OPEN AND CLOSED-LOOP PROPERTIES

OF AN R.F. ACCELERATED BEAM

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1. OPEN-LOOP PROPERTIES OF AN R.F. ACCELERATED BEAM

1.1 The Synchrotron Motion Equations

We are going to look at various forms of beam-controlled acceleration. To do this we put the system of beam plus R.F. field in the form of a possible element in a servo-system, with some inputs and outputs. This element is the part that we cannot easily change : the servo-system has to be designed around it.

For outputs we shall take detectable quantities $\Delta\varphi$, phase of R.F. relative to beam, and ΔR , beam radial displacement*.

For inputs one can consider the things that affect the process of acceleration :

- Ω_1 deviation of frequency programme from ideal value (correctly linked to B).
- V peak accelerating volts (per turn, say) on the cavities.
- $V_{\dot{B}}$ volts appropriate to the rate of rise of field.

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*Betatron oscillations are disregarded.

We shall only consider Ω_1. The others are of interest, particularly as one ought to study the effect of perturbation in them, but let us concentrate first on our basic servo-system which uses only Ω_1 as input point (Ref 5).

We can linearize and simplify the equations for the synchroton motion to :

$$\left.\begin{aligned} \frac{d}{dt}\Delta\varphi &= a\Delta R + \Omega_1 \\ \frac{d}{dt}\Delta R &= b\Delta\varphi \end{aligned}\right\}$$

The quantities a and b are interpreted as follows (see fig I).

Particles at ΔR say positive have higher energy than they would have at $\Delta R = 0$. They go round the machine faster and tend to arrive earlier, if we a re below transition energy, conversely if above. This means a changes continuously from negative, before transition, to positive after.

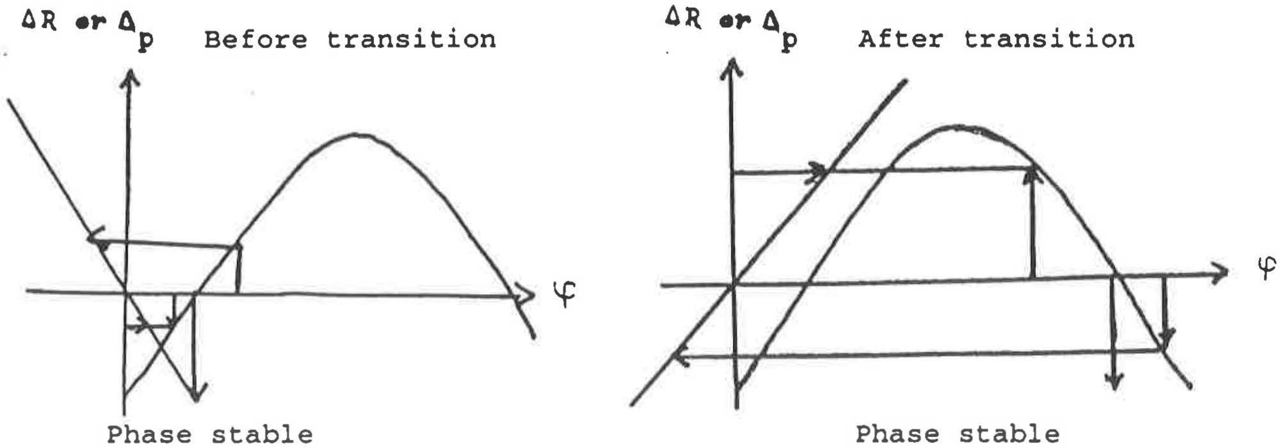
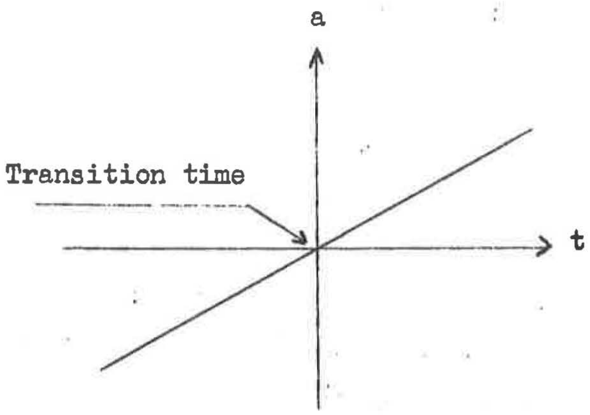
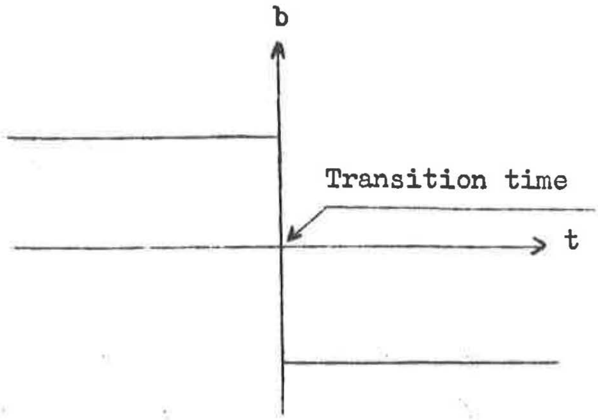


