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KINEMATICAL RECONSTRUCTION OF $K_2^o \longrightarrow \pi^+\pi^-\pi^o$ FOR K_2^o DECAYS INSIDE THE VACUUM REGION OF THE CERN HEAVY LIQUID BUBBLE CHAMBER

I) INTRODUCTION

A set of modifications to GRIND have been written to permit kinematic fitting of charged decay modes of K_2^o mesons and in particular the modes :

$$K_2^o \longrightarrow \pi^+\pi^-\pi^o$$

taking account of the fact that in the X 4 experiment the decay vertex is <u>not</u> seen, but, in fact, the charged tracks are only seen after they leave the vacuum pipe a significant distance beyond the decay point. Although the problems involved are similar to those connected with the $K_2^o \longrightarrow 2\ \pi^o \longrightarrow 4\ \gamma$ reconstruction, there are certain essential new complications arising from the curvature of the π^+, π^- tracks which create significant difficulties. In particular unlike the $K_2^o \longrightarrow 4\ \gamma$ situation, the measured directions of the $\pi^{\pm}$ tracks are not those to which the momentum-energy constraints must be applied since the changes of direction of these tracks as they curve inside the vacuum pipe are very large. The consequence of this is, that the momentum-energy constraints applied by GRIND must be turned off and all constraints, kinematic and geometrical, must be applied through appropriate external routines.

It should be emphasized right off that the formulae and procedures which have been used are not absolutely exact, but only good approximations used because more exact calculations would have been too involved and time-consuming to set up and program. In other words this program is in a certain sense a compromise between an optimal calculation and one, which could be set up in a reasonable time. For example, the most glaring inaccuracy comes from the fact that the position errors of points are not considered in the usual GRIND : consequently it was expedient (as has also been done in the

$K_2^o \longrightarrow 2\ \pi^o$ GRIND) to ignore the position errors and consider only errors in momenta and directions of tracks. Thus in making a geometrical fit requiring the intersection of tracks, errors in their absolute positions are ignored and are in effect taken up by larger adjustments of track directions and momenta. Since very major changes of GRIND would have been necessary to take account of the position errors, it was not considered worthwhile to go into this. As will be seen further, there are also some approximations in the calculations to simplify things. By fitting Monte Carlo events, it has been checked that no significant errors are introduced in the fit by the approximations although these checks have not been exhaustive and perhaps at some moment when time permits further studies may be worthwhile.

One essential difficulty deserves special mention. It is a basic assumption of the usual fitting programs like GRIND that a set of accurate initial values for the variables are available, and that linearizing the constraint equations about these initial values is a good approximation. This assumption, usually valid in Hydrogen chamber analysis, is much less justified in heavy liquid work. Indeed it is remarkable how far off the measurements often are, and the consequence is that although a reasonable fit with a reasonable χ^2 in principle may exist, the program is unable to converge to it. In the present program, much effort has been devoted to making the program efficient, i.e. getting to the proper χ^2 if a reasonable exists in spite of poor starting conditions, but there is still a fraction of $\sim$ 10 % of the events which fail because of inefficiency of fitting program. This class of events is readily recognizable from the nature of the output and can be separated easily from the real failures (i.e. events for which no acceptable χ^2 exists).

II) THE INTERSECTION OF TWO CHARGED TRACKS

A) Orbit Equations

The first fit which the program attempts is the intersection in space of the two charged tracks. We therefore begin with a discussion of this problem. Consider a charged particle in a magnetic field $\vec{B}$ (assumed constant).

Let the particle pass through the point $\vec{r}_1$ with direction given by unit vector $\vec{e}_1$ and let R_1 be the radius of curvature of the orbit. Effectively these quantities $\vec{r}_1$, $\vec{e}_1$, R_1 are just what are given by the geometrical reconstruction program (say DRAT). It can be shown easily that the parametric equations of the helical orbit of the particle are given by

$$\vec{r} = \vec{r}_1 + \frac{(\vec{e}_1 \cdot \vec{B})}{|\vec{e}_1 \times \vec{B}|} \frac{\vec{B}}{|\vec{B}|} R_1 \theta_1 + R_1 \left\{ \text{Sin } \theta_1 \frac{\vec{B} \times (\vec{e}_1 \times \vec{B})}{|\vec{B}||\vec{e}_1 \times \vec{B}|} \pm (1 - \cos \theta_1) \frac{(\vec{e}_1 \times \vec{B})}{|\vec{e}_1 \times \vec{B}|} \right\} \quad (1)$$

where $\pm$ refers to the sign of charge of the particle, and θ_1 is the angle measured in the plane (perpendicular to $\vec{B}$) between the radius vector from the centre to the point $\vec{r}$ and the initial radius vector to the point $\vec{r}_1$ (i.e. $\theta_1 = 0$ when $\vec{r} = \vec{r}_1$). Note that the subscript 1 ($\vec{r}_1$, θ_1, R_1 etc) denotes one charged particle, the subscript 2 denoting the other.

