic process described before. Moreover, let $d \geq 3$. We choose the sequence of initial measures as*

$$\mu^\varepsilon = Z_\varepsilon^{-1} \exp \sum_{x\in\mathbb{T}_\varepsilon} \{[\beta + \varepsilon\lambda_0(\varepsilon x)]\eta(x)\}$$

and let $m_0(z)$ be such that for any $\delta > 0$ and any f smooth and of compact support

$$\lim_{\varepsilon\to 0} Prob\{|z_0^\varepsilon(f) - \int dz f(z) m_0(z)| > \delta\} = 0$$

Then there is a symmetric matrix D satisfying (as a matrix)

$$D > 1 \tag{4.7}$$

such that for any $t \geq 0$ and $\delta > 0$

$$\lim_{\varepsilon\to 0} Prob\{|z_t^\varepsilon(f) - \int dz f(z) u(z,t)| > \delta\} = 0$$

where $m(z,t)$ is the unique smooth solution of the d-dimensional nonlinear diffusion equation (4.6) *(the viscous Burgers equation) with initial condition $m_0(z)$.*

Remark 1: The result of the theorem is espressed as a law of large numbers, but with an extra factor ε^{-1}. Hence, as remarked before, it cannot hold in dimension less than 3, because the fluctuations are too big. Actually this is not the only reason for the restriction; in fact the multiscale analysis on which it is based fails in dimension less than 3. Moreover (see [4]) the diffusion coefficient for the ASEP is expected to be infinite in dimension 1 and 2.

Remark 2: Consider the symmetric exclusion process with $\bar{p}_e = \bar{p}_{-e} = 1/2(p_e + p_{-e}) = 1$. The limit satisfies the diffusion equation with diffusion matrix 1. Hence the inequality (4.7) shows that there is a contribution to the diffusion coming from the asymmetry of the jumps. Since such an asymmetry may be interpreted as a "deterministic" motion added to the symmetric diffusion, its contribution to the diffusion is the analog of the viscosity for the Hamiltonian systems where, of course, there is no symmetric part due to the randomnes.

Remark 3: The strict analog of the incompressible limit is actually the case $\theta = 1/2$, for which $v = 0$ and the "velocity" is of order ε. The case considered in [14] is slightly more general because of the diverging transport term in the limit equation (4.6). It has been possible to manage it (because v is constant in space) by considering a frame of reference moving with speed $\varepsilon^{-1}v$. Hence this is an example (indeed very simple) in which one can give sense to the Navier-Stokes correction also in presence of a diverging Euler contribution.

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