Figure I

One knows that equations like (1) have bounded oscillating solutions if a and b are of opposite signs (otherwise the particles go rapidly to infinites). So we make b changes discontinuously.





This is done by suitable choice of operating point :

$\cos \varphi_s$	+	−
φ_s	30°	150°

1.2 Frequency Tolerances

One reason beam control was designed for the CPS right from the start is the question of frequency tolerances, so let us look at them.

Suppose $\Omega_1 \neq 0$ is constant, and we look at the possible steady state condition $\Delta\dot{R} = \Delta\dot{\varphi} = 0$

$$\Delta R = -\frac{\Omega_1}{a}$$

What does this mean quantitatively ? If we are well above transition, a arises just from circumference considerations. With mean radius 100 m, + 10^{-4} frequency error will make - 1 cm displacement.

One does not need to be terribly close to transition for the situation to be say 10 times worse :

$$\Omega = 10^{-5} \rightarrow \Delta R \; 1 \text{ cm}$$

It is difficult to be more precise on this point : to put the equations in the form we have them and to make simple arguments about the way they behave, we have to treat the coefficients (a, b and the more fundamental quantities from which they are derived) as constants. This is not too bad * if they change slowly, but near transition they change fast. I am not sure that anyone has really calculated what frequency tolerances would make it possible to pass transition without beam control, but there may be something in the early literature.

A qualitative picture of what happens at transition in such circumstances is shown in Fig. 1 of Reference 1.

Constant-gradient machines are easier, partly because one does not have to pass transition energy, partly because their ratio aperture / mean radius is much bigger.

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* But misleading if you try to calculate the adiabatic damping.

1.3 The Synchrotron Equations in the Form of Transfer Functions

In the usual way we replace $\frac{d}{dt}$ by p, or sometimes by $j\omega$ and one can represent this system by * :

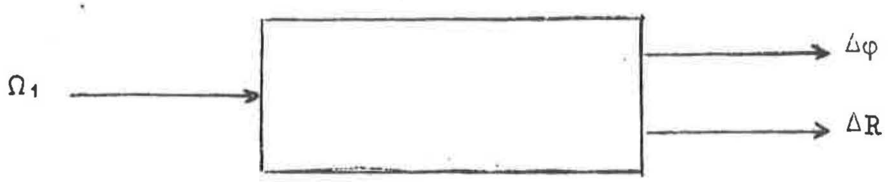


Figure II

with transfer properties which come directly from (1) :

$$\left.\begin{aligned} \Delta\varphi &= \frac{p}{p^2 + \omega_\varphi^2}\,\Omega_1 \\ &= \mu_1\,\Omega_1 \\ \Delta R &= \frac{o}{p^2 + \omega_\varphi^2}\,\Omega_1 \\ &= \mu_2\,\Omega_1 \end{aligned}\right\} \qquad (2)$$

where we have simplified the expressions a little by putting $\omega_\varphi = \sqrt{-ab}$, $\omega\varphi$ is, of course, just the frequency of synchrotron oscillations, which we could easily have obtained directly from (1)

* Reminder : Ω's for RF frequency and ω's for frequencies involved in the synchrotron motion.

