INTRODUCTION TO NONCOVARIANT GAUGES

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ABSTRACT

The most important single attribute of noncovariant gauges is their ghost-free nature. Although noncovariant gauges have been an integral part of quantum field theory for many decades, their effectiveness in the quantization of non-Abelian theories and their broad range of applicability have only recently been appreciated by theorists at large. The purpose of this review is to explain and illustrate the essential characteristics of some typical noncovariant gauges, such as the axial gauge, the planar gauge, the light-cone gauge and the temporal gauge. Our aim is to acquaint the reader not only with the basic properties of these ghost-free gauges, but also with their deficiencies and advantages over covariant gauges, their computational idiosyncracies and dominant areas of application.

Contents

I. Introduction

A. Overview

After playing second fiddle to their covariant counterparts for many a decade, noncovariant gauges are finally making a name for themselves by acquiring an ever-increasing share of the flourishing, if risky, "gauge market". There are sound reasons for this popularity, the most important one being the decoupling of fictitious particles, or ghosts, from the theory. As a result, all Feynman diagrams involving ghost loops can be shown to vanish, a circumstance which simplifies perturbative calculations. There is another reason why ghost-free gauges are popular. Some of today's most sophisticated models, like superstring theories in the light-cone gauge, are more tractable, and certain field-theoretic properties, such as the ultraviolet finiteness of supersymmetric Yang-Mills theory, are more transparent in a noncovariant gauge.

A powerful and indispensable tool in theoretical discussions, from quantum electrodynamics to gravity and superstring theories, is the principle of gauge invariance. One of the earliest references to gauge invariance dates back over fifty years to the pioneering work of Weyl who exploited this principle in the quantization of the Maxwell-Dirac field. Curiously enough, this quantization was performed in the temporal gauge which is one of the ghost-free gauges to be reviewed in this project.

To quantize a theory with gauge symmetry it is necessary to eliminate the unphysical gauge degrees of freedom. The standard procedure is to break the gauge symmetry by adding a gauge condition on the field variables. The explicit form of this gauge condition is, within the confines of a given

theory, largely dictated by computational convenience. Even so, the number of gauges is vast: some are linear and covariant, others nonlinear; some are homogeneous but noncovariant, others inhomogeneous, and so forth. Fortunately we can divide the majority of gauges into two categories. The first category consists of covariant gauges like the Feynman gauge and the Landau gauge whose reliability has been tested in numerous computations. The second category contains the noncovariant gauges, including the familiar Coulomb gauge and the gauges to be studied in this paper, namely, the axial gauge, the planar gauge, the light-cone gauge and the temporal gauge.

The purpose of this article is to study the essential features of these four gauges, all of which belong to the "axial" type and are defined in terms of a fixed, noncovariant vector. We caution the reader not to regard this review as the final word on ghost-free gauges, but merely as a guide to the literature, and to keep an open mind especially about issues currently under attack. Among the unsettled problems are the proper use of the Coulomb gauge in non-Abelian theories, the correct implementation of the temporal gauge in the context of path integrals, and the overall role played by the principal-value prescription in the treatment of spurious singularities.

Finally, a comment about the limitations of this project. We have omitted, except for occasional mention, such important topics as the Coulomb gauge and stochastic identities. Nor is there any detailed discussion about phenomenological aspects or the impact of fermions. We also decided, for the sake of brevity, to present the axial, planar and light-cone gauges in the elegant and convenient path-integral formalism, and the bulk of the material on the temporal gauge in the canonical formalism.

B. The gauge zoo

1. Covariant gauges

The success of covariant gauges extends over many years and there is no denying that even nowadays the majority of calculations in quantum field theory are performed in such popular covariant gauges as the Landau gauge and the Feynman gauge. Of course there are compelling reasons for this popularity. Technical problems are under control and there exist elegant procedures - for example, in the framework of dimensional regularization - for computing covariant-gauge Feynman integrals.

Especially prominent among the covariant-gauge choices has been the Feynman gauge which can be deduced from the generalized Lorentz gauge‡

$$\partial^\mu A^a_\mu(x) = B^a(x), \qquad (1.1)$$

where B^a is an arbitrary function which is independent of the gauge field, and where the gauge-fixing part of the Lagrangian density is given by

$$L_{fix} = -\frac{1}{2\lambda}(\partial^\mu A^a_\mu)^2 \; ; \qquad (1.2)$$

λ is the gauge parameter which is taken to be real. For $\lambda \to 0$, system (1.1)-(1.2) yields the Landau gauge, and for $\lambda \to 1$ we recover the Feynman gauge.

In order to help us pinpoint both the similarities and differences between covariant and noncovariant gauges, we recall the following dominant features of covariant-gauge Feynman integrals.

C1: The divergent parts of all one-loop integrals are local functions of the external momenta. (Their finite parts may, of course, be

‡Unless otherwise specified, we shall work in the context of Yang-Mills theory (Klein, 1939; Yang and Mills, 1954; Shaw, 1954), where $A^a_\mu(x)$ denotes the gauge field.

nonlocal functions of the external momenta and masses.)

C2: The divergent parts of one-loop integrals give rise to simple poles only.

C3: Naive power counting is valid.

C4: A Wick rotation from Minkowski space to Euclidean space may always be performed without crossing a pole, because Feynman's $i\epsilon$-prescription places the poles of a typical propagator like $(q^2 - m^2 + i\epsilon)^{-1}$, $\epsilon > 0$, in the second and fourth quandrants of the complex q_0-plane. Thus $q_0 = \pm\sqrt{\vec{q}^2 + m^2 - i\epsilon}$ gives two poles, $q_0^{(\pm)} = \pm\sqrt{\vec{q}^2 + m^2} \mp i\epsilon'$, with $\epsilon' \simeq \frac{1}{2}\epsilon\,(\vec{q}^2 + m^2)^{-1/2}$, where m is a mass parameter. (See Fig. 1.1)

C5: Covariant-gauge integrals preserve Lorentz invariance which permits application of the efficient tensor method. For the integral $I_{\mu\nu} = \int d^{2\omega}q\, q_\mu q_\nu\, [q^2 (q-p)^2]^{-1}$, for instance, symmetry considerations and Lorentz invariance dictate an ansatz of the form: $I_{\mu\nu} = A(p^2)\,\delta_{\mu\nu} + B(p^2)\,p_\mu p_\nu$. The coefficients A, B are determined by multiplying $I_{\mu\nu}$ first with $p_\mu p_\nu$, then contracting μ and ν, and finally solving the resulting two equations for A, B.

These five properties have been tested extensively and are firmly established. In fact, most of them are also shared by noncovariant gauges (Sect. B.2).

Covariant gauges possess three major advantages: they preserve relativistic invariance, they are easy to apply, particularly in conventional theories like quantum electrodynamics, and there exists a uniform prescription for the momentum-space singularities of the propagators, known as Feynman's $i\epsilon$-prescription. But there are also disadvantages in using covariant gauges. The principal drawback is the need for ghost particles which complicate perturbative calculations, especially in non-Abelian theories. Another disadvantage surfaces in the treatment of sophisticated models such as supersymmetric Yang-Mills and superstring theories, which are awkward to handle in a covariant gauge, yet become amazingly tractable in noncovariant gauges like the light-cone gauge. It is this limited range of applicability that has led to the current fascination with noncovariant gauges.

2. Noncovariant gauges

One of the oldest noncovariant gauges is the Coulomb gauge, or radiation gauge,

$$\frac{\partial}{\partial x^k} A^k(x) = 0, \quad k = 1,2,3 \tag{1.3}$$

which has been applied literally by generations of physicists, chiefly in quantum electrodynamics. In non-Abelian models, the dominant noncovariant gauge is the general axial gauge specified by

$$n^\mu A^a_\mu(x) = 0, \quad \mu = 0,1,2,3; \; n^2 = n_0{}^2 - \vec{n}^2 \; ; \tag{1.4}$$

$n_\mu = (n_0, \vec{n})$ is an arbitrary constant vector which defines a preferred axis in space, hence the name "axial" gauge. Different functional forms of the gauge-fixing part L_{fix} of the Lagrangian density, coupled with special values of n^2, give rise to some particularly convenient axial-type gauges, such as the pure axial gauge ($n^2<0$), the planar gauge ($n^2<0$), the light-cone gauge ($n^2=0$) and the temporal gauge ($n^2>0$). (See also Tables B and C.) These gauges form the nucleus of the present review.

The pure axial gauge, also called the homogeneous axial gauge, is specified by

$$n^\mu A^a_\mu(x) = n\cdot A^a(x) = 0, \quad n^2 < 0 \quad , \tag{1.5}$$

with

$$L_{fix} = -\frac{1}{2\alpha}(n\cdot A^a)^2 \; , \quad \alpha \to 0 \; ^‡, \tag{1.6}$$

where α is the gauge parameter. Similarly, the planar gauge is defined by

$$n\cdot A^a(x) = B^a(x), \; n^2 < 0 \; , \tag{1.7}$$

$$L_{fix} = -\frac{1}{2\alpha}\, n\cdot A^a \,(\partial^2/n^2)\, n\cdot A^a \; , \quad \alpha = -1 \; , \tag{1.8}$$

and the light-cone gauge by

$$n\cdot A^a(x) = 0 \quad , \quad n^2 = 0 \; , \tag{1.9}$$

$$L_{fix} = -\frac{1}{2\alpha}(n\cdot A^a)^2 \; , \quad \alpha \to 0 \; . \tag{1.10}$$

Noncovariant gauges possess three major advantages:

(i) ghosts decouple from physical S-matrix elements (although ghosts are required in the discussion of the Becchi-Rouet-Stora identities);

(ii) some aspects of field theory become more transparent in a noncovariant gauge, such as the proof of ultraviolet finiteness of supersymmetric Yang-Mills theory in the light-cone gauge;

(iii) certain sophisticated models like superstring theories are more tractable in a ghost-free gauge.

‡See the footnote connected with eq. (4.2).

But noncovariant gauges also possess disadvantages. Feynman integrals are trickier to handle and higher-order loop calculations become more demanding.

It may come as a surprise, but ghost-free gauges share many of the properties of covariant gauges, provided a sensible prescription is used for the unphysical singularities of $(q\cdot n)^{-1}$. This is certainly true for the special gauges in eqs. (1.5), (1.7) and (1.9) whose Feynman integrals possess these characteristics to one-loop order:

NC1: Their divergent parts are local functions of the external momenta.

NC2: They yield at most simple poles.

NC3: They obey naive power counting.

NC4: Feynman integrals in the pure axial gauge and planar gauge satisfy the covariant-gauge property C5, but Feynman integrals in the light-cone gauge do not.

We shall examine these and other properties later in the relevant chapters.