The parameter θ_1 can be replaced by the length L_1 measured along the trajectory

$$L_1 = \frac{(R_1 \theta_1) |\vec{B}|}{|\vec{e}_1 \times \vec{B}|} \quad (2)$$

Unfortunately (1), expressed either in terms of θ_1, or L_1, is awkward for calculation. The mixture of terms linear in θ_1, and trigonometric functions of θ_1 leads to transcendental equations which are unpleasant. To simplify we will assume that over the distance of extrapolation of the tracks, the bending angle is not too large, i.e. $\theta_1 << 1$.

$$\text{Then Sin } \theta_1 \approx \theta_1 = \frac{L_1}{R_1} \frac{|\vec{e}_1 \times \vec{B}|}{|B|}$$

$$1 - \cos \theta_1 \approx \frac{\theta_1^2}{2} = \frac{L_1^2}{2 R_1^2} \frac{|\vec{e}_1 \times \vec{B}|^2}{B^2} \quad (3)$$

We can then rewrite (1) in terms of L_1, using the approximations (3) :

$$\vec{r} = \vec{r_1} + \left[\frac{(\vec{e_1} \cdot \vec{B})\,\vec{B} + \vec{B} \times (\vec{e_1} \times \vec{B})}{B^2} \right] L_1 \pm \frac{L_1^2}{2 R_1} (\vec{e_1} \times \vec{B}) \frac{|\vec{e_1} \times \vec{B}|}{B^2} \qquad (4)$$

or

$$\vec{r} = \vec{r_1} + \vec{e_1} L_1 + \vec{f_1} L_1^2 \qquad (5)$$

$$\text{where } \vec{f_1} = \pm \frac{(\vec{e_1} \times \vec{B})\,|\vec{e_1} \times \vec{B}|}{2 R_1 B^2}$$

Equation (5) is easily seen to go into the usual straight line limit as $R_1 \to \infty$, $f_1 \to 0$. The $L_1^2 f_1$ term comes from the effect of the track curvature. For typical situations with our data, we have to extrapolate tracks through bending angles of order 200 mr with radii of order 50 cm, and the error in the extrapolated position from the above approximation (3) is of order of a few hundred microns which is quite negligible in comparison to effects of measurement error.

We now further specialize the result (5) by taking $\vec{B}$ in the z-direction of the chamber coordinate system : $B = |B|\,k$, and we express R_1 in terms of the momentum P_1

$$R_1 = \frac{P_1}{0.3} \frac{|\vec{e_1} \times \vec{B}|}{B^2} \qquad \text{where} \quad P_1 \text{ is in } \frac{\text{MeV}}{c}, \quad B \text{ is in kGauss} \qquad (6)$$

With these conditions the components of $\vec{f_1}$ are

$$f_1 = (u_1, v_1, 0) \quad \text{where}$$

$$u_1 = \pm 0.15 \frac{|B|}{P_1} m_1 = \pm 0.15 \frac{|B|}{P_1} \cos\lambda_1 \sin\varphi_1 \qquad (7\,a)$$

$$v_1 = \mp 0.15 \frac{|B|}{P_1} l_1 = \mp 0.15 \frac{|B|}{P_1} \cos\lambda_1 \cos\varphi_1 \qquad (7\,b)$$

$$\underline{l_1 = \cos\lambda_1 \cos\varphi_1} \qquad \underline{m_1 = \sin\varphi_1 \cos\lambda_1} \; ; \; \underline{n_1 = \sin\lambda_1}$$

are the usual direction cosines of the initial point r_1.

In summary we can write (5) in component form :

$$\begin{aligned} x &= x_i + \ell_i L_i + u_i L_i^2 \\ y &= y_i + m_i L_i + v_i L_i^2 \\ z &= z_i + n_i L_i \end{aligned} \qquad (8)$$

for the i^{th} charged track passing through point x_i, y_i, z_i with direction cosines ℓ_i, m_i, n_i.

B) Energy Loss Correction

Before a fit is attempted, an energy loss correction is made to the momentum of each charged track, to take account of the energy loss in the pipe aluminium wall and freon.

Let the equation of the pipe outer wall be

$$(y + 0.38)^2 + (z + 59.5)^2 = (2.25)^2 \qquad (9)$$

To go upstream along the track L_i must take on negative values, hence let $S_i = - L_i$, and substitute (8) with S_i into (9).

$$(y_i - m_i S_i + v_i S_i^2 + 0.38)^2 + (z_i - n_i S_i + 59.5)^2 = (2.25)^2 \qquad (10)$$

Solving for S_i (done numerically in the program) gives directly the freon path length,

$$S_{freon} = S_i$$

The aluminium path length is obtained from the wall thickness t (t = 0.25 cm) as follows

$$S_{alum} = \left| \frac{t}{\left(\frac{d\rho}{dS_i}\right)} \right|$$

where

$$\rho = \sqrt{(y + 0.38)^2 + (z + 59.5)^2} \qquad (11)$$

$$\rho = \sqrt{(y_i - m_i S_i + v_i S_i^2 + 0.38)^2 + (z_i - n_i S_i + 59.5)^2}$$

and the absolute value bracket is used because $\frac{d\varphi}{dS_i}$ is negative.

In calculating the energy loss, the aluminium loss is converted into an effective freon loss by multiplying S_{alum} by 1.8.

Effective total : $S_{freon} = S_{freon} + 1.8\ S_{alum}$

Finally it may be noted that not unfrequently the initial trajectory point x_i, y_i, z_i already falls inside of the pipe surface. In this case $S_{freon} = 0$, and S_{alum} is calculated as in (11) with S_i set equal to zero.

Besides the energy loss correction, the extrapolation of the track back leads to some corrections in the assumed errors. In particular, to both azimuth and dip errors we quadratically add multiple scattering errors of the form

$$\Delta\varphi = \Delta\lambda = \frac{15}{P\beta}\sqrt{\frac{S}{x_o}} \qquad (12)$$

where $\Delta\varphi$, $\Delta\lambda$ are the added errors in radians, P, β are the momentum (MeV/c) and velocity of the particle, S is the distance travelled in freon or aluminium and x_o is the radiation length. Actually for this we use x_o = 11 cm, the freon radiation length, and let $S = S_{freon} + 1.2\ S_{alum}$. It can be shown that (12) overestimates somewhat the error, but since other factors underestimate them, we accept it.