Before considering closed loops let us just look at some of the features of the transfer functions.

At the frequency $\omega = \omega_\varphi$ the denominators vanish, so one can have some $\Delta\varphi$ and ΔR without any input Ω_1. The system has an oscillatory transient of constant amplitude, so no damping. The damping of phase oscillations which you probably know occurs when one accelerates in a proton synchrotron is due to the time-variation of the coefficients, which we have decided to neglect, and does not appear in the transfer functions.

In the limit of low frequencies, $\omega \to 0$, one has DC characteristics :

$$\left.\begin{aligned} \Delta\varphi &\to 0 \\ \Delta R &\to \frac{b}{\omega_\varphi^2}\,\Omega_1 \end{aligned}\right\} \qquad (3)$$

corresponding to what we already mentioned in connection with the RF frequency tolerances. One can see from (2) that (3) is also in good approximation true for any frequencies of perturbation low compared with ω_φ.

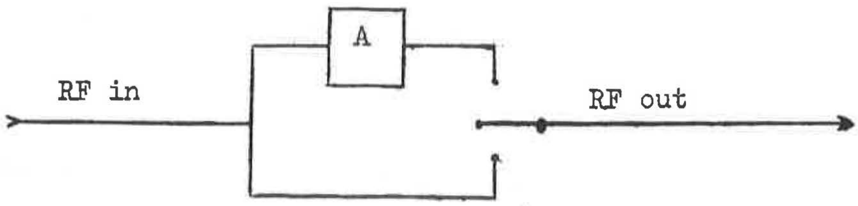
We may also look at the response to a unit impulse (delta function) in Ω_1. Since the Laplace transform of the unit impulse is just 1, all we need to do is look up the inverse Laplace transforms of :

$$\mu_1 = \frac{p}{p^2 + \omega_\varphi^2} \quad \text{and} \quad \mu_2 = \frac{b}{p^2 + \omega_\varphi^2}$$

so we find the response :

$$\Delta\varphi = \cos \omega_\varphi t \quad ; \quad \Delta R = \frac{b}{\omega_\varphi} \sin \omega_\varphi t$$

To understand this it is only necessary to see what is the physical interpretation of a unit impulse in Ω_1. We jump the frequency Ω_1 to infinity for zero time, in such a way that the integral is 1. This is just equivalent to jumping the phase of the RF system by one radian.



A = one radian phase advanced network

Figure III

There is no instantaneous effect on the protons, so they find themselves suddenly one radian away from φ_s and start oscillating freely from this initial condition.

A unit step function in Ω_1 can be dealt with by the usual techniques too. It also results in undamped oscillations, and the only other interesting fact that it discloses is the DC shift in ΔR which we have already calculated twice.

Clearly one would like to have a beam-control system which eliminates these undamped transients. One reason for this is because our equations of motion are not, in fact, linear ; consequently such oscillations of the bunches as a whole will in time be converted to oscillations of the particles within the bunch, so increasing the phase-spread and energy spread of the beam.

The last thing I want to mention before considering a closed loop system is the question of noise. Suppose Ω_1 contains noise * : it is, I think, physically obvious that a system whose transient response is an undamped sinusoidal is bad from the point of view of response to noise. The noise arising in any short interval of time will produce a transient which lasts forever, and all later noise will add statistically to it, so the amplitude can be expected to increase with time without limit. So one would like to produce a system in which the transient response is rapidly damped, so that noise would only build up the amplitude to some finite level, for at any instant of time only the recently-arrived noise would be effective.

It is perhaps worth remarking that a delta function (unit impulse) has a uniform Fourier spectrum, the same as has white noise ; so the transient response to a delta function can in fact enable one to calculate rather directly the response to white noise. More about noise without beam control is in Ref. 2.