3. Some interesting gauges

The next three tables contain some high-profile gauges, as well as a few lesser known ones. Using the same notation as in Section III we represent the homogeneous gauge condition by $F^a = 0$, the inhomogeneous gauge condition by $F^a = B^a(x)$, where B^a is a function of x, and the gauge fixing part of the Lagrangian density by L_{fix}. For the gauge parameter we shall adhere, whenever possible, to the following convention: in covariant gauges, we denote the gauge parameter by λ, and in noncovariant gauges by α.

Table A. Principal Covariant Gauges

1. Generalized Lorentz gauge [1,2,3,5]:

$$F^a = \partial^\mu A^a_\mu(x) = B^a(x), \quad \mu = 0,1,2,3,$$

$$L_{fix} = -\frac{1}{2\lambda}(\partial^\mu A^a_\mu)^2 \quad .$$

a. The choice $\lambda \to 0$ gives the Landau gauge (or transverse Landau gauge)[1,4].

b. The choice $\lambda \to 1$ leads to the Feynman gauge[4].

c. The generalized Lorentz gauge with $B^a = 0$ is sometimes called the Fermi gauge.

2. 't Hooft gauges ('t Hooft, 1971; Abers and Lee, 1973; Itzykson and Zuber, 1980; Ryder, 1985):

$$F^a = \partial^\mu A^a_\mu - i\,\xi\,(v, t^a \phi) = B^a,$$

$$L_{fix} = -\frac{1}{2\xi}(\partial^\mu A^a_\mu - i\,\xi\,(v, t^a\phi))^2 \quad ,$$

where ξ is the gauge parameter (for historical reasons we use the letter ξ rather than λ); $v/\sqrt{2}$ is the vacuum expectation value of the Higgs field ϕ, and t^a are generators.

a. The choice $\xi \to 0$ yields the renormalizable Landau gauge.

b. The choice $\xi \to \infty$ gives the unitary gauge[1,5]. See also Weinberg (1973).

3. Background field gauge (De Witt, 1967b, 1967c; 't Hooft, 1975; Abbott, 1981; Capper and MacLean, 1981; McKeon et al., 1985b; Sohn, 1986):

$$F^a = \partial^\mu Q^a_\mu(x) + g f^{abc} A^b_\mu Q^{c\mu} = B^a(x),$$

where Q^a_μ and A^a_μ denote quantum fields and background fields, respectively.

$$L_{fix} = -\frac{1}{2\lambda}(\partial^\mu A^a_\mu + g f^{abc} A^b_\mu Q^{c\mu})^2 .$$

Table B. Principal Noncovariant Gauges

1. Coulomb gauge, or radiation gauge[1-5] (See also Heckathorn (1979), Muzinich and Paige (1980), Adkins (1986)):

$$F^a = \partial^k A_k^a(x) \;, \quad k = 1,2,3,$$

$$L_{fix} = -\frac{1}{2\alpha}(\partial^k A_k^a)^2 \;, \quad \alpha \to 0 \;.$$

2. a. Axial gauge, or pure axial gauge, or homogeneous axial gauge (Section IV).

$$F^a = n^\mu A_\mu^a(x) \;, \quad n^2 < 0,\; n^2 = n_0^2 - \vec{n}^2 \;,$$

$$L_{fix} = -\frac{1}{2\alpha}(n^\mu A_\mu^a)^2 \;, \qquad \alpha \to 0 \;.$$

   b. Inhomogeneous axial gauge:

$$F^a = n^\mu A_\mu^a(x) - B^a(x), \quad n^2 < 0 \;,$$

$$L_{fix} = -\frac{1}{2\alpha}(n^\mu A_\mu^a)^2 \;.$$

3. Planar gauge (Section V):

$$F^a = n^\mu A_\mu^a(x) - B^a(x),\; n^2 < 0 \;,$$

$$L_{fix} = -\frac{1}{2\alpha n^2}\, n\cdot A^a\, \partial^2\, n\cdot A^a \;, \qquad \alpha = -1.$$

4. Light-cone gauge (Sections VI and VII):

$$F^a = n^\mu A_\mu^a(x) \;, \qquad n^2 = 0 \;,$$

$$L_{fix} = -\frac{1}{2\alpha}(n^\mu A_\mu^a)^2 \;, \qquad \alpha \to 0 \;.$$

5. Temporal gauge, or Heisenberg-Pauli gauge, or Weyl gauge (Section VIII):

$$F^a = n^\mu A_\mu^a = A_0^a,\; n^2 > 0,\; n_\mu = (1,0,0,0)$$

$$L_{fix} = -\frac{1}{2\alpha}(n^\mu A_\mu^a)^2, \quad \alpha \to 0.$$

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[1]Abers and Lee (1973); [2]Coleman (1975); [3]Faddeev and Slavnov (1980); [4]Huang (1982); [5]Itzykson and Zuber (1980).

Table C. Other Gauges

1. **Abelian gauge** ('t Hooft, 1981; Min et al., 1985).

2. **Dirac gauge** (Dirac, 1959; see also Fradkin and Tyutin (1970)).

3. **Flow gauges** (Chan and Halpern, 1986).

4. **Fock-Schwinger gauge, or coordinate gauge** (Fock, 1937; Schwinger, 1951; Cronström, 1980; Shifman, 1980; Durand and Mendel, 1982; Kummer and Weiser, 1986):

$$F^a = (x^\mu - z^\mu) A_\mu^a \;, \quad z \text{ is "gauge parameter"}.$$

5. **Nonlinear gauge conditions** (Dirac, 1951; Nambu, 1968; 't Hooft and Veltman, 1972; Fujikawa, 1973; Shizuya, 1976; Zinn-Justin, 1984).

6. **Poincaré gauge** (Schwinger, 1973; Dubovikov and Smilga, 1981; Brittin et al., 1982; Skagerstam, 1983):

$$F^a = x^\mu A_\mu^a(x).$$

7. **'t Hooft-Veltman gauge** ('t Hooft and Veltman, 1972; Mann et al., 1984; McKeon et al., 1985a):

$$F = \partial\cdot A + \lambda A^2 \;, \quad \lambda \text{ gauge parameter},$$

$$L_{fix} = -\frac{1}{2}(\partial\cdot A + \lambda A^2)^2 \;.$$

8. **Wess-Zumino gauge** (Wess and Zumino, 1974; Gates et al., 1983).

C. Outline

Section II contains some elementary definitions from group theory and the theory of gauge fields, while Section III reviews the general notion of a gauge constraint. An important tool is the Faddeev-Popov determinant which is derived in the axial gauge and the Lorentz gauge.

Section IV deals with the axial gauge. After some theoretical considerations emphasizing the decoupling of ghosts, we discuss the principal-value prescription and evaluation of axial-gauge Feynman integrals. In the second half we obtain a Ward identity, look at renormalization and unitarity, and compute the gluon self-energy in the general axial gauge, $\alpha \neq 0$, and the pure axial gauge, $\alpha = 0$. The Section concludes with a description of Einstein gravity in the pure axial gauge.

The planar gauge in Section V is characterized by $\alpha = -1$ and by a gauge-fixing part which differs substantially from that in the axial gauge. We witness once again the decoupling of ghosts, but also draw attention to the nontransversality of the Yang-Mills self-energy and the importance of ghosts in the context of Becchi-Rouet-Stora (BRS) invariance.

Sections VI and VII are devoted to the light-cone gauge. Section VI begins with a brief history and some basic definitions, and then focuses on the main problem: the correct treatment of the unphysical singularities arising from factors like $(q \cdot n)^{-1}$ in the gluon propagator. We explain why the principal-value prescription is unsuitable for the light-cone gauge and suggest an alternative method. A fascinating feature of the new prescription is the appearance, in the gluon self-energy and three-gluon vertex, of <u>nonlocal</u> expressions that require the introduction of nonlocal BRS-invariant counterterms. The usefulness of the light-cone gauge and its tremendous

range of applicability are further underscored in Section VII by detailed examples from gravity, superstrings and supersymmetric Yang-Mills theory.

Section VIII starts with a review of the history and main attributes of the temporal gauge. This capricious gauge continues to baffle investigators for a variety of reasons, one difficulty being the correct implementation of Gauss' law. We study the quantization of gauge theories in the temporal gauge both in the canonical and path-integral formalisms, and also consider some recent pragmatic approaches. Our philosophy for this Section is to inform the reader of the pros and cons of the temporal gauge, but at the same time refrain from extolling the virtues of any particular viewpoint.

The feasibility of performing two-loop calculations in the light-cone gauge is explored in Section IX, where we also comment on stochastic quantization and stochastic identities. The article concludes in Section X.

There are three appendices. Appendix A lists a few axial-gauge integrals, while Appendix B summarizes the tensor components of $T_{\mu\nu\rho\sigma}$ appearing in Section IV. Finally, Appendix C contains a collection of both massive and massless integrals in the light-cone gauge.

We shall adhere, whenever possible, to the notation of Bjorken and Drell (1964) and work in natural units $\hbar = c = 1$. Space-time indices are denoted by Greek letters $\mu,\nu,\sigma,\ldots$, ranging over 0,1,2,3. Internal symmetry indices are represented by Latin letters $a,b,c,\ldots$, ranging over $1,2,\ldots,N^2-1$, for SU(N), N being the dimension of the symmetry group in the adjoint representation. No distinction is made between upper and lower <u>Latin</u> indices. We use a metric tensor $g_{\mu\nu}$ whose diagonal elements in Minkowski four-space are given by (+1,-1,-1,-1).

II. Basic Definitions

In this Section we establish our notation and review some definitions from the theory of gauge fields. The subject of gauge fields has grown tremendously in significance during the last decade and a half and now permeates essentially every area of modern quantum field theory.

The study of gauge theories is aided considerably by the use of Lie groups, among which the compact simple and semi-simple Lie groups are of particular interest. We recall the following definitions from the theory of groups:

(i) A Lie group G is a group of operators which depend on a set of continuous parameters.

(ii) A Lie group G is compact if the parameters of G vary over a finite, closed region.

(iii) A Lie group G is said to be simple if it has no nontrivial invariant subgroup; G is called semi-simple if it has no invariant Abelian subgroup.

Instead of dealing with the whole Lie group, it is often advantageous to work with the corresponding Lie algebra, defined by the group generators and their commutation relations.

Next we introduce the notion of gauge field. A gauge field A_μ is a vector field which may be expressed as $A_\mu = \sum_a t^a A^a_\mu$, where $a = 1,\ldots,N^2-1$, for SU(N), and $\mu = 0,1,2,3$; A^a_μ are the components of A_μ, while t_a denote the generators in the adjoint representation of the gauge group G. The latter is usually taken to be a simple compact Lie group. The generators t_a are linear operators satisfying the commutation relations

$$[t_a,t_b] = t_a t_b - t_b t_a = \sum_c f_{ab}{}^c\, t_c, \quad a,b,c = 1,\ldots,N^2-1, \qquad (2.1)$$

where $f_{ab}{}^c = f_{abc}$ are totally anti-symmetric structure constants of G, and N labels the dimension of the group. A_μ takes its values in the adjoint representation of G.