C) The One Constraint Fit

We now consider the problem of finding the best intersection point of the two-charged tracks. It is evident that this is a one-constraint fit and the problem is to express the constraint equation in convenient form. We assume that the energy loss correction has been made and set up the equations (8) using the corrected values of the momenta in the calculations of the u_i and v_i of (7). The x_i, y_i, z_i are still the measured initial points of the trajectories and hence there is an approximation in that the momentum applicable inside the aluminium pipe is also used in the freon from the pipe wall up to the first measured point. The error is however very small since (i) there is very little path length of extrapolation in the freon (ii) the energy loss correction is generally quite small.

Solving (8) for L_i in terms of z

$$L_i = \frac{z - z_i}{n_i} \tag{13}$$

and substituting into the other equations

$$\begin{aligned} x &= x_i + \ell_i \left(\frac{z - z_i}{n_i}\right) + u_i \left(\frac{z - z_i}{n_i}\right)^2 \\ y &= y_i + m_i \left(\frac{z - z_i}{n_i}\right) + v_i \left(\frac{z - z_i}{n_i}\right)^2 \end{aligned} \tag{14}$$

If the orbits are to intersect, at the point of intersection the x's of two trajectories must be the same :

$$x_1 + \ell_1 \left(\frac{z - z_1}{n_1}\right) + u_1 \left(\frac{z - z_1}{n_1}\right)^2 = x_2 + \ell_2 \left(\frac{z - z_2}{n_2}\right) + u_2 \left(\frac{z - z_2}{n_2}\right)^2 \tag{15}$$

We can rewrite (15) in the form

$$az^2 + bz + c = 0$$

where $a = \frac{u_1}{n_1^2} - \frac{u_2}{n_2^2}$

$$\begin{aligned} b &= \frac{\ell_1}{n_1} - \frac{2\, z_1 u_1}{n_1^2} - \frac{\ell_2}{n_2} + \frac{2\, z_2 u_2}{n_2^2} \\ c &= x_1 - \frac{z_1 \ell_1}{n_1} + \frac{z_1^2 u_1}{n_1^2} - x_2 + \frac{z_2 \ell_2}{n_2} - \frac{z_2^2 u_2}{n_2^2} \end{aligned} \tag{16}$$

and solve for z :

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{17}$$

assuming that we know the proper choice of solution, this gives us the z value of the intersection of the x - z projections of the orbits.

Call this value z defined by (17) g. Then the constraint consists in requiring that at z = g the g values of the two orbits coincide :

$$y_1 + m_1\left(\frac{g - z_1}{n_1}\right) + v_1\left(\frac{g - z_1}{n_1}\right)^2 - y_2 - m_2\left(\frac{g - z_2}{n_2}\right) - v_2\left(\frac{g - z_2}{n_2}\right)^2 = 0 \quad (18)$$

(18) is the actual constraint equation used in the program.

We now consider in more detail the difficulties and possible criticisms of this procedure :

a) Calculational difficulties

At the outset, the terms $\frac{z_i u_i}{n_i^2}$, $\frac{z_i^2 u_i}{n_i^2}$ in (16) were found to becmme very large and give rise to large round-off errors. This was immediately repaired by translating the z-coordinate origin, so as to make z_1, z_2 much smaller than the 60 cm values they had. In particular the z origin was placed at the z_i value of the track with the least dip, so that in (16) the z_1 or z_2 is zero corresponding to whichever is the smaller of n_1 or n_2. This solution completely removed the round-off problem. It may be added at this point that the round-off error problem was further looked into at a later time, and it was found that the difficulty was connected with the fact that the quantities a, b, c in (16) contain factors which can become very large but which when combined in (17) cancel out. A more intelligent procedure would therefore have been to replace (17) by the explicit formula found from substituting the equations (16) into it:

$$z \equiv g = \frac{z_1 + z_2}{2} + \frac{1}{2(a_1 - a_2)}\left[- r + (z_1 - z_2)(a_1 + a_2) \pm \sqrt{\quad} \right] \quad (19)$$

where $r_i = \frac{\ell_i}{n_i}$ $a_i = \frac{u_i}{n_i^2}$ $r = r_1 - r_2$

$$\sqrt{\quad} = \sqrt{r^2 + 4\left[a_1 a_2 (z_1 - z_2)^2 - (a_1 - a_2)(x_1 - x_2) + (z_1 - z_2)(a_1 r_2 - a_2 r_1)\right]}$$

It is clear from (19) that except for the $\frac{z_1 + z_2}{2}$ terms all the z_i dependence is through the term $(z_1 - z_2)$ and independence of coordinate origin is explicit. In some development work (19) was used instead of

(16), (17) but made no difference even for very small n_i. Hence the present program still uses (16) and (17).

b) How do we choose the sign in the solution for g ?

Because of the low Q value of the τ decay and the relatively high momenta of the events, it is a general characteristic that the charged tracks tend to go very much in the forward direction. Under these conditions, the curvature plays only a small role in the x - z plane. This is also obvious from the fact that the curvature term is proportional to $u_i \sim \mathrm{Sin}\,\varphi_i$, and φ_i is small, when the tracks go forward. Consequently, since if there were no curvature there would be no sign ambiguity one can suppose that we should find the proper sign by taking the limit of zero curvature ($u_i \longrightarrow 0$). It is evident from (16), (17) that this leads to the choice of sign : sign = sign of b. It turns out in practice that this choice is almost always correct, but there are exceptions. In the present program, the proper choice is made by comparison of the two solutions for g with an approximate intersection point obtained prior to the fitting procedure. This approximate intersection is found using the charged tracks and the gamma rays and is used to obtain directions and curvatures for the tracks to be used as initial values in the fits. The method of calculation of this point will be discussed further on.

c) What happens if the dips of the charged tracks are nearly the same ?