2. CLOSED-LOOP PROPERTIES OF AN RF ACCELERATED BEAM

2.1 Feed-back from radial position of the beam

The first closed loop system that I shall consider is a type which we do not use on the CPS but is the first type to be proposed and used on a synchrotron (Ref. 3).

* Noise on Ω_1 means, of course some random F.M. of the acceleration frequency programme ; this is not the same as a noise voltage on the cavities in addition to the RF sinusoid.

If the main thing that worries us is the big radial excursions that result from small errors in the frequency programme, the obvious thing is to servo the radial position of the beam.

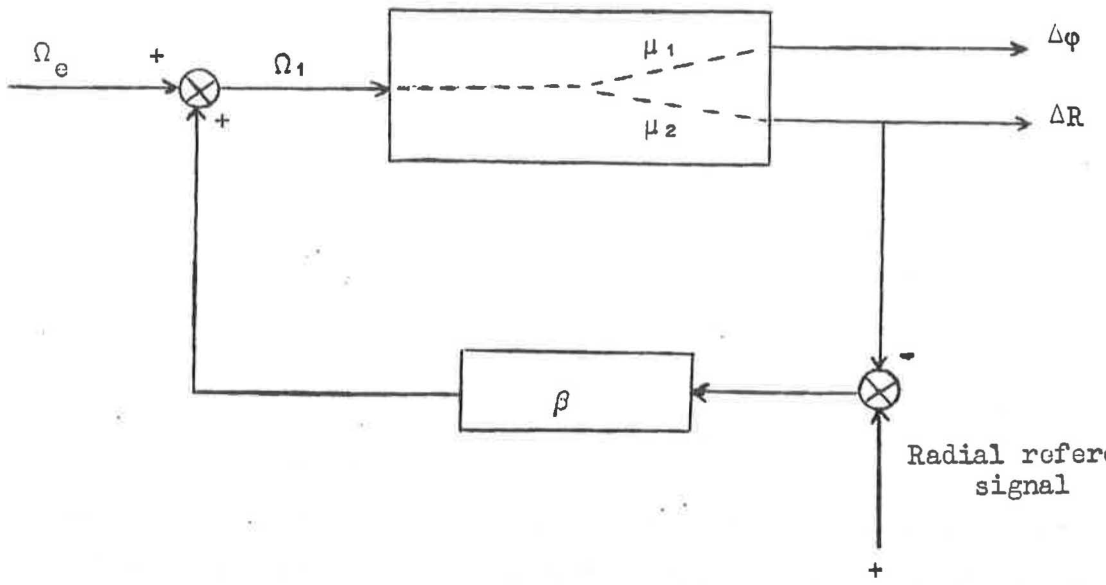


Figure IV

The radial position of the beam is compared with some reference value (which may well be zero if we want to accelerate in the middle of the aperture), and the difference is fed back in such a way as to change the frequency. There are many ways of drawing the diagram for such a set-up : note that (to avoid changing the formulae that we already have) we have kept Ω_1 to mean the frequency error applied to the beam and introduced Ω_e for the error of the frequency programme.

The radial error signal ΔR that we detect and compare and feed back is of course the average over the individual protons in the beam, and it is only this average behaviour that we shall be able to influence directly * by the servo-system that we consider.

The response of ΔR to programme errors Ω_e for this closed-loop system is given by the usual formula

$$\frac{\Delta R}{\Omega_e} = \frac{\mu_2}{1 + \beta \mu_2} \tag{4}$$

where we have $\mu_2 = \dfrac{b}{p^2 + \omega_\varphi^2}$ and where β is within practical limits, whatever we care to make it.

It is clear that to improve the situation with respect to programme frequency tolerances we must have a high open-loop gain $\beta \mu_2$ at least for all low frequencies (low compared with ω_φ), so one has the usual approximate relation for systems with high open-loop gain :

$$\frac{\Delta R}{\Omega_e} \simeq \frac{1}{\beta}$$

To give an example, one might make β such that ΔR is, say, 1 cm for a programme frequency error of 1 %, so giving reasonable frequency tolerances.