Furthermore, if the generators commute,

$$[t_a,t_b] = 0 \,, \qquad (2.2)$$

G is called a commutative, or Abelian, Lie group and the associated field A_μ an Abelian gauge field. Conversely, if the generators do not commute, i.e. if the structure constants in (2.1) differ from zero, we call G a non-commutative or non-Abelian Lie group and the corresponding field A_μ a non-Abelian gauge field.

Of special interest to the theorist are the transformation properties of these gauge fields. Suppose we are given a Lagrangian density L of an N-multiplet $(\phi_a) = \Phi$ of scalar fields , $a = 1,\ldots,N$, which transforms according to an irreducible representation of a compact simple Lie group G (Itzykson and Zuber, 1980):

$$\phi \to \phi' = U(g)\phi, \quad U^{-1}(g) = U^{\dagger}(g) \,, \qquad (2.3)$$

where g(x) is the generic element of G and U(g) is an N x N unitary matrix. It usually suffices to work with the infinitesimal transformation

$$g = g_0 + \omega^a t^a \,,$$

where g_0 is the identity, ω^a are arbitrary infinitesimal gauge functions and t_a group generators, $a = 1,2,\ldots,N^2-1$, for SU(N). If ω^a depends on the space-time variable x^μ, the gauge group is called local; if ω^a is independent of x^μ, one speaks of a global gauge group. If ω^a is x-dependent, we must introduce a gauge field A_μ, which transforms like

$$A_\mu(x) \to {}^{g}A_\mu(x) = g(x)\, A_\mu\, g^{-1}(x) + (\partial_\mu\, g(x))\, g^{-1}(x) \tag{2.4}$$

and leads to a gauge theory. To say that a certain dynamical theory is a gauge theory simply means that the defining Lagrangian density L is invariant under the gauge transformations (2.3) and (2.4).

The concept of gauge symmetry plays an essential role in quantum field theory. Consider, for instance, the theories of quantum electrodynamics (QED) and quantum chromodynamics (QCD). Quantum electrodynamics is an Abelian gauge theory, since its Lagrangian density L_{QED},

$$L_{QED} = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\Psi}(i\gamma\cdot\partial + e\gamma\cdot A)\Psi - m\bar{\Psi}\Psi\ , \quad \gamma\cdot\partial = \gamma^\mu\,\partial_\mu ,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ , \tag{2.5}$$

is invariant under the Abelian gauge transformations

$$\begin{aligned} \Psi(x) &\to \exp(ie\,\omega(x))\,\Psi(x)\ , \\ \bar{\Psi}(x) &\to \bar{\Psi}(x)\exp(-ie\,\omega(x))\ , \\ A_\mu(x) &\to A_\mu(x) + \partial_\mu\omega(x)\ , \end{aligned} \tag{2.6}$$

where $A_\mu(x)$ is the photon field, $\Psi(x)$ the spinor field and e, m denote, respectively, the charge and mass of $\Psi(x)$; $\omega(x)$ is the gauge parameter connected with the transformation (2.6). Here the group of transformations G is U(1), the group of unitary transformations in one dimension.

As a second example consider the Yang-Mills Lagrangian density for a massless vector field A^a_μ:

$$L_{YM} = -\frac{1}{4}(F^a_{\mu\nu})^2, \qquad a = 1,\ldots,N^2\text{-}1, \quad \mu = 0,1,2,3 \tag{2.7}$$

where the field strength $F^a_{\mu\nu}$ reads

$$F^a_{\mu\nu} = \partial_\mu\, A^a_\nu - \partial_\nu\, A^a_\mu + gf^{abc}\, A^b_\mu\, A^c_\nu\ ; \tag{2.8}$$

g is the strong coupling constant which sets the scale between gauge fields and matter fields (Huang, 1982), and f^{abc} are the structure constants introduced in (2.1). The Lagrangian density (2.7) is invariant under the <u>finite</u> gauge transformation (2.4); the corresponding <u>infinitesimal</u> gauge transformation reads

$$\delta A^a_\mu(x) = \partial_\mu\omega^a(x) + gf^{abc}\,\omega^b(x)\,A^c_\mu(x)\ . \tag{2.9}$$

For an elegant geometrical definition of the gauge field we refer the reader to Faddeev and Slavnov (1980), or to Konopleva and Popov (1981). This completes our very brief review of some basic gauge theory concepts.

III. Choosing a Gauge

Gauge symmetry plays an essential role in many theoretical models from QED to supergravity and supersymmetric string theories. According to the preceding Section, invariance of the Lagrangian density under a set of gauge transformations implies a certain freedom in defining the fields. The central question is, therefore, what are the implications of this "gauge freedom", for either canonical quantization or path-integral quantization?

In the context of canonical quantization the general procedure is to construct a complete set of canonical coordinates and momenta whose values at an initial time $t = t_0$ determine their values at some future time t (see, for instance, Coleman (1975), Kummer (1976), or Lee (1976)). However, if there is a "gauge freedom", it is impossible to find such a complete set of coordinates and conjugate momenta, since one may always choose a gauge transformation which vanishes at $t = t_0$ but is different from zero for $t > t_0$. In the case of the photon field, for example, not all the components of $A_\mu(x)$ are dynamical variables, and it becomes impossible to construct an independent set of canonical coordinates and momenta (Faddeev, 1976).

In the framework of path-integral quantization, characterized by functional integration over the field $A_\mu(x)$, the gauge degrees of freedom manifest themselves in a different manner. Due to gauge invariance, there now exist infinitely many fields ${}^gA_\mu(x)$ which are physically equivalent to $A_\mu(x)$ and are related by transformations of the form eq. (2.4). Integration over these gauge-equivalent fields ${}^gA_\mu$ produces an infinite volume factor, which is proportional to $\int \prod_x dg(x)$ in group space, and whose presence in the generating functional leads to ill-defined Green functions.

For a consistent quantization in either formalism it is clearly mandatory to eliminate the troublesome gauge degrees of freedom. This may be achieved by imposing on the system an auxiliary constraint, called a gauge condition, or choice of gauge, of the form

$$F^a[A^b_\mu(x);\ \phi(x)] = 0 \qquad , a = 1,\ldots,N^2-1 \ ; \tag{3.1}$$

N is the dimension of the group, and F^a a local functional of A^b_μ and ϕ, with values in the Lie algebra, where ϕ denotes all other fields (Itzykson and Zuber, 1980). The gauge condition (3.1) represents the equation of a hypersurface and may be covariant like the Feynman gauge, or noncovariant, such as the planar gauge or light-cone gauge; condition (3.1) is usually linear like the Coulomb gauge, but it may also be nonlinear.

The gauge constraint (3.1) has to fulfill two important criteria (Faddeev and Slavnov, 1980). First, it must be satisfied by the transformed fields ${}^gA$ and ${}^g\phi$, namely

$$F^a[{}^gA^b_\mu(x);\ {}^g\phi(x)] = 0 \ , \tag{3.2}$$

and, second, for a given A^b_μ and ϕ, system (3.2) must yield a unique solution $g(x)$ subject to certain boundary conditions. The second criterion implies the nonvanishing of the Jacobian determinant (we take $\phi = 0$, for simplicity) with respect to infinitesimal transformations:

$$\det(M_F) = \det\left[\frac{\delta\ F^a(x)}{\delta\ \omega^b(y)}\right] \neq 0 \ , \tag{3.3a}$$

or

$$\det(M_F) = \det\left[\frac{\delta\ F^a[A]}{\delta\ A^c_\mu(x)}\ D^\mu_{x,cb}\right] \neq 0, \tag{3.3b}$$

with

$$D^\mu_{x,cb} = \partial^\mu_x\ \delta_{cb} + gf_{cbd}\ A^\mu_d(x),\ \partial^\mu_x = \frac{\partial}{\partial x_\mu}\ . \tag{3.4}$$

For infinitesimal transformations $g(x) \simeq g_0 + \omega(x)$, where g_0 is the identity transformation, the Jacobian matrix M is given by

$$M_{ab}(x,y) = \frac{\delta F^a[{}^gA(x)]}{\delta \omega^b(y)}\bigg|_{g(x) \simeq g_0} , \tag{3.5}$$

$$= \frac{\delta F^a}{\delta A^c_\mu} D^{cb}_{x,\mu} \, \delta^4(x-y) .$$

We illustrate the above formulae for the Lorentz gauge and the axial gauge.

Example 1: Let us first compute det (M_F) in the Lorentz gauge

$F[A_\mu] = \partial^\mu_y A_\mu(y) = 0$, in the case of QED, an Abelian gauge theory.

Under an infinitesimal gauge transformation, the field A_μ transforms as

$$\delta A_\mu = \partial_\mu \, \omega(x) , \tag{3.6}$$

where $\omega(x)$ is a local gauge function; noting that $\delta F/\delta A_\mu(x) = \partial^\mu_y \, \delta(x-y)$ we obtain from eq. (3.3b)

$$\det(M_F) = \det \left(\frac{\delta F}{\delta A_\mu} D_{x,\mu}\right) = \det (\partial^\mu \partial_\mu) = \det(\partial^2) . \tag{3.7}$$

In QED, the factor det(∂^2) is a constant which can be readily absorbed into an overall normalization constant N (cf. eq. (3.13)).

Next consider the Yang-Mills field A^a_μ, a non-Abelian gauge field, with

$$\delta A^a_\mu = D^{ab}_\mu \, \omega^b(x) = \partial_\mu \, \omega^a(x) + gf^{abc} \, \omega^b(x) \, A^c_\mu(x) . \tag{3.8}$$

Here $\partial F^a/\partial A^c_\mu = \delta^{ac}\partial^\mu$, so that

$$\det(M_F) = \det \left(\frac{\delta F^a}{\delta A^c_\mu} D^{cb}_\mu\right) ,$$

$$= \det [\delta^{ac}\partial^\mu (\delta^{cb}\partial_\mu + gf^{cbd} A^d_\mu)] ,$$

$$= \det [\partial^2 \delta^{ab} + gf^{abd} \partial^\mu_x A^d_\mu(x) + gf^{abd} A^d_\mu(x)\partial^\mu_x]$$

$$= \det [\partial^2 \delta^{ab} + gf^{abd} A^d_\mu(x) \partial^\mu_x] , \tag{3.9}$$

since $\partial^\mu_x A^d_\mu = 0$. Unlike the Abelian case, det (M_F) depends now on the gauge field and is no longer a constant. As we shall see later (cf. eq. (3.18)), the factor det (M_F) leads to ghost particles.