Because of measurement errors it is evident that the intersection point of the tracks in the x - z plane may shift very far in one direction or another for slight changes in the dips or may not exist at all if the discriminant of (17) becomes negative. This is a substantial problem if the initial values have large errors. For that reason it is essential to have a procedure prior to the fit which uses all available information (i.e. pion tracks and gamma directions) to find an approximate decay point and which calculates for all quantities initial values such that the tracks come from that point. Once this is done the problem of pion tracks of nearly equal dips essentially disappears. This "initialization procedure" already mentioned in the previous paragraph will be discussed further.

III) THE GAMMA GEOMETRIC CONSTRAINTS

It is straightforward to add to the constraint forcing the two charged tracks to meet further constraints requiring the gamma lines of flight to go through the intersection point of the pion tracks. Thus for the ith gamma ray one has the equations

(we let i = 1, 2 stand for charged tracks)
i = 3, 4 " " γ rays)

$$\varphi_i - \tan^{-1}\left(\frac{y_i - y_g}{x_i - x_g}\right) = 0 \qquad (20\ a)$$

$$\lambda_i - \mathrm{Sin}^{-1}\left[\frac{z_i - g}{\sqrt{(z_i - g)^2 + (x_i - x_g)^2 + (y_i - y_g)^2}}\right] = 0 \qquad (20\ b)$$

where g is the value of z given by (17), x_g is the corresponding value of x.

$$x_g = x_1 + \ell_1\left(\frac{z_1 - g}{n_1}\right) + u_1\left(\frac{z_1 - g}{n_1^2}\right)^2 \qquad (21\ a)$$

or

$$x_g = x_2 + \ell_2\left(\frac{z_2 - g}{n_2}\right) + u_2\left(\frac{z_2 - g}{n_2}\right)^2 \qquad (21\ b)$$

Since g was calculated from the requirement of an intersection in the x - z plane of tracks 1 and 2 (the charged tracks) (21 a) or (21 b) must give the same value of x_g and one can use either equation.

However, until the fit is completed, the corresponding quantities y_g for tracks 1 and 2 do not agree (unless the charged tracks precisely intersect before the fit is completed) and we use for y_g in eq. (20 a) and (20 b) the following average

$$y_g = \frac{1}{2}\left\{ y_1 + y_2 + m_1\left(\frac{z_1 - g}{n_1}\right) + m_2\left(\frac{z_2 - g}{n_2}\right) + v_1\left(\frac{z_1 - g}{n_1}\right)^2 + v_2\left(\frac{z_2 - g}{n_2}\right)^2 \right\} \qquad (22)$$

It should be noted that the form of the constraint equations (20) is particularly convenient for GRIND since they are linear in φ_i, λ_i and have not difficulty with singularities.

IV) THE KINEMATIC CONSTRAINTS

A) Extrapolation to vertex

Having discussed the geometric constraints, we are now ready to discuss the kinematic constraints.

Unlike the X 4 situation, the momentum energy constraints at the K^o decay vertex are not properly applied by GRIND since the measured pion directions are not these which apply at the vertex.

It is therefore necessary to "turn off" the GRIND constraints and put in our own. In the original planning of the program, it was envisaged that the "turning off" would not be done by any change in GRIND routines but by a trick as follows : in the GRIND set-up the K-mass, momentum and the two gamma momenta would be left as unknown, and the proper momentum-energy constraints would be put in externally through a CONSTR (I) routine. Since there are four energy-momentum equations at the K^o decay vertex and four unknowns (K mass, K momentum, two gamma momenta) the unproper GRIND constraints would only have the effect of producing fictitious values for these unknowns but would not insert any unproper constraints: Provided now that the externally put in constraints did not use the fictitious GRIND quantities, and that these fictitious quantities were not the ones printed out at the end, the program should then work properly. This long explanation is given only by way of motivating the fact that the true K-momentum and the two true gamma momenta are not part of the GRIND variables. They are calculated externally to the standard GRIND package in the CONSTR (I) routines, and their errors are also calculated externally. The results of this are then fed into the proper banks for printout at the end of the fit.

Before considering in detail the kinematic constraints, we discuss the necessary extrapolation of the pion tracks to obtain the proper directions for substitution in the conservation laws. For this purpose we consider again the exact orbit equation (1) as the directions obtained by differentiating the approximate equations (5) are not precise enough. If one expands the equations (1) to higher order than second order in L, namely to fourth order in L and differentiates $\vec{r}$ with

respect to L to obtain the direction cosines one finds for these at the vertex the values

$$\ell_i' = \ell_i + \xi_i m_i L_i - \frac{\xi_i L_i^2}{2} \ell_i - \frac{\xi_i^3 L_i^3 m_i}{6} \cdots \quad (23\ a)$$

$$m_i' = m_i - \xi_i \ell_i L_i - \frac{\xi_i^2 L_i^2}{2} m_i + \frac{\xi_i^3 L_i^3 \ell_i}{6} \cdots \quad (23\ b)$$

$$n_i' = n_i \quad (23\ c)$$

where ℓ_i', m_i', n_i' are the direction cosines of the vertex, ℓ_i, m_i, n_i are the direction cosines of the first measured point of the track

$$\xi_i \equiv \frac{2 u_i}{m_i} = \frac{-2 v_i}{i} = \pm 0.3 \frac{|B|}{P_i} \quad (24)$$

and $L_i = \frac{g - z_i}{n_i}$ is the distance along the track from the point x_i, y_i, z_i to the vertex point whose z coordinate (see eq. 17) is g. Obviously the fact that $n_i' = n_i$ comes from taking the field along the z direction. One can easily show from (23) that $\ell_i'^2 + m_i'^2 + n_i'^2 = \ell_i^2 + m_i^2 + n_i^2$ up to (but not including) terms of order L_i^4. To guarantee that all direction cosines are properly normalized we amend the ℓ_i', m_i' from (23) by

$$\ell_i' \Rightarrow \ell_i' \frac{\sqrt{\ell_i^2 + m_i^2}}{\sqrt{\ell_i'^2 + m_i'^2}}$$

$$m_i' \Rightarrow m_i' \frac{\sqrt{\ell_i^2 + m_i^2}}{\sqrt{\ell_i'^2 + m_i'^2}} \quad (25)$$

Thus we now have the direction cosines at the vertex in terms of the quantities used in the fit.