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* We already mentioned that the phase-spread and energy-spread of particles around the average can be influenced indirectly (by way of the non-linearities) : adversely by transients, favourably by a beam control system that suppresses transients.

Working for the moment in relative units for Ω etc., we have then to make :

$$\beta = \frac{1\ \%}{\text{cm}}$$

We have already seen that for low frequency components in the perturbation, we have :

$$\mu = \frac{1\ \text{cm}}{\sim 10^{-4}} \quad \text{(at high energies, and more near transition)}$$

So the open-loop is a hundred or more for a useful value of β. What is the transient response of such a system ?

Putting our expression for μ_2 into (4) we get :

$$\frac{\Delta R}{\Omega_e} = \frac{b}{p^2 + \omega_\varphi^2 + \beta b}$$

If β is just a simple coefficient one can conveniently write thus :

$$\frac{\Delta R}{\Omega_e} = \frac{b}{p^2 + \omega_r^2}$$

with $\omega_r^2 = \omega_\varphi^2 + \beta b$

This is of the same form as we had before we chosed the loop (2), but with a higher apparent resonant frequency.

We have seen that to get anything useful out of the feedback we must have $\beta\mu_2$ large compared with one at low frequencies ; and this amounts to saying βb large compared with ω_φ^2 ; so the effect on the transient behaviour is just to raise the free oscillation frequency of the system from ω_φ to ω_r, and this will be a substantial factor. The feedback has not altered the fact that the transient response contains an undamped oscillation, and does not help with respect to the noise problem nor the other bad effects of transients.

We are very much in the situation of someone who does not like the elastic oscillations of a mechanical system, so they add a more powerful spring to the existing one : this increases the resonant frequency, and makes the system much stiffer against DC or low-frequency forces, but does not produce any damping.

Evidently it would be interesting to consider the case where the transfer function β of the return path is more complicated than a simple constant coefficient : in particular, one might put an integration in :

$$\beta = K\, p^{-1}$$

this means the feedback goes to infinity at zero frequency, and, as is well known, reduces to zero the DC error ΔR associated with a DC or step-function in Ω_e.

The stability and transient response of Fig. IV with this type of β is an interesting exercise in servo-theory but we do not have time to discuss it here.

2.2 Phase-Lock

We now consider a system in which we measure $\Delta\varphi$ and feed this back into the RF frequency, so our block diagram becomes :

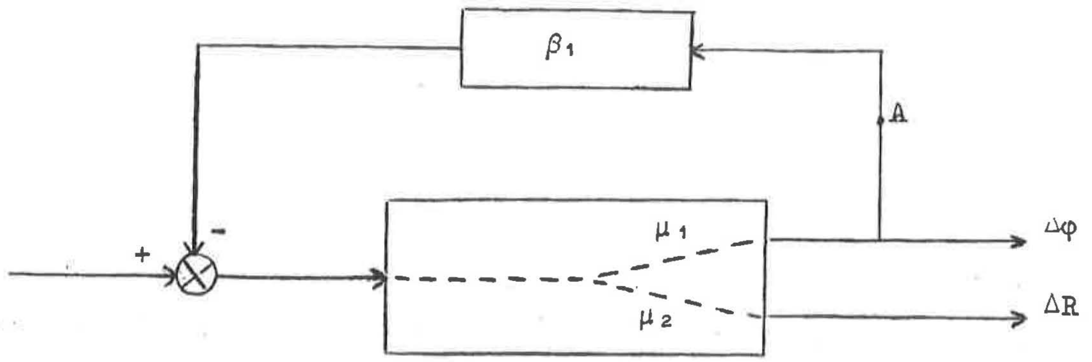

Figure V

We look first at the open-loop gain, which is :

$$\beta_1 \mu_1 = \frac{\beta_1 p}{p^2 + \omega_\varphi^2} \tag{5}$$

One sees immediately that (unless we include a stage of integration in β, to make β go to infinity like p^{-1} or faster as p approaches zero) this quantity $\beta_1 \mu_1 \to 0$ as $p \to 0$.