Example 2: We evaluate det (M_F) in the axial gauge $F^a[A^b_\mu] = n^\mu A^a_\mu = 0$, specified by the noncovariant vector n_μ. Here

$$\frac{\partial F^a}{\partial A^c_\mu} = \delta^{ac} n^\mu , \tag{3.10}$$

so that

$$\det (M_F) = \det (\delta^{ac} n^\mu [\delta^{ab}\partial_\mu + f^{cbd} A^d_\mu]) ,$$

$$= \det (n\cdot\partial \, \delta^{ab} + gf^{abc} \, n\cdot A^c) ,$$

$$= \det (M_{axial}) . \tag{3.11}$$

Since $n\cdot A^a = 0$, we find that $\det (M_{axial}) = \det (n\cdot\partial \, \delta^{ab})$, the gauge field having <u>decoupled</u>.

The factor det (M_F), frequently denoted by $\Delta_F[A]$ in the literature, can also be introduced by requiring that

$$\int \prod_x dg(x) \, \Delta_F[{}^gA] \prod_{a,x} \delta(F^a[{}^gA_\mu(x)]) = 1, \tag{3.12}$$

with the interpretation that

$$\Delta_F[A] = \det (M_F) = \left\{ \int \prod_x dg(x) \prod_{a,x} \delta(F^a[{}^gA_\mu(x)]) \right\}^{-1}$$

"compensates" for an infinite volume factor arising from integration over the gauge group. We shall not pursue this approach further, since it is discussed extensively in the literature. (See, for example, Itzykson and Zuber (1980); Taylor (1976); Lee (1976); Coleman (1975); Faddeev and Slavnov

Let us incorporate condition (3.1) into the generating functional of the Green functions. Suppose $L(x)$ is a Lagrangian density invariant under a simple compact Lie group (Lee, 1976), and let $J^a_\mu(x)$ be an external c-number vector source function for the field A^a_μ. The generating functional may then be written as

$$Z[J^a_\mu] = e^{iW[J^a_\mu]} = N \int D(A) \det(M_F) \prod_x \delta(F^a[A])$$
$$\exp i \int d^4x \, [\, L(x) + J^{\mu b} A^b_\mu(x)] \, , \tag{3.13}$$

where $D(A) = \prod_\mu \prod_x \prod_a d\,A^a_\mu(x)$, $a = 1,\ldots,N^2-1$, is a local gauge-invariant measure (Capper <u>et al</u>., 1973), and $W[J^a_\mu]$ generates connected Green functions. The dependence on ϕ (cf. eq. (3.1)) has been dropped in eq. (3.13) for convenience. The normalization factor N should be such that $W[J^a_\mu]$ vanishes for $J^a_\mu = 0$ (Coleman, 1975).

The generating functional Z may be cast into "practical form" by rewriting both the Jacobian determinant $\det(M_F)$ and the delta functional $\delta(F^a[A])$ as exponentials of an <u>action</u>. Concerning $\delta(F^a)$ it is advantageous to replace (3.1) by

$$F^a[A^b_\mu(x)] = B^a(x) \, , \tag{3.14}$$

where $B^a(x)$ takes its values in the Lie algebra, so that

$$Z[J^a_\mu] = N \int D(A) \det(M_F) \prod_x \delta(F^b - B^b) \exp i \int d^4x (\, L(x) + J^{\mu c} A^c_\mu) \, . \tag{3.15}$$

Since (3.15) is independent of B^b, we may apply 't Hooft's technique ('t Hooft, 1971) and integrate over B^b with the help of a judiciously chosen weight function $\sigma[B^b]$,

$$\sigma[B^b] = \exp(-\frac{i}{2\lambda} \int d^4x \, [B^a(x)]^2) \, , \tag{3.16}$$

in which case

$$Z[J^a_\mu] = N \int D(A) \det(M_F) \exp i \int d^4x \, [L(x) - \frac{1}{2\lambda} (F^b[A])^2 + J^{\mu b} A^b_\mu] \, , \tag{3.17}$$

with λ a real parameter.

The nonlocal functional $\det(M_F)$ can be exponentiated in a variety of ways. A particularly elegant representation, based on the anti-commuting c-number fields $\eta_a(x)$ and $\bar\eta_a(x)$, reads (Faddeev and Slavnov, 1980; Itzykson and Zuber, 1980; Coleman, 1975):

$$\det(M_F) = \int D(\bar\eta) \, D(\eta) \, e^{i \int d^4x \, \bar\eta_a(x) \, M_{ab} \, \eta_b(x)} \, , \tag{3.18}$$

where the phase of the exponent $(\bar\eta \, M \, \eta)$ is conventional (Lee, 1976). The fields η and $\bar\eta$ represent ghost particles and obey Fermi statistics (Feynman, 1963; De Witt, 1967a,b,c; Faddeev and Popov, 1967; Mandelstam, 1968). Here these ghosts are scalar particles, but in quantum gravity, for example, they are oriented vector particles. The purpose of ghost particles is to eliminate the unphysical polarizations arising from closed loops: in short, they restore the unitarity of the scattering matrix and the transversality of the scattering amplitudes. For more details we refer the reader to Faddeev and Popov (1967), Fradkin and Tyutin (1970), Coleman (1975), Lee (1976), Faddeev and Slavnov (1980), and Itzykson and Zuber (1980).

Substitution of (3.18) into (3.17) yields the generating functional

$$Z[J_\mu, \bar\zeta, \zeta] = N \int D(A) \, D(\bar\eta) \, D(\eta) \exp i \int d^4x \, L'(x), \tag{3.19}$$

where

$$L'(x) = L_{inv} + L_{fix} + L_{ghost} + L_{ext} = L'(A, \bar\eta, \eta; \lambda, g) \, ,$$

with

$$L_{inv} = -\frac{1}{4} (F^a_{\mu\nu})^2 \, ,$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu ,$$

$$L_{fix} = - \frac{1}{2\lambda} (F^a[A^b_\mu])^2 ,$$

$$L_{ghost} = \tilde{\eta} M \eta ,$$

$$L_{ext} = J^{\mu a} A^a_\mu + \bar{\zeta}^a \eta^a + \tilde{\eta}^a \zeta^a .$$

M is given by eq. (3.5), and $\bar{\zeta}$ and ζ are anti-commuting c-number sources for the fields η and $\tilde{\eta}$, respectively. Formula (3.19) gives rise to well behaved Green functions. As mentioned already, the term $- \frac{1}{2\lambda} (F^a[A])^2$ breaks the gauge symmetry, while the factor det (M_F) "compensates" for an infinite volume factor which arises from integrating over points in the manifold of A^a_μ. Variation of A^a_μ in $Z[J_\mu, \bar{\zeta}, \zeta]$ leaves Z invariant, $\delta Z = 0$, and leads to Ward identities which will be studied in the next Section.

IV. The Axial Gauge

A. General considerations

1. Introduction

As mentioned in Section I, the axial gauge ($n \cdot A = 0$) was originally introduced by Kummer in 1961 in a paper dealing with the quantization of the free electromagnetic field. A year later Arnowitt and Fickler (1962) used the axial gauge in the form $A^a_3(x) = 0$ to investigate the quantization of non-Abelian gauge theories. With its help, Arnowitt and Fickler were able to examine not only consistency between the Lagrange and Heisenberg equations of motion, but also consistency between the equations of constraint and the dynamical equations. They also found that the axial gauge permitted a solution of the constraint equations, in contrast to the Coulomb gauge $\partial^k A^a_k = 0$, $k = 1,2,3$, where the constraints can only be solved approximately. Shortly thereafter, Schwinger (1963) studied the axial gauge in an article dealing with the equivalence between the Lorentz and Coulomb gauge formulations of non-Abelian field theories. In 1964, Yao carried out the first quantization of electrodynamics in the gauge $A_3(x) = 0$ and then used it to demonstrate that the assumption of manifest Lorentz covariance was not essential in proving the spin-statistics theorem.

Despite their technical advantages, interest in axial-type gauges remained marginal until the early seventies, when more and more researchers began to exploit the absence of fictitious particles in noncovariant gauges (Gross and Wilczek, 1974; Fradkin and Tyutin, 1970; Mohapatra, 1971, 1972; Delbourgo et al., 1974; Kainz et al., 1974; Kummer, 1975). Encouraged by the ghost-free formulation of QCD (Crewther, 1976; Frenkel and Meuldermans, 1976; Frenkel and Taylor, 1976; Konetschny and Kummer, 1977; Amati et al., 1978;

Ellis et al., 1978, 1979; Humpert and van Neerven, 1981a, 1981b), people wasted little time in applying the axial gauge to other non-Abelian models, notably gravity (Matsuki, 1979; Capper and Leibbrandt, 1982b, 1982c; Winter, 1984; Capper and Maclean, 1982; Delbourgo, 1981) and supergravity (Matsuki, 1980).

The purpose of this Section is to examine the principal features of the pure (or homogeneous) axial gauge and then illustrate them with specific examples from Yang-Mills theory, Einstein gravity and supergravity. Due to lack of space, we shall review the axial gauge only in the path-integral formalism. Discussions in the framework of canonical quantization may be found, for example, in Schwinger (1963), Burnel (1982b), Huang (1982), Cheng and Li (1984), Bassetto et al. (1984), and Cheng and Tsai (1986). See particularly the recent paper by Simões and Girotti (1986) on the quantization of non-Abelian gauge theories in a "completely fixed" axial gauge. These authors analyze in detail the residual gauge invariance in the axial gauge generated by local x^3-independent gauge transformations.

2. Decoupling of ghosts

Consider the Yang-Mills Lagrangian density for a massless vector field A^a_μ in the presence of an external c-number source $J^a_\mu(x)$, which depends only on the space-time variables x^μ:

$$L_{YM} = L_{inv} + L_{fix} + L_{ext} + L_{ghost} , \tag{4.1}$$

$$L_{inv} = -\frac{1}{4}(F^a_{\mu\nu})^2 ,$$

$$L_{fix} = \frac{1}{2\alpha} (n^\mu A^a_\mu)^2 ,$$

$$L_{ghost} = \bar{\eta}^a n^\mu D^{ab}_\mu \eta^b ,$$

$$L_{ext} = J^{a\mu} A^a_\mu ,$$

where the fields η^a and $\bar{\eta}^a$ represent fictitous particles and obey Fermi statistics, and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu ,$$

$$D^{ab}_\mu = \delta^{ab} \partial_\mu + g f^{abc} A^c_\mu ,$$

while g and f^{abc} have the same meaning as in eq. (2.8). The axial gauge is then specified by

$$n^\mu A^a_\mu(x) = 0, \quad n^2 < 0 , \tag{4.2}$$

with $n_\mu = (n_0, \vec{n})$. (For a distinction between $n^2 < 0$ and $n^2 > 0$, see the papers by Delbourgo and Phocas-Cosmetatos (1979), Humpert and van Neerven (1981b), Burnel (1982a, 1982b, 1983), Burnel and van der Rest-Jaspers (1983).) Taking the limit $\alpha \to 0$† we obtain the pure (homogeneous) axial gauge.