B) The π^o constraint

In order to apply the π^o mass constraint we must have values of the gamma momenta. Since the gamma momentum is not used as a GRIND variable in the fit, it must be calculated externally (in the CONSTR (I) routines). We use the transverse momentum conservation to do this.

Thus if $P_{\gamma 3}$ and $P_{\gamma 4}$ are the gamma momenta we have

$$P_{\gamma 3}\, m_3 + P_{\gamma 4}\, m_4 = - P_1\, m_1' - P_2\, m_2' \qquad (26\ a)$$

$$P_{\gamma 3}\, n_3 + P_{\gamma 4}\, n_4 = - P_1\, n_1 - P_2\, n_2 \qquad (26\ b)$$

where the quantities on the right side are transverse momenta of the pions. From (26) we solve for $P_{\gamma 3}$, $P_{\gamma 4}$

$$P_{\gamma 3} = \frac{m_4\,(P_1\, n_1 + P_2\, n_2) - n_4\,(P_1\, m_1' + P_2\, m_2')}{m_3\, n_4 - m_4\, n_3} \qquad (27\ a)$$

$$P_{\gamma 4} = \frac{-\, m_3\,(P_1\, n_1 + P_2\, n_2) + n_3\,(P_1\, m_1' + P_2\, m_2')}{m_3\, n_4 - m_4\, n_3} \qquad (27\ b)$$

We are now ready to write down the pion constraint

$$P_{\gamma 3} + P_{\gamma 4} - \sqrt{(m_{\pi^o})^2 + P_{\gamma 3}^{\,2} + P_{\gamma 4}^{\,2} + 2\, P_{\gamma 3}\, P_{\gamma 4} \cos\theta} = 0 \qquad (28)$$

where $\cos\theta = \ell_3\, \ell_4 + m_3\, m_4 + n_3\, n_4$ (29)

i.e. θ is the angle between the gamma rays.

One point here is noteworthy. Equ. (28) is not the only possible form of the pion constraint: indeed in early versions other forms were used such as

$$m_{\pi^o}^{\,2} - 2\, P_{\gamma 3}\, P_{\gamma 4}\,(1 - \cos\theta) = 0 \qquad (30)$$

However (28) has the useful property not shared by (30) that it cannot be satisfied except with positive values of $P_{\gamma 3}$, $P_{\gamma 4}$. Since it may occasionally happen because of errors of measurement that $P_{\gamma 3}$, $P_{\gamma 4}$ from (27) are negative, this feature is of some relevance.

C) The K° Constraint

The K° constraint is most easily written if one remembers that the vector sum of the secondary momenta lies along the x-axis; hence only x - components of momenta need be added to get the total momentum. Thus the constraint can be written

$$(E_1 + E_2 + P_{\gamma 3} + P_{\gamma 4})^2 - (P_1 \ell'_1 + P_2 \ell'_2 + P_{\gamma 3} \ell_3 + P_{\gamma 4} \ell_4)^2 - m_K^2 = 0 \qquad (31)$$

where E_1, E_2 are the total pion energies.

Incidentally equ. (31) is also used to calculate the K-mass for the fit in which only the π° kinematic constraint is invoked leaving the mass of the incident particle floating. The momentum of the K after the final π° and K° constraints are satisfied is calculated from the formula

$$P_K = P_1 \ell'_1 + P_2 \ell'_2 + P_{\gamma 3} \ell_3 + P_{\gamma 4} \ell_4 \qquad (32)$$

V) INITIALIZATION

It has already been pointed out that it is valuable in many cases (and probably harmful in none) to initialize the variables rather than use the measured quantities. This procedure is also carried out in the X 4 GRIND in that a geometrical intersection point is found via a special routine prior to the fit. The necessity of this arises from the fact that GRIND, assuming, as is the case in hydrogen chamber physics, that precise starting values are available from the measurements, linearizes the constraint equations in solving each step. The convergence of the procedure (in which at the end the true rather than the linearized constraint equations must be satisfied) depends upon whether the linearization is a good approximation.

Clearly, by initializing the variables so as to start with them as close as possible to their final values, one optimizes the chance of a successful convergence.

The general philosophy of initialization in the program is that one splits up the fits using as the initial value for each subsequent fit the fitted value from the previous fit. In particular one first carries out the intersection of the charged tracks, a 1-C procedure where (18) is equation of constraint, using initial variables obtained in a manner to be described further along. One then uses the fitted directions and curvatures of the charged tracks, and γ-ray directions obtained by constructing lines from the charged track intersection to the gamma apices as initial values for a complete geometric fit in which all outgoing tracks and gamma rays are made to intersect. This complete geometric fit has the five constraints (18) and (20 a), (20 b) for each gamma ray. (Evidently if only a vertex is measured for one of the gammas there are only three constraints). One then continues by using the fitted information from the geometric fit as initial values for a 6-C fit using besides the geometric constraints (18), (20) the π^o kinematic constraint (28). Finally the result of this is used as initial value for the 7-C fit which in addition to the previous constraints embodies the K^o constraint (31).