So this form of feedback does nothing about our problem of DC frequency tolerances : the $\frac{\Delta R}{\Omega_e}$ at DC is just as bad as it was without the feedback.

Before abandoning phase lock on these grounds, let us look at the transient response of this system. One finds very easily :

$$\frac{\Delta\varphi}{\Omega_1} = \frac{\mu_1}{1 + \beta_1 \mu_1} = \frac{p}{p^2 + \beta_1 p + \omega_\varphi^2} \qquad (6)$$

$$\frac{\Delta R}{\Omega_1} = \frac{\mu_2}{1 + \beta_1 \mu_1} = \frac{b}{p^2 + \beta_1 p + \omega_\varphi^2} \qquad (7)$$

Now these are interesting, as we have managed to introduce a p term into the denominators, and this changes the characteristics from those of a resonator without damping to those of a damped resonator. We can, and in the CPS phase-lock system we in practice do, make β_1 large enough to have this damping much stronger than critical damping. The condition for this is $\beta_1 >> \omega_\varphi$.

The CPS phase-lock system has β_1 about 2.10^7 s^{-1}, while ω_φ is never higher than 5.10^4 radians/sec, so this is very largely satisfied.

Under these conditions we can factorize this denominator :

$$p^2 + \beta_1 p + \omega_\varphi^2 = (p + \alpha_1)(p + \alpha_2) \qquad (8)$$

with approximately :

$$\alpha_1 = \beta_1 \quad (>> \omega_\varphi)$$

$$\alpha_2 = \frac{\omega_\varphi^2}{\beta_1} \quad (<< \omega_\varphi)$$

This is a system with two simple real decaying time-constants. Perhaps it is interesting to consider an electrical analogue : I take an L-C resonant circuit and damp it by putting a very small resistor across it :

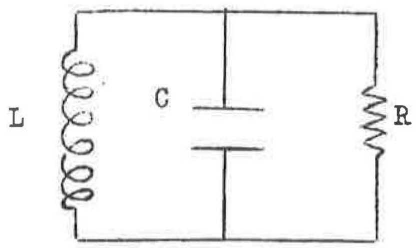

Figure VI

Nearly all the charge in the capacitor disappears very quickly, with time-constant RC. On the other hand the current in the inductor will continue to flow for a long time, the time-constant being $\frac{L}{R}$.

In our case any phase error of the beam nearly all disappears very fast, because of the strong feedback : this justifies the expression phase-lock for such a system. On the other hand any radial displacement of the beam tends to persist with a long time-constant, because the feedback results in there being only a very small $\Delta\varphi$ to move the beam across the chamber.

On the basis of (6), (7) and (8) and a table of Laplace transforms the response of $\Delta\varphi$ or ΔR to an impulse or step-function in Ω_1 can very easily be written down if one wants them.

The fact that most of the transient in $\Delta\varphi$ is very brief is particularly interesting from the point of view of the noise problem.

With the equations we have used there is no limit to β_1 but in practice there will be delays and phase-shifts which must be taken into account at high frequencies, and looked at in relation to the stability margins of the system. This has been done by Schnell (Ref. 4). There is one very interesting point about this : for frequencies substantially above ω_φ one can, in good approximation, neglect ω_φ^2 in any of the expressions that we have used, and, in particular, in (5), the loop gain, which becomes :

$$\frac{\beta_1}{p}$$

this does not contain any reference to the properties of the beam. So the problem of the stability of our phase-lock servo in the high frequency region where the delays and phase-shifts in β_1 begin to enter is just the same whether we servo onto the beam or servo onto some other RF signal. In the CPS, things are arranged so that when not servoed onto the beam one is servoed onto the programme RF instead.

The fact that one has to make a system with reasonable stability margins does of course mean that β_1 must be made to fall off suitably at high frequencies, and sets a limit to how big one can make it at low frequencies.

One must admit that the high frequency part of the transient response, if one writes it down on the basis of β_1 being a constant coefficient without frequency dependance or phase shift, will certainly be completely misleading in practice.