As stressed several times before, the principal advantage of the axial gauge arises from the effective decoupling of the fictitious particles in the theory. According to Taylor (1986), it is convenient to distinguish between the decoupling of <u>closed</u> ghost lines and the decoupling of <u>open</u> ghost lines#.

†This limit is connected with the representation of the delta functional (Abers and Lee, 1973; Dittrich and Reuter, 1986),

$$\delta[n\cdot A^a] = \lim_{\alpha\to 0} (2\pi\alpha)^{-1/2} \exp\left[-i \int d^4z \frac{1}{2\alpha} (n\cdot A^a)^2\right] .$$

A second way of implementing the axial gauge condition (4.2) is to employ a gauge-fixing term of the form $L_{fix} = C^a\, n\cdot A^a$, where $C^a(x)$ is a Lagrange multiplier field (see, for instance, Delbourgo et al., 1974; Kummer, 1975, 1976; Capper et al., 1986; Antoniadis and Floratos, 1983).

#The author is grateful to Professor J.C. Taylor for providing him with the following analysis in terms of <u>open</u> and <u>closed</u> ghost lines.

While closed ghost lines may occur in any Feynman diagram, open ghost lines occur only in some of the terms entering the BRS identities. We shall illustrate the decoupling of ghosts by two distinct arguments. (A third argument is given in Section V, near eq. (5.11).)

Let us first consider the ghost Lagrangian

$$L_{ghost} = \bar{\eta}^a n^\mu D_\mu^{ab} \eta^b , \tag{4.3}$$

$$D_\mu^{ab} = \delta^{ab} \partial_\mu + g f^{abc} A_\mu^c , \tag{4.4a}$$

$$n^\mu D_\mu^{ab} = \delta^{ab} n\cdot\partial + g f^{abc} n\cdot A^c . \tag{4.4b}$$

Since the ghost vertex is proportional to n_μ, the gluon propagator $G_{\mu\nu}$ satisfies $n^\mu G_{\mu\nu} = 0$, for $\alpha = 0$. Hence ghosts decouple[#] in <u>any</u> Feynman diagram, whether the ghost lines are open or closed. This simple argument applies both to the axial gauge and the light-cone gauge ($n^2 = 0$).

In order to see at what stage of the computation the decoupling process actually occurs, it is useful to work with the Faddeev-Popov determinant det (M_F) (cf. eqs. (3.11), (3.13)),

$$\det(M_F) = \det(n\cdot\partial\, \delta^{ab} + g f^{abc} n\cdot A^c) . \tag{4.5}$$

Following Frenkel (1976), we initially write

$$\det(M_F) = \exp(\mathrm{Tr}\, \ell n\, M_F) , \tag{4.6}$$

$$= \exp[\mathrm{Tr}\, \ell n\, n\cdot\partial + \mathrm{Tr}\, \ell n\, (1 + g(n\cdot\partial)^{-1} f^c\, n\cdot A^c)] ,$$

$$= \det(n\cdot\partial) \exp[\mathrm{Tr}\, \ell n\, (1 + g(n\cdot\partial)^{-1} f^c\, n\cdot A^c)],$$

and then apply the formula (Abers and Lee, 1973; Itzykson and Zuber, 1980)

$$\mathrm{Tr}\, \ell n\, (1 + L) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr}\, (L^n) \tag{4.7}$$

[#]A simplistic argument would be that implementation of the constraint $n\cdot A^a = 0$ in eqs. (4.4b) and (4.3) leads to $L_{ghost} = \bar{\eta}^a \delta^{ab} n\cdot\partial\, \eta^b$.

to obtain

$$\det(M_F) = \det(n\cdot\partial) \exp \sum_{n=1}^{\infty} \frac{(-g)^{n+1}}{n} \mathrm{Tr}\, [(n\cdot\partial)^{-1} f^c\, n\cdot A^c]^n .$$

Since the trace "Tr" includes integration over coordinates, we have explicitly

$$\det(M_F) = \det(n\cdot\partial) \exp \left\{ \sum_{n=1}^{\infty} \frac{(-g)^{n+1}}{n} \int d^4x_1 \ldots d^4x_n \right.$$

$$\mathrm{Tr}\, [n^\mu G(x_1 - x_2)\, f^a A_\sigma^a (x_2)\, n^\sigma G(x_2 - x_3)$$

$$\left. \ldots G(x_n - x_1)\, f^b A_\mu^b (x_1)] \right\} , \tag{4.8}$$

where x_i, $i = 1,\ldots,n$, are Euclidean coordinates and $G(x_i - x_j)$ satisfies

$$n\cdot\partial\, G(x_i - x_j) = \delta^4 (x_i - x_j),\ i,j = 1,\ldots,n;$$

in 2ω-dimensional momentum space,

$$G^{ab}(q) = \frac{-i}{(2\pi)^{2\omega}} \frac{\delta^{ab}}{q\cdot n} . \tag{4.9}$$

The factor $\det(n\cdot\partial)$ in eq. (4.8) is inconsequential and may be absorbed into the normalization constant, such as N in eq. (3.13). As for the exponential factor in eq. (4.8), each term inside the summation symbol gives rise to a single connected ghost loop of order n (Frenkel, 1976; Matsuki, 1979), with n external gauge bosons attached to the ghost loop. In 2ω-dimensional momentum space, this graph yields integrals which are proportional to

$$I = \int \frac{d^{2\omega}q}{(q\cdot n)^h} ,\ h = 0,1,\ldots,n, \tag{4.10}$$

and thereby vanish in the context of dimensional regularization (Frenkel, 1976; Matsuki, 1979). Note that the letter q in eq. (4.10) differs from the q in Fig.(4.1). The net result is an effective decoupling of the scalar ghost particles in the generating functional $Z[J^a_\mu]$ and hence from the gauge field $A^a_\mu(x)$. The argument between eqs. (4.5) and (4.10) applies only to closed ghost loops but is valid both for the axial gauge and the planar gauge (Taylor, 1986). Although this analysis was carried out for the particular gauge choice $F^a[A] = n\cdot A^a = 0$, it holds equally well for the more general condition

$$F^a = n\cdot A^a = B^a(x) ,$$

where B^a, an arbitrary function of space-time, is independent of the gauge field.

3. Feynman rules

The Feynman rules in the axial gauge follow from the Yang-Mills Lagrangian density (4.1). In the general axial gauge, where $\alpha\neq 0$, the bare gauge field propagator reads

$$G^{ab}_{\mu\nu}(q,\alpha) = \frac{-i\,\delta^{ab}}{(2\pi)^{2\omega}(q^2+i\epsilon)}\left[g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q\cdot n} + q_\mu q_\nu \frac{(n^2+\alpha q^2)}{(q\cdot n)^2}\right], \quad \epsilon > 0 . \qquad (4.11)$$

Letting $\alpha \to 0$ in eq. (4.11), we get the bare gauge field propagator in the pure axial gauge ($n^2 < 0$):

$$G^{ab}_{\mu\nu}(q,\alpha=0) = \frac{-i\,\delta^{ab}}{(2\pi)^{2\omega}(q^2+i\epsilon)}\left[g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q\cdot n} + q_\mu q_\nu \frac{n^2}{(q\cdot n)^2}\right], \quad \epsilon > 0 , \qquad (4.12)$$

while the ghost propagator is given by:

$$G^{ab}(q) = \frac{-i\,\delta^{ab}}{(2\pi)^{2\omega}\, q\cdot n} . \qquad (4.13)$$

A prescription for the unphysical poles of $(q\cdot n)^{-\beta}$, $\beta = 1,2$, will be discussed in Section IV.B.

The Lagrangian density (4.1) also implies the following axial-gauge vertices (Itzykson and Zuber, 1980):

(i) three-gluon vertex

$$V^{abc}_{\mu\nu\rho}(p,q,r) = +\,g\,f^{abc}(2\pi)^{2\omega}\,\delta^{2\omega}(p+q+r)[g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-r)_\mu + g_{\rho\mu}(r-p)_\nu] . \qquad (4.14)$$

(ii) four-gluon vertex

$$W^{abcd}_{\mu\nu\sigma\rho}(p,q,s,r) = -\,i\,g^2(2\pi)^{2\omega}\,\delta^{2\omega}(p+q+r+s)\Big[f^{eab}f^{ecd}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + f^{eac}f^{edb}(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma}) + f^{ead}f^{ebc}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu})\Big] . \qquad (4.15)$$

We also note, for completeness, the following

(iii) ghost-ghost gluon vertex

$$U^{abc}_{\mu}(p,k,q) = -i\,g\,f^{abc}\,n_\mu\,(2\pi)^{2\omega}\,\delta^{2\omega}(k+p-q) . \qquad (4.16)$$

B. Axial-gauge integrals

1. Prescription for unphysical poles

The three propagators in eqs. (4.11)-(4.13) contain the notorious factor $(q\cdot n)^{-1}$ leading to integrals of the form

$$\int \frac{dq}{(q-p)^2\, q\cdot n}\ ,\quad \int \frac{dq\, q_\mu}{q^2(q-p)^2(q\cdot n)^2}\ ,\ \text{etc.},\qquad d^{2\omega}q = dq, \tag{4.17}$$

where 2ω is the dimensionality of complex space-time and $\omega = 2$ corresponds to Minkowski 4-space, with metric (+1, -1, -1, -1). The central question is how to handle the unphysical poles arising from $(q\cdot n)^{-1}$ when $q\cdot n = 0$. One reasonably successful approach has been to employ the principal-value (PV) prescription (Yao, 1964; Gel'fand and Shilov, 1964; Frenkel and Taylor, 1976; Kazama and Yao, 1979; Konetschny, 1983; West, 1983)

$$PV\, \frac{1}{q\cdot n)^\beta} = \frac{1}{2}\lim_{\mu\to 0}\left[\frac{1}{(q\cdot n + i\mu)^\beta} + \frac{(-1)^\beta}{(-q\cdot n + i\mu)^\beta}\right]\ ,\quad \begin{matrix}\mu > 0\ ,\\ \beta = 1,2,3\ldots,\end{matrix} \tag{4.18a}$$

$$= \frac{1}{2}\lim_{\mu\to 0}\left[\frac{1}{(q\cdot n + i\mu)^\beta} + \frac{1}{(q\cdot n - i\mu)^\beta}\right]\ , \tag{4.18b}$$

which respects both power counting (Kummer, 1975) and unitarity (Konetschny and Kummer, 1976).