The above strategy has, with experience, undergone some slight modifications.

i) If the 5-C fit, using the 1-C fit results as input values fails (i.e. does not converge satisfactorily) we try again using as input values those determined in the initialization routine. This procedure is useful in the case where the two charged tracks are emitted at only a small angle with respect to each other, and consequently their fitted intersection has a large uncertainty in its location.

ii) If a fit fails, so that it provides no initial values for the next fit, the measured values of the variables are used for the initial values of the next fit.

iii) Because of measurement errors in the pion tracks, the correction for energy loss in the freon and pipe wall is often uncertain. Consequently after a successful geometric (5-C) fit is made, the energy loss correction is repeated prior to the next fits.

We now come to a discussion of the initialization routine used before any fits, to set up starting values for the variables. This consists of going upstream along one of the two charged pion tracks, say the positive, following it as though its measured parameters were exact and taking equally spaced points along this trajectory as possible. K^o decay vertex points. For each such point, one then forces the other (negative) track and the gamma rays to go through that point and calculates the corresponding χ^2. This procedure is followed until a minimum χ^2 is reached. The same procedure is then carried out moving back along the other (negative) track upstream until a minimum ^{2} for the intersection of the positive track and the two gammas is reached. An average is then made of the two optimal points along the two trajectories and this is taken as an initial decay point, from which initialized directions and momenta are calculated for the first GRIND fit. These remarks are somewhat sketchy and we now go into further detail.

i) Calculation of the χ^2

Given a possible decay point x, y, z it is easy to draw lines to the gamma apices, calculate the corresponding directions and by comparison with the measured quantities and their errors, form a χ^2. This is a little less trivial for a charged track since to a given decay point and track point, there is not a unique direction and curvature to compare with the measured values; but, in fact, a continu s of such values. In forming this contribution to the χ^2 one must therefore find the particular combination of direction and curvature which minimizes the χ^2. This problem can be readily solved and we give the results here and leave the derivation for the Appendix.

Thus if we construct a curved track from the decay point x, y z to the initial track point x_i, y_i, z_i, the contribution to the χ^2 from the φ_i and $1/p_i$ measured values of azimuth and reciprocal momentum with squared errors $\left(\Delta\frac{1}{p_i}\right)^2$ $(\Delta\varphi_i)^2$:

$$\chi^2\left(\varphi_i, \frac{1}{p_i}\right) = \frac{\left\{\frac{.3|B|}{p_i\cos\lambda_i} \mp \frac{2}{(S_{ixy})^2}\left[(x_i-x)\sin\varphi_i - (y_i-y)\cos\varphi_i\right]\right\}^2}{\left[\frac{.3|B|}{\cos\lambda_i}\right]^2 \left(\Delta\frac{1}{p_i}\right)^2 + 4\,(\Delta\varphi_i)^2\left[\frac{(x_i-x)\cos\varphi_i + (y_i-y)\sin\varphi_i}{(S_{ixy})^2}\right]^2} \tag{33}$$

where $S_{ixy} \equiv \sqrt{(x - x_i)^2 + (y - y_i)^2}$

and the upper sign applies to the positive track, and the lower to the negative track. Furthermore the values of $1/p_i$ and φ_i which give this minimum χ^2, $\left(\frac{1}{p_i}\right)_{i\ \text{optimal}}$, $(\varphi_i)_{\text{optimal}}$

are given by

$$(\varphi_i)_{\text{optimal}} = \varphi_i + \chi^2\left(\varphi_i, \frac{1}{p_i}\right) \frac{(\Delta\varphi_i)^2\left\{\mp \frac{2}{(S_{ixy})^2}\left[(x_i-x)\cos\varphi_i + (y_i-y)\sin\varphi_i\right]\right\}}{\left[\frac{.3|B|}{p_i\cos\lambda_i} \mp \frac{2}{(S_{ixy})^2}\left[(x_i-x)\sin\varphi_i - (y_i-y)\cos\varphi_i\right]\right]} \tag{34}$$

$$\left(\frac{1}{p_i}\right)_{i\ \text{optimal}} = \frac{1}{p_i} - \frac{\chi^2\left(\varphi_i, \frac{1}{p_i}\right)\left(\Delta\frac{1}{p_i}\right)^2 \frac{.3|B|}{\cos\lambda_i}}{\left[\frac{.3|B|}{p_i\cos\lambda_i} \mp \frac{2}{(S_{ixy})^2}\left[(x_i-x)\sin\varphi_i - (y_i-y)\cos\varphi_i\right]\right]} \tag{35}$$

Just for completeness, we exhibit the other contributions to the χ^2. First from the dip error in the charged track

$$\chi^2(\lambda_i) = \frac{\left[\lambda_i - \sin^{-1}\frac{z_i - z}{S_{ic}}\right]^2}{(\Delta\lambda_i)^2} \tag{36}$$

where $S_{ic} = \sqrt{(z_i - z)^2 + (2\,R_i\alpha_i)^2}$ (37)

$$R_i = \frac{p_i\cos\lambda_i}{.3|B|} = \text{radius of the orbit}$$

$$\alpha_i = \sin^{-1}\left(\frac{S_{ixy}}{2\,R_i}\right) \tag{38}$$

Here, although there is a slight interplay between the curvature $1/R_i$ and the dip, this is negligible since $\frac{S_{ixy}}{2R_i} \ll 1$, so that $(2R_i\alpha_i) \approx S_{ixy}$, a fixed number. Hence what one substitutes for R_i or p_i in (37), (38), the measured value or the optimal value of (35) makes no difference.