Since we are soon going to add another loop, and want to avoid too much complication, we shall write down the "medium-frequency" approximation to some of our expressions. This is done by assuming β_1 large and taking only the middle term of our denominators. Then (6) and (7) become :

$$\text{medium frequencies} \quad \begin{cases} \dfrac{\Delta\varphi}{\Omega_1} = \dfrac{\mu_1}{1 + \beta_1 \mu_1} \approx \dfrac{1}{\beta_1} \\[2ex] \dfrac{\Delta R}{\Omega_1} = \dfrac{\mu_2}{1 + \beta_1 \mu_1} \approx \dfrac{b}{\beta_1 \, p} \end{cases} \qquad (9)$$

With $\beta_1 >> \omega_\varphi$ this approximation is good over a wide band centred (logarithmically) on ω_φ.

2.3 Phase-Lock with Radial Position Feedback

We have shown that phase-lock, with as high as possible a feedback coefficient β_1, is good from the point of view of noise, but we still have to do something about reducing the response of ΔR to zero frequency or very low frequency perturbations in Ω_1. This is done by adding a radial servo loop.

Let us look at things physically for a moment, Suppose we detect a non-zero ΔR (say positive) and want to bring the beam back to $\Delta R = 0$. To do this we shall have to give the beam a bit less acceleration for a while.

There are only two reasonably direct ways of doing this, and both were considered for the CPS :

- a) Reduce the RF amplitude.
- b) Change the $\Delta\varphi$.

The second is the one we use, and the only one I shall discuss.

Since we already have a phase servo, the way chosen to change $\Delta\varphi$ with a radial error signal, with some coefficient β_2 is fed in at the point marked A on Fig. V.

First it is convenient to draw the phase-lock system radial loop open, arranged in a different way from Fig. V.

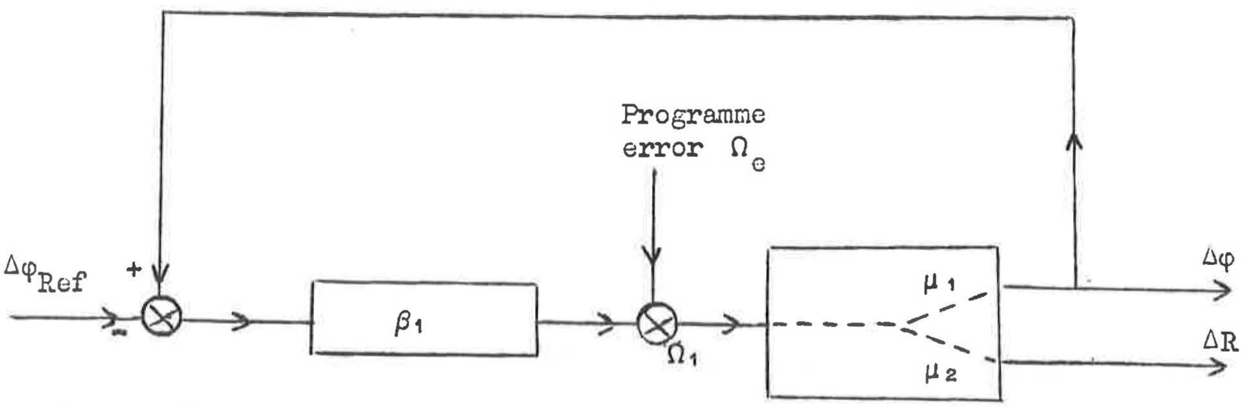

Figure VII

The transfer function from the input $\Delta\varphi$ to the output ΔR is :

$$\frac{\Delta R}{\Delta\varphi_{ref}} = \mu_R = \frac{\beta_1 \mu_2}{1 + \beta_1 \mu_1}$$

We call this μ_R because we are going to treat it as the forward gain of the radial loop.