The PV prescription (4.18) allows us to compute, in principle, all axial-type integrals, either in Minkowski space or Euclidean space. In Minkowski space, one first combines $(q\cdot n + i\mu)^\beta$ with the remaining terms in the denominator, and then repeats the procedure for $(-1)^\beta/(-q\cdot n + i\mu)^\beta$, replacing $+n_\mu$ by $-n_\mu$, as advocated by Konetschny (1983). Alternatively, one may assume from the very outset that the integrals (4.17) are defined over

Euclidean space and apply (4.18b) in the form:

$$PV\, \frac{1}{q\cdot n} = \lim_{\mu\to 0} \frac{q\cdot n}{(q\cdot n)^2 + \mu^2}\ ,\qquad \mu > 0\ , \tag{4.19a}$$

$$PV\, \frac{1}{(q\cdot n)^2} = \lim_{\mu\to 0}\frac{(q\cdot n)^2 - \mu^2}{[(q\cdot n)^2 + \mu^2]^2}$$

$$= \lim_{\mu\to 0}\left(1 + 2\,\mu^2 \frac{\partial}{\partial\,\mu^2}\right)\frac{1}{(q\cdot n)^2 + \mu^2}\ , \tag{4.19b}$$

with similar expressions for $\beta = 3,4\ldots$. In this article we advocate the Euclidean-space approach, since it is simpler and more reliable than Minkowski-space methods.

2. Evaluation of axial-gauge integrals

Consider the four-dimensional divergent integral, defined over Euclidean four-space,

$$I(p,n) = \int_{-\infty}^{+\infty} \frac{d^4q}{(2\pi)^4}\, M(q,p,n)\ , \tag{4.20}$$

where $M(q,p,n)$ is typically a function of $(q-p)^2$, $q\cdot n$, q_μ, etc., with $q\cdot n = q_4n_4 + \vec{q}\cdot\vec{n}$ and $n^2 = n_4{}^2 + \vec{n}^2$, $n_\mu = (n_4, \vec{n})$, $\mu = 1,2,3,4$. The main steps in the computation of (4.20) may be summarized as follows:

(i) Define (4.20) over 2ω-space, i.e. work with dimensional regularization:

$$I(p,n) = \int \frac{dq}{(2\pi)^{2\omega}}\, M(q,p,n)\ ,\quad d^{2\omega}q = dq\ .$$

(ii) For integrands containing multiple factors of n^μ, like $\frac{1}{q\cdot n(q-p)\cdot n}$, $\frac{1}{(q\cdot n)^2(q-p)\cdot n}$, etc., employ the decomposition formula

$$\frac{1}{q\cdot n(q-p)\cdot n} = \frac{1}{p\cdot n}\left(\frac{1}{(q-p)\cdot n} - \frac{1}{q\cdot n}\right), \quad p_\sigma \neq 0 .$$

(iii) Replace $(q\cdot n)^{-\beta}$ by the principal-value prescription (4.18b):

$$\frac{1}{(q\cdot n)^\beta} \to \mathrm{PV}\,\frac{1}{(q\cdot n)^\beta} = \frac{1}{2}\lim_{\mu\to 0}\left(\frac{1}{(q\cdot n + i\mu)^\beta} + \frac{1}{(q\cdot n - i\mu)^\beta}\right),$$

and keep μ different from zero until <u>all</u> parameter integrations have been completed.

(iv) It is convenient to parametrize the propagators according to

$$\frac{1}{A^N} = \frac{1}{\Gamma(N)}\int_0^\infty d\alpha\, \alpha^{N-1}\, e^{-\alpha A}, \quad A > 0 .$$

(v) Integrate over momentum space by using the generalized Gaussian integrals (Capper and Leibbrandt, 1982b):

$$\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \exp\left[-\alpha\, q^2 - 2\beta\, q\cdot p - \gamma(q\cdot n)^2\right]$$

$$= \left(\frac{\pi}{\alpha}\right)^\omega \frac{\alpha^{1/2}}{(\alpha + \gamma\, n^2)^{1/2}} \exp\left[\frac{\beta^2 p^2}{\alpha} - \frac{\gamma\,\beta^2\,(p\cdot n)^2}{\alpha(\alpha + \gamma\, n^2)}\right]; \qquad (4.21a)$$

$$\int \frac{d^{2\omega}q\; q_\mu}{(2\pi)^{2\omega}} \exp\left[-\alpha\, q^2 - 2\beta\, q\cdot p - \gamma\,(q\cdot n)^2\right]$$

$$= -\left(\frac{\pi}{\alpha}\right)^\omega \frac{\beta\,\alpha^{-1/2}}{(\alpha + \gamma\, n^2)^{1/2}}\left(p_\mu - n_\mu \frac{\gamma\, p\cdot n}{\alpha + \gamma\, n^2}\right) \exp\left[\frac{\beta^2 p^2}{\alpha} - \frac{\gamma\,\beta^2 (p\cdot n)^2}{\alpha(\alpha + \gamma\, n^2)}\right], \qquad (4.21b)$$

where α, β, $\gamma \in [0,1]$ are Feynman parameters. Similar formulae containing $q_\mu q_\nu$, $q_\mu q_\nu q_\sigma$, etc., in the numerator may be deduced from eq. (4.21a) by operating, respectively, with $\partial^2/\partial p_\mu\, \partial p_\nu$, $\partial^3/\partial p_\mu\, \partial p_\nu\, \partial p_\sigma$, etc., on both sides of eq. (4.21a).

(vi) Integrate over Feynman parameters by following, for instance, the outline in Leibbrandt (1975: steps (v)-(viii), Section II. B.1).

The procedure (i)-(vi) permits us to evaluate the divergent and finite components of axial-gauge integrals. For example, the divergent part of $\int dq\, q_\mu\, [(q-p)^2\, q\cdot n]^{-1}$ reads

$$\mathrm{div}\int \frac{dq\; q_\mu}{(q-p)^2\; q\cdot n} = \frac{2p\cdot n}{n^2}\left(p_\mu - \frac{p\cdot n}{n^2}\, n_\mu\right)\bar{I}, \qquad (4.22)$$

where

$$\bar{I} = \text{divergent part of} \int dq\, [q^2\, (q-p)^2]^{-1}, \qquad (4.23)$$

$$\int \frac{dq}{q^2\, (q-p)^2} = \frac{i\,(-\pi)^\omega\; \Gamma(2-\omega)\; [\Gamma(\omega-1)]^2}{(p^2)^{2-\omega}\;\Gamma(2\omega-2)}, \quad \omega \neq 2;$$

thus in Euclidean space $\bar{I} = \pi^2/(2-\omega)$, while in Minkowski space $\bar{I} = i\pi^2/(2-\omega)$. Other massless axial-gauge integrals are given in Appendix A and in Capper and Leibbrandt (1982b).

Integrals containing several q_μ's in the numerator may also be computed by the elegant <u>tensor method</u> (Kainz <u>et al</u>., 1974; Capper, 1979; Tkachov, 1981; Jones and Leveille, 1982; Leibbrandt, 1984b), provided certain basic integrals are already known. Lee and Milgram (1983) have derived a formula for $\int dq\, (q^2)^\mu\, [(q-p)^2]^\nu\, (q\cdot n)^\sigma$, $\mu,\nu,\sigma \in \not{C}$, in terms of Meijer functions by using a mixture of dimensional and analytic regularization (see also Lee and Milgram (1985b)).

Although the PV prescription (4.18) leads to consistent one-loop <u>integrals</u>, both in the axial and planar gauge, it is by no means an ideal technique (Wu, 1979; Bassetto and Soldati, 1986; Cheng and Tsai, 1986; Lee and Milgram, 1985a), and should not be applied indiscriminately to just any gauge. For example, the PV technique is known to be inappropriate for the temporal gauge (Sect. VIII) and to give wrong results in the light-cone gauge (Sect. VI). Difficulties with the PV prescription have also been encountered in the treatment of infrared divergences (Gastmans and Meuldermans, 1973; Marciano and Sirlin, 1975; Gastmans *et al.*, 1976)*.

*The author is grateful to Professor A. Burnel for bringing these references to his attention.

C. Ward identity

In the axial gauge the Ward identity for the self-energy, with $L_{fix} = -(2\alpha)^{-1}(n\cdot A^a)^2$, is derived from the generating functional for complete Green functions

$$Z[J^b_\mu] = N \int D(A)\, \bar{Z} , \qquad (4.24)$$
$$\bar{Z} = \exp i \int d^4z \left[-\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\alpha}(n\cdot A^a)^2 + J^{\mu a} A^a_\mu \right] ,$$
$$D(A) = \prod_x \prod_a \prod_\mu d A^a_\mu(x) ,$$

where L_{ghost} has been omitted since the fictitious particles were shown to decouple (Section IV. A.2.). The effect of the gauge transformation

$$\delta A^a_\mu(x) = (\delta^{ac}\partial_\mu + g f^{abc} A^b_\mu)\, \omega^c(x) \qquad (4.25)$$

on Z gives $\delta Z = 0$ and leads to (Capper and Leibbrandt, 1982a)

$$i N \int D(A)\, \bar{Z} \left[\frac{1}{\alpha} n\cdot\partial^x\, n\cdot A^a - \partial^{x\mu} J^a_\mu + g f^{bca} J^{\mu b} A^c_\mu \right] = 0 , \qquad (4.26)$$

with $\partial^x_\mu = \partial/\partial x^\mu$. Differentiating (4.26) functionally with respect to the external current $J^e_\nu(y)$ and equating J^a_μ to zero, we obtain in coordinate space

$$< 0|T \left\{ -\frac{1}{\alpha} n\cdot\partial^x\, n\cdot A^a(x)\, A^p_\alpha(y) + \partial^x_\alpha\, \delta^{2\omega}(x-y)\, \delta^{ap} - g f^{pca}\, \delta^{2\omega}(x-y)\, A^c_\alpha(y) \right\} |0> = 0 , \qquad (4.27)$$

where T is the time-ordering operator. Fourier-transforming eq. (4.27) with the help of the definitions

$$\delta^{2\omega}(x-y) = (2\pi)^{-2\omega} \int d^{2\omega}q\, e^{iq\cdot(x-y)} ,$$

$$< 0|T[A^a_\mu(x)\, A^b_\nu(y)]|0> = \int d^{2\omega}q\, e^{iq\cdot(x-y)}\, D^{ab}_{\mu\nu}(q) ,$$

we arrive at the Ward identity

$$\frac{1}{\alpha} n^\mu \, q\cdot n \, D^{ab}_{\mu\nu}(q) + i(2\pi)^{-2\omega} q_\nu \, \delta^{ab} - g(2\pi)^{-2\omega} f^{abc} B^c_\nu(q) = 0. \qquad (4.28)$$