Finally, the χ^2 from the γ-rays have dip and azimuth contributions:

$$\chi^2(\lambda_i) = \frac{\left[\lambda_i - \sin^{-1}\left(\frac{z_i - z}{z_i}\right)\right]^2}{(\Delta\lambda_i)^2} \qquad (39)$$

where $S_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$

$$\chi^2(\varphi_i) = \frac{\left[\varphi_i - \tan^{-1}\left(\frac{y_i - y}{x_i - x}\right)\right]^2}{(\Delta\varphi_i)^2} \qquad (40)$$

ii) Minimizing the χ^2

From formulas (33), (36), (39), (40) for a given point on the trajectory of one of the pions taken as exactly known, one calculates the χ^2 contributions from the gammas and the other pion. This is done at equally spaced intervals along the trajectory. One then looks for a χ^2 which is less than both the preceding one and the following one. If one denotes the preceding χ^2 as $\chi^2(-1)$, the least χ^2 as $\chi^2(0)$, and the following one as $\chi^2(+1)$ and denote the distance between points where the χ^2 is found by ΔS, the true minimum χ^2 is located at a position

$$S = \frac{\left[\chi^2(-1) - \chi^2(+1)\right]\Delta S}{2\left[\chi^2(+1) + \chi^2(-1) - 2\chi^2(0)\right]} \qquad (41)$$

with respect to the point where the least χ^2 was found, where (41) is easily obtained if χ^2 is assumed to have a quadratic behaviour in the neighbourhood of its minimum.

iii) Combining results for the two tracks

Following the above procedures, one finds generally one point on each of the two pion tracks which minimizes the χ^2 as just discussed. What is then done is to take a point half-way in between those, calculates the χ^2 for it and uses formula (41) to find a minimal χ^2 position which is then the initial decay from which all initial variables are calculated. Because the first GRIND fit made is the 1 C fit where only the pion tracks are made to intersect, only the pion track contributions to the χ^2 are used in the final calculations from (41). If, as sometimes is the case, one of the pion tracks does not give a χ^2 minimum, but the other one does, then the vertex is taken on the track which gives a χ^2 minimum. All these χ^2 are printed out and if they are large ($\gtrsim 30$) they give a sure indication that no geometric fit (5 c fit) is possible.

VI) GENERAL REMARKS ON THE PROGRAM

i) If only a vertex is available for one of the gammas, then 2 geometric constraints are removed buth otherwise everything goes on as before. Thus the highest order fit possible is a 5 c fit. If both gammas have measured directions but a pion momentum is missing, in principle 1 constraint is lost and the fit should go through. In fact such events have been processed successfully, but very few so far have been attempted and there is no information on the success rate for such fits.

ii) What about failures ?
Most failures are connected with inability to obtain even a geometric (5 C) fit and are evident from large χ^2 values from the initialization. Presumably they occur either when the tracks and gammas are really unassociated, or one of the

pions scatters in the pipe wall. Occasionally events may pass the 5 C fit hit not the 6 C fit: this usually occurs in situations where due to errors of measurement, perhaps pion scatterings etc. Eqs. (27) for the gamma momenta give negative values. In many cases even when this occurs at the start of the 6 C fit the fit does converge satisfactorily. Sometimes however there is no convergence. It is not impossible that in some such cases the failure is purely technical in that a satisfactory χ^2 exists, but the program fails to converge to it, because it has start values which are too far off. Fortunately the occurrence of this sort of failure is rare.

Another failure which occurs in going from the 5 C to the 6 C fit can arise if the two chosen gamma rays are brems of each other, or both brems of an electron secondary (i.e. if by mistake a π e ν event has been mistaken for a $\pi^+ \pi^- \pi^\circ$ event). In that case the two gamma momenta tend to become large and, of opposite sign, a behaviour easily understood when one remembers that they are calculated from transverse momentum balance.

Besides, the usual GRIND standard failures (high χ^2, too many cutsteps etc.) there are several specific to this program, which we explain here: "CN 1 - 1" starting point of one of pion tracks lies at a radial distance of less than 1.8 cm from the pipe axis (outer wall of pipe is at radius = 2.25 cm). This is only a warning, but the program still goes ahead and processes the event. "CN 1 - 2" one of tracks does not, upon extrapolation intersect the pipe. This is a consequence of mismeasurement or large track scattering, and prevents fitting "CN 1 - 4" - Eq. 17 has no real roots. This kills that particular fit. Since the introduction of the initialization procedure, this error has rare occurrence.

Appendix

Derivation of Initialization Formulas

A) Statement of problem and geometry

Given a point x, y, z and a charged track with measured parameters $(\lambda_i, \varphi_i, \frac{1}{p_i})$ starting at point x_i, y_i, z_i, find a circular trajectory passing through x, y, z and x_i, y_i, z_i which minimizes the quantity χ^2.

$$\chi^2 = \frac{(\lambda_i - \lambda)^2}{(\Delta\lambda)^2} + \frac{(\varphi_i - \varphi)^2}{(\Delta\varphi)^2} + \frac{(\frac{1}{p_i} - \frac{1}{p})^2}{(\Delta\frac{1}{p})^2} \qquad (a)$$

where λ, φ , $\frac{1}{p}$ are the parameters of the trajectory evaluated at the point x_i, y_i, z_i.