Now close the radial loop :

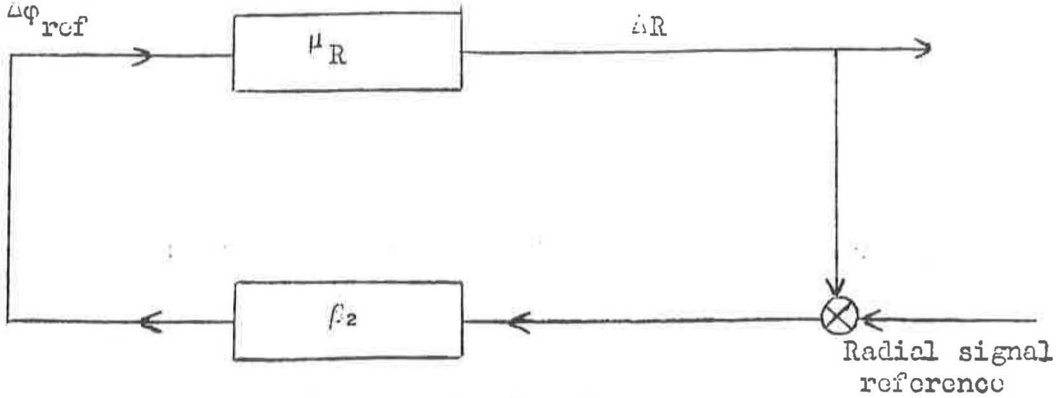

Figure VIII

The open-loop gain of this radial loop we look at first, for if it is large we can use the large-loop gain approximation, which is convenient :

$$\beta_2 \mu_R = \frac{\beta_2 \beta_1 \mu_2}{1 + \beta_1 \mu_1}$$

In the DC case, zero frequency, μ_1 is zero and μ_2 is $-\frac{1}{a}$, so we have :

$$\beta_2 \mu_R = - \beta_2 \beta_1 \frac{1}{a}$$

One sees that β_2 will need to be switched from positive before transition to negative after, because a is negative before, positive after. This is physically obvious when one goes back and considers what is the purpose of β_2.

Let us check whether this zero-frequency open-loop gain of the radial loop is in fact large. Working for the moment in millimetres and kHz, the CPS phase-lock loop has approximately :

$$\beta_1 = 50 \text{ kHz/degree}$$

And the radial error signal is fed back with about :

$$\beta_2 = 2 \text{ degrees/mm}$$

so $\beta_1 \beta_2$ is about 100 kHz/mm. The bunch-frequency change with radius, a, is biggest at injection, where it is about : 1 kHz/mm. So we have DC open loop gain radial ~ 100 at injection.

It is more (because a is less) at other energies, and, in particular, tends to infinity as transition is approached.

This very well justifies working in the approximation that radial loop gain is high at zero frequency. You will remember that we had, for programme errors, before closing the radial loop :

DC Radial loop open $\quad \frac{\Delta R}{\Omega_1} = -\frac{1}{a}$

Closing the radial loop we just divide this by the radial open loop gain and get :

DC Radial loop closed $\quad \frac{\Delta R}{\Omega_1} = \frac{1}{\beta_1 \beta_2} \sim 1 \text{ mm/100 kHz}$

independent of a.

The radial open-loop gain (9) in the medium frequency region becomes,:

$$\frac{\Delta R}{\Omega_1} = \frac{b\,\beta_2}{p}$$

If I put in $p = j\omega$ and the numbers, and calculate the ω at which this falls to $1/j$ I find at injection an ω corresponding to about 2 kHz, which is fairly well below the synchrotron oscillation frequency. At higher energies, except the immediate neighbourhood of transition, this is even more true.

Thus the open-loop gain of the radial servo is small, and it has little effect on the behaviour of the system, except at the very lowest frequencies where we need it.

As one may suppose, the radial response when we try to change radius is slow with this system (order of 10 ms time-constant at top energy). This is less of a disadvantage than one might guess, because there is risk of overloading the system if one tries to move the radius too fast.

Suppose I put 20 mm into the radial reference : the first thing that happens is that φ_{ref} jumps 40° and pretty soon after $\Delta\varphi$ servoes onto this and the beam starts moving radially at a rate determined by the RF working at 40° away from stable phase. Evidently the physical limits mean that we could not go much faster (especially if it is towards the outside). So there is not much interest in considering trying to get a faster time-constant in the radial response.

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