The term $B^c_\nu(q)$, which is the Fourier-transform vacuum expectation value of $A^c_\nu(y)$, corresponds to a massless tadpole and vanishes in the context of dimensional regularization (Capper and Leibbrandt, 1973). Hence (4.28) reduces to

$$\frac{1}{\alpha} q\cdot n \, n^\mu G^{ab}_{\mu\nu}(q) + i(2\pi)^{-2\omega} q_\nu \, \delta^{ab} = 0 , \qquad (4.29)$$

where $G^{ab}_{\mu\nu}(q) = G_{\mu\nu}(q)\,\delta^{ab}$ is the bare α-dependent propagator to one-loop order, given in eq. (4.11). Multiplication of (4.29) by

$$(G^{ab}_{\mu\nu})^{-1} = (G^{ab}_{o\mu\nu})^{-1} - \Pi^{ab}_{\mu\nu}$$

leads to the Ward identity

$$q^\mu \, \Pi^{ab}_{\mu\nu}(q) = 0 , \qquad (4.30)$$

with $(G_{o\mu\nu})^{-1} = i\,(q^2 g_{\mu\nu} - q_\mu q_\nu + \frac{1}{\alpha} n_\mu n_\nu)$; $\Pi^{ab}_{\mu\nu}$ denotes the one-loop gluon self-energy,

$$\Pi^{ab}_{\mu\nu}(q,\alpha\to 0) = g^2 \delta^{ab} C_{YM} \, [(- \frac{11}{3} + \frac{4}{3}\frac{\alpha \, p^2}{n^2})(p_\mu p_\nu - g_{\mu\nu} p^2)$$

$$+ \frac{4\alpha}{3(n^2)^2} (p\cdot n \, p_\mu - p^2 n_\mu)(p\cdot n \, p_\nu - p^2 n_\nu)] \, \bar{I} .$$

D. Renormalization and unitarity

Renormalization of Yang-Mills theory in the pure (homogeneous) axial gauge was established by Konetschny and Kummer# (1975, 1977) and Kummer (1975, 1976) over ten years ago, and has contributed significantly to placing the pure axial gauge on an equal footing with covariant gauges. We do not intend to review here the literature in detail, since the original papers are sufficiently explicit, but shall confine ourselves to a few short remarks.

Working to order $O(g^2)$, Kummer demonstrated as early as 1975 the following identity between the divergent parts of the wave function renormalization constants Z_A and the renormalization constants for the 3-vertex and 4-vertex, Z_3 and Z_4, respectively:

$$(Z_A)_{div} = (Z_3)_{div} = (Z_4)_{div} .$$

This equality implies, among other things, the gauge independence of $(Z_A)_{div}$ (Kummer, 1975). In the same vein, Beven and Delbourgo (1978), verifying a general theorem of Konetschny and Kummer (1977), studied the equality of the infinite parts of the renormalization constants in various gauges.

General axial gauges, with $n^2 \neq 0$, require the appearance of noncovariant n_μ-dependent counterterms. Such counterterms may possess both finite and infinite parts. Whereas the infinite parts of these counterterms can be shown to be covariant in the gauge $n\cdot A^a = 0$ (Konetschny and Kummer, 1977), it was noted by Leibbrandt et al. (1982) that in inhomogeneous gauges of the planar type, Lorentz-noncovariant infinite counterterms are admitted by the solution of the Slavnov-Taylor identities which depend on the gauge

#In order to avoid any danger possibly related to the limit $\alpha \to 0$, these authors assumed a gauge-fixing term of the form $L_{fix} = C^a \, n\cdot A^a$, with C^a an auxiliary Lagrange multiplier field.

parameter α and on the noncovariant vector n_μ, and surface already at the one-loop level. It is interesting to note in this connection that n_μ-dependent counterterms contributing to a "nonmultiplicative" renormalization of the wave-function are already apparent in the gauge $n \cdot A^a = 0$ (Konetschny, 1978). The general renormalization program requires the addition of a finite number of local counterterms to the renormalized action so that physically observable S-matrix elements are finite and the symmetries are conserved (Itzykson and Zuber, 1980). Therefore, proofs of renormalization based upon this program have no difficulties accommodating noncovariant counterterms as well (for $n \cdot A^a = 0$, see Konetschny and Kummer, 1975, 1977). Of course, a "multiplicative" renormalization as in covariant gauges, cannot be performed in general.

In 1976, Konetschny and Kummer established unitarity in the pure axial gauge ($\alpha = 0$) by working with the imaginary part of the S-matrix and then proving cancellation of the unphysical degrees of freedom. Their proof is conceptually simpler than for covariant gauges, because there are no fictitious particles to contend with#. The presence of spurious poles from $(q \cdot n)^{-1}$, on the other hand, poses certain technical challenges, resulting for instance in the modification of the Cutkosky cutting rules (Cutkosky, 1960). Naively speaking, unitarity is guaranteed by the use of the principal-value prescription. According to Konetschny and Kummer (1976), "...the principal value is real by definition and therefore does not contribute to the imaginary part involved in the unitarity equation."

#In non-Abelian theories, ghost fields were originally introduced for the sole purpose of preserving unitarity in the framework of covariant-gauge quantization.

Discussions of other relevant topics, such as the gauge-independence of matrix-elements of operators between physical bound-states (Kummer, 1980), of Lorentz invariance and gauge-independence of the S-matrix, etc., can be found in the cited literature, notably in Frenkel (1976), Konetschny and Kummer (1975, 1976, 1977) and in Konetschny (1978).

E. Applications

We illustrate the use of the axial gauge in quantum chromodynamics and pure Einstein gravity. Other applications can be found in the listed references.

1. Quantum chromodynamics

(a) Gluon self-energy in the pure axial gauge ($\alpha = 0$)

As our first example we consider the gluon self-energy $\Pi^{ab}_{\mu\nu}$ to one-loop order (Fig. 4.7). From the Lagrangian density

$$L = -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\alpha}(n^\mu A^a_\mu)^2 \quad , \qquad (4.31)$$

we obtain for the gluon loop in Fig.(4.7a) (Frenkel and Meuldermans, 1976)

$$\Pi^{ab}_{\rho\rho'}(p) = \frac{1}{2}\,(\text{factors}) \int dq\, V^{acd}_{\rho\sigma\tau}(p,q,-(p+q))\, G^{cc'}_{\sigma\sigma'}(q)$$

$$V^{bc'd'}_{\rho'\sigma'\tau'}(p,q,-(p+q))\, G^{dd'}_{\tau\tau'}(p-q) \quad , \qquad (4.32)$$

while the contribution from Fig. (4.7b), corresponding to a tadpole diagram, vanishes in dimensional regularization (Capper and Leibbrandt,1973). $G^{ab}_{\sigma\sigma'}(q)$ is the bare gluon propagator in the limit as $\alpha \to 0$, eq. (4.12), and $V^{abc}_{\rho\sigma\tau}$ denotes the three-gluon vertex given by (Capper and Leibbrandt, 1982a)

$$V^{abc}_{\rho\sigma\tau}(p,q,-(p+q)) = g\, f^{abc}(2\pi)^{2\omega}[-g_{\rho\sigma}(p+q)_\tau + g_{\sigma\tau}(2q-p)_\rho + g_{\tau\rho}(2p-q)_\sigma] \quad . \qquad (4.33)$$

Multiplying out the integrand and using the decomposition formula from Section IV.B, we can rewrite $\Pi^{ab}_{\rho\rho'}(p)$ as a sum of integrals whose dependence on n_μ in the denominator is either proportional to $(q\cdot n)^{-m}$ or $[(p-q)\cdot n]^{-m}$, $m = 0,1$, or 2. Since a shift of the integration variable from q_μ to $(p-q)_\mu$ replaces $[(p-q)\cdot n]^{-m}$ by $(q\cdot n)^{-m}$, we see that all self-energy integrals are only proportional to $(q\cdot n)^{-m}$, $m = 0$, 1, or 2. Using the appropriate formulae in Appendix A, together with the tadpole integrals

$$\int dq/q^2 = 0 \ , \quad \int dq/(q\cdot n)^2 = 0 \ , \text{ etc.},$$

we find for the one-loop gluon self-energy in the pure axial gauge (Frenkel and Meuldermans, 1976; Capper and Leibbrandt, 1982a)

$$\Pi^{ab}_{\mu\nu}(p) = -(11/3)g^2\,\delta^{ab}\,C_{YM}\,(p_\mu p_\nu - p^2 g_{\mu\nu})\,\bar{I} \ , \qquad (4.34)$$

where $f^{acd} f^{bcd} = C_{YM}\,\delta^{ab}$ and $\bar{I}$ is defined in (4.23). Clearly, $\Pi^{ab}_{\mu\nu}(p)$ is <u>transverse</u>, in agreement with the Ward identity (4.30).

(b) Gluon self-energy in the general axial gauge ($\alpha \neq 0$)

Computation of the gluon self-energy in the general axial gauge, $\alpha \neq 0$, is identical in procedure to the $\alpha = 0$ case, differing solely in the degree of complexity. With $L_{fix} = -\frac{1}{2\alpha}(n\cdot A^a)^2$ and $V^{abc}_{\rho\sigma\tau}$ the same as in eqs. (4.31) and (4.33), respectively, the extra complexity arises from the α-dependent term in the propagator $G^{ab}_{\mu\nu}$,

$$G^{ab}_{\mu\nu}(q,\alpha\neq 0) = \frac{-i\,\delta^{ab}}{(2\pi)^{2\omega}(q^2+i\epsilon)}\left[g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q\cdot n} + q_\mu q_\nu \frac{(n^2+\alpha q^2)}{(q\cdot n)^2}\right] ,$$

$$\epsilon > 0 \ . \qquad (4.12)$$

The final expression for the divergent part of the gluon self-energy reads

(Capper and Leibbrandt, 1982a)

$$\Pi^{ab}_{\mu\nu}(p,\alpha\neq 0) = g^2\delta^{ab}\, C_{YM}[(-\frac{11}{3} + \frac{4\alpha p^2}{3n^2})(p_\mu p_\nu - g_{\mu\nu}p^2)$$

$$+ \frac{4\alpha}{3(n^2)^2}(p\cdot n\, p_\mu - p^2 n_\mu)(p\cdot n\, p_\nu - p^2 n_\nu)]\, \bar{I} \,. \qquad (4.35)$$

Although (4.35) satisfies the transversality condition $p^\mu\, \Pi^{ab}_{\mu\nu}(p,\alpha\neq 0) = 0$, in accordance with the identity (4.30), $\Pi^{ab}_{\mu\nu}$ now depends on α as well as n_μ and will require more complicated counterterms.

2. Pure Einstein gravity

In view of the fiendish complexity of the gravitational interaction, the number of explicit calculations in noncovariant gauges is even sparser than in traditional covariant gauges like the Feynman gauge (Capper et al., 1973). One of the earliest studies in the pure axial gauge was carried out by Matsuki (1979) who analyzed the behaviour of infrared gravitons in ordinary Einstein gravity. He demonstrated that the associated ghost fields decouple from the graviton field, as expected, and that the dominant infrared divergences exponentiate in the spirit of Bloch-Nordsieck and then vanish in the graviton scattering amplitude. In the present graviton self-energy example, we wish to acquaint the reader with some of the subtleties symptomatic of the axial gauge, emphasizing particularly its ultraviolet behaviour and the associated Ward identities to one-loop order.