First of all considering dips we have that

$$\lambda = \sin^{-1}\left(\frac{z_i - z}{S_{ic}}\right) \qquad (b)$$

where S_{ic} is the path length along the curved trajectory from x, y, z to x_i, y_i, z_i. If R is the radius of the trajectory it is easy to see that

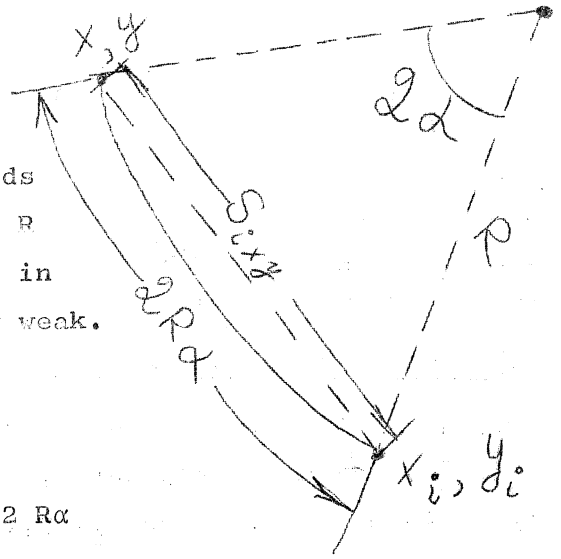
$$S_{ic} = \sqrt{(z_i - z)^2 + (2\,R\alpha)^2} \qquad (c)$$

where α is the half angle subtended by this path length at the centre of the circle

$$\alpha = \sin^{-1}\left(\frac{S_{ixy}}{2\,R}\right) \qquad (d)$$

and S_{ixy} is the straight line distance from x, y to x_i, y_i in the x - y plane.

$$S_{ixy} = \sqrt{(x_i - x)^2 + (y_i - y)^2} \qquad (e)$$



Now λ, as given in (b) depends on the assumed radius R through the R dependence of the quantity S_{ic}, but in fact this R dependence is extremely weak. This is clear from the fact that in typical situations with real events $2\alpha \approx 200$ mr, $\alpha \approx_S 100$ mr $<< 1$ r. Hence in (d) $\alpha \approx \frac{S_{ixy}}{2R}$ and the term $2R\alpha$ in (c) just becomes S_{ixy}. Thus (b) becomes

$$\lambda \approx \sin^{-1}\left(\frac{z_i - z}{S_{ixy}}\right) \qquad (f)$$

where there is no dependence on R. Consequently the contribution of the first term to the χ^2 is readily computed from (b) where it makes little difference what value of R is used (for example the measured value is perfectly adequate).

The calculation of the last two terms in the χ^2 is less trivial because there is a relation between φ and p (or R) and one has to find the appropriate combination which minimizes the sum of those two terms. To do this we solve the problem in a general form and then apply the result specifically to our situation.

B) Consider a system where there are two measured quantities ξ, η with measured values ξ_m, η_m and which satisfy a relation of the form

$$f(\xi, \eta) = 0 \qquad (g)$$

One wishes to minimize the quantity

$$\chi^2 = \frac{(\xi - \xi_m)^2}{\sigma_\xi^2} + \frac{(\eta - \eta_m)^2}{\sigma_\eta^2} \qquad (h)$$

Subject to the condition (g)

Let $u = \xi - \xi_m$ $\quad v = \eta - \eta_m$ so that

$$\chi^2 = \frac{u^2}{\sigma_\xi^2} + \frac{v^2}{\sigma_\eta^2} \qquad (i)$$

Now assume that the measurements are fairly accurate, so that u, v are small and linearize $f(\xi, \eta)$

$$f(\xi,\eta) = f(\xi_m, \eta_m) + \frac{\partial f}{\partial \xi} u + \frac{\partial f}{\partial \eta} v = 0 \qquad (j)$$

Thus we can replace (g) by a linear relation

$$a\ u + b\ v = C$$

where $a = \frac{\partial f}{\partial \xi}$ $\quad b = \frac{\partial f}{\partial \eta}$ $\quad C = - f(\xi_m, \eta_m)$ (k)

Differentiating (i) to get minimum for χ^2

$$\frac{d\chi^2}{2} = 0 = \frac{udu}{\sigma_\xi^2} + \frac{vdv}{\sigma_\eta^2} \qquad (l)$$

From (k) one easily finds

$$v = \frac{c - au}{b} \quad \text{and} \quad dv = -\frac{a}{b}\ du$$

Substituting this into (b), one easily solves for u

$$u = \frac{a\ c\ \sigma_\xi^2}{a^2 \sigma_\xi^2 + b^2 \sigma_\eta^2} \qquad (m)$$

Then

$$v = \frac{b\ c\ \sigma_\eta^2}{a^2 \sigma_\xi^2 + b^2 \sigma_\eta^2} \qquad (n)$$

$$\chi^2_{min} = \frac{c^2}{a^2 \sigma_\xi^2 + b^2 \sigma_\eta^2} \qquad (o)$$

Finally we can also write

$$u = \frac{a}{c}\ \sigma_\xi^2\ \chi^2_{min} = \xi - \xi_m \qquad (p)$$

$$v = \frac{b}{c}\ \sigma_\eta^2\ \chi^2_{min} = \eta - \eta_m \qquad (q)$$

C) Application to present problem

These results are easily applied to the specific problem of interest. Let $\xi, \eta \Rightarrow \varphi, \frac{1}{p}$.

We have to write the appropriate relation (g) between φ and $\frac{1}{p}$. By geometry one can show that if one puts a circle of radius R through the points x, y and x_i, y_i the φ at x_i, y_i and the radius R are related by

$$\frac{1}{R} = \frac{0.3|B|}{p} = \pm \frac{2}{(S_{ixy})^2} \left[(x_i - x) \sin\varphi - (y_i - y) \cos\varphi \right] \qquad (r)$$

Then the quantities a, b, c from the previous section are given by

$$a = \frac{\partial f}{\partial \varphi} = \mp \frac{2}{(S_{ixy})^2} = \left[(x_i - x) \cos\varphi_i + (y_i - y) \sin\varphi_i \right]$$

$$b = \frac{f}{\frac{1}{p}} = 0.3 \mid B \mid$$

$$c = - \left[\frac{.3|B|}{p_i} \mp \frac{2}{(S_{ixy})^2} \left[(x_i - x) \sin\varphi_i - (y_i - y) \cos\varphi_i \right] \right]$$

Substitution of these quantities into (o), (p), (q) immediately gives the results (33), (34), (35)

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