In quantum gravity, the axial gauge condition reads

$$n^\mu \phi_{\mu\nu}(x) = f_\nu(x) \quad , \; n^2\neq 0 \,, \qquad (4.36)$$

where f_ν is an arbitrary vector function, which does not affect the final result.

where the physical graviton field $\phi_{\mu\nu}$ is defined by

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa\, \phi_{\mu\nu} \,, \quad \kappa = 32\pi G, \quad G \text{ is Newton's constant,}$$

$g_{\mu\nu}$ being the metric tensor, and $\delta_{\mu\nu}$ the flat-space metric "tensor". The appropriate Lagrangian density reads

$$L = L_{Ein} + L_{fix} + L_{ghost} \,, \qquad (4.37)$$

$$L_{Ein} = 2\kappa^{-2}\sqrt{-g}\, R,$$

$$L_{fix} = -(2\alpha)^{-1}(n^\mu \phi_{\mu\nu})^2 \,,$$

$$L_{ghost} = \eta_\mu(x)[n_\rho\partial_\mu + n\cdot\partial\, \delta_{\mu\rho} + \kappa(n^\nu \phi_{\nu\rho}\partial_\mu + \phi_{\rho\mu} n\cdot\partial + n^\nu\phi_{\mu\nu,\rho})]\, \xi_\rho(x) \;;$$

η_μ and ξ_ρ are ghost fields, and

$$g = \det g_{\mu\nu} \,,$$

$$R = g^{\mu\nu}R_{\mu\nu} \,,$$

$$R_{\mu\nu} = \Gamma^\rho_{\mu\rho,\nu} - \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} + \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\mu\rho} \,,$$

$$\Gamma^\sigma_{\beta\gamma} = 2^{-1}g^{\sigma\omega}(g_{\beta\omega,\gamma} + g_{\omega\gamma,\beta} - g_{\beta\gamma,\omega}) \,.$$

An important difference between quantum gravity and Yang-Mills theory concerns the gauge parameter α and the decoupling of ghosts. Unlike Yang-Mills theory, ghost fields remain coupled to the graviton field even in the axial gauge where $\alpha = 0$. For this reason it is important to keep α different from zero till the end of the computation. This "restriction" on α makes the use of the pure axial gauge in gravity highly nontrivial. By comparison, the lack of decoupling of the vector ghost fields is harmless, since the various ghost loops are proportional to integrals of the form $\int dq(q\cdot n)^{-m}$ which vanish in dimensional regularization (Matsuki, 1979). The non-decoupling of the ghosts from the graviton field $\phi_{\mu\nu}$ is, therefore, without consequence and justifies the deletion of L_{ghost} from the Lagrangian density (4.37). Computation of the graviton self-energy involves now "only" a single diagram, the graviton-graviton loop shown in Fig. (4.8).

The bare graviton propagator in momentum space follows from $(L_{Ein} + L_{fix})$ in (4.37) and reads (Capper and Leibbrandt, 1982b):

$$G_{\lambda\beta,\rho\sigma}(q,\alpha) = \frac{i}{2(2\pi)^{2\omega}(q^2+i\epsilon)} \left\{ 2\, I^1_{\lambda\beta,\rho\sigma} - \frac{1}{\omega-1} I^2_{\lambda\beta,\rho\sigma} - 2\alpha(q^2/q\cdot n)^2 \left[T^6_{\lambda\beta,\rho\sigma} + \frac{q^2 n^2}{(q\cdot n)^2} T^9_{\lambda\beta,\rho\sigma} - 4\, T^{10}_{\lambda\beta,\rho\sigma} \right] \right\} , \tag{4.38a}$$

where

$$I^1_{\mu\nu,\rho\sigma} = \frac{1}{4} (d_{\mu\kappa} d_{\nu\lambda} + d_{\mu\lambda} d_{\nu\kappa})(d_{\rho\kappa} d_{\sigma\lambda} + d_{\rho\lambda} d_{\sigma\kappa}) ,$$

$$I^2_{\mu\nu,\rho\sigma} = d_{\mu\kappa} d_{\nu\kappa} d_{\rho\lambda} d_{\sigma\lambda} , \quad d_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{q\cdot n} q_\mu n_\nu ,$$

while the tensors $T^i_{\mu\nu,\rho\sigma}$, $i = 1,2,\ldots,14$ are listed in Appendix B (Matsuki, 1979). To calculate the divergent part of the graviton self-energy $\Pi_{\mu\nu,\lambda\beta}(p)$, it suffices to work with the propagator

$$G_{\lambda\beta,\rho\sigma}(q,\alpha=0) = \frac{i}{2(2\pi)^{2\omega}(q^2+i\epsilon)} (2 I^1_{\lambda\beta,\rho\sigma} - I^2_{\lambda\beta,\rho\sigma}) , \tag{4.38b}$$

and with the three-graviton vertex $V_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3}(p_1,p_2,p_3)$, Fig. (4.9), given in eq. (2.14) of Capper and Leibbrandt (1982b).

Application of these Feynman rules leads to the following expression for the infinite real part of the graviton self-energy (Capper and Leibbrandt, 1982c):

$$\Pi_{\mu\nu,\rho\sigma}(p,\alpha=0) = \Pi^{trans}_{\mu\nu,\rho\sigma} + \Pi^{non\text{-}trans}_{\mu\nu,\rho\sigma} , \tag{4.39}$$

$$\Pi^{trans}_{\mu\nu,\rho\sigma}(p,\alpha=0) = \frac{(p^2)^2\kappa^2\bar{I}}{120} F_{\mu\nu,\rho\sigma}(y, T^i) ; \tag{4.40a}$$

$F_{\mu\nu,\rho\sigma}$ is a function of $y = (p^2n^2)^{-1}(p\cdot n)^2$ and of the tensors $T^i_{\mu\nu,\rho\sigma}$, $i = 1,\ldots,14$; the non-transverse portion reads

$$\Pi^{non\text{-}trans}_{\mu\nu,\rho\sigma}(p,\alpha=0) = \frac{4}{3} y^2 \left[- T^1_{\mu\nu,\rho\sigma} + T^2_{\mu\nu,\rho\sigma} - T^5_{\mu\nu,\rho\sigma} + (2y)^{-1} T^8_{\mu\nu,\rho\sigma} - 2T^{12}_{\mu\nu,\rho\sigma} + 4T^{13}_{\mu\nu,\rho\sigma} - 2(y)^{-1} T^{14}_{\mu\nu,\rho\sigma} \right] (p^2)^2 \kappa^2 \bar{I} . \tag{4.40b}$$

Note that $\Pi_{\mu\nu,\rho\sigma}$, a <u>local</u> function of p_μ, is <u>non-transverse</u> even for $\alpha = 0$, in contrast to the Yang-Mills self-energy, eq. (4.34). The question has been raised whether it is possible to recover the transversality of $\Pi_{\mu\nu,\rho\sigma}$ for $\alpha \neq 0$ by choosing, for example, a gauge-breaking term like (Capper and Leibbrandt, 1982c)

$$L_{fix} = -\frac{1}{2\alpha} n^\mu \phi_{\mu\nu} \frac{\partial^2}{n^2} n^\sigma \phi_{\sigma\nu} , \quad \partial^2 = \partial^\mu \partial_\mu , \quad n^2 \neq 0 .$$

The answer is negative: there does <u>not</u> appear to exist a real value for α for which the infinite real part of the graviton self-energy is transverse. On the other hand, Winter (1984) has recently shown that the <u>imaginary</u> component of the graviton self-energy <u>is</u> transverse.

The non-transversality of the infinite real part $\Pi_{\mu\nu,\rho\sigma}$ emerges logically from a study of the appropriate Ward identity. The gravitational Ward identity in the axial gauge can be derived from the generating functional

$$Z[J_{\mu\nu}] = \tilde{N} \int D(\phi)\, \bar{Z} , \tag{4.41}$$

$$\bar{Z} = \exp[i \int d^4z \, (L_{Ein} + L_{fix} + J_{\mu\nu}\phi^{\mu\nu})] ;$$

here $J_{\mu\nu}(x)$ is an external c-number source, $\tilde{N}$ a normalization factor and $D(\phi) = D(\phi_{\mu\nu})$. Application of the gauge transformation

$$\delta\phi_{\mu\nu}(x) = A_{\mu\nu\rho}(x)\ \xi_\rho(x), \quad \xi_\rho \text{ arbitrary gauge parameter,} \tag{4.42}$$
$$A_{\mu\nu\rho}(x) = \kappa^{-1}(\delta_{\nu\rho}\partial_\mu + \delta_{\mu\rho}\partial_\nu) + (\phi_{\rho\nu}\partial_\mu + \phi_{\rho\mu}\partial_\nu + \partial_\rho\phi_{\mu\nu}),$$

to $Z[J_{\mu\nu}]$ implies $\delta Z = 0$, and gives

$$\tilde{N} \int D(\phi) \Big[\kappa^{-1} B_{\lambda\beta\rho}(x)\ \delta(x-y) + C_{\lambda\beta\rho}(x)\ \delta(x-y)$$
$$- i(\alpha\kappa)^{-1} n^\mu B_{\mu\nu\rho}(x)\ n^\gamma\phi_{\gamma\nu}(x)\ \phi_{\lambda\beta}(y)$$
$$- \frac{1}{\alpha}\ n^\mu C_{\mu\nu\rho}(x)\ n^\gamma\phi_{\gamma\nu}(x)\ \phi_{\lambda\beta}(y)\ \exp[i\int d^4z\ (\ L_{Ein} + L_{fix} + J_{\mu\nu}\phi^{\mu\nu})\Big] = 0,$$

$$B_{\mu\nu\rho} = \delta_{\nu\rho}\partial_\mu + \delta_{\mu\rho}\partial_\nu\ , \qquad \phi_{\mu\nu,\rho} = \partial\phi_{\mu\nu}\ /\partial x^\rho\ , \text{ etc.,} \tag{4.43}$$

$$C_{\mu\nu\rho} = \phi_{\rho\nu,\mu} + \phi_{\rho\nu}\partial_\mu + \phi_{\rho\mu,\nu} + \phi_{\rho\mu}\partial_\nu - \phi_{\mu\nu,\rho}\ ,$$

leading eventually to the gravitational Ward identity

$$(\delta_\rho{}^\mu p^\nu + \delta_\rho{}^\nu p^\mu)\ \Pi_{\mu\nu,\lambda\beta}(p) - F_{\lambda\beta,\rho}(p) = 0\ , \tag{4.44}$$

shown diagrammatically in Fig. (4.10).

The new function $F_{\lambda\beta,\rho}(p)$, defined by

$$< 0|T[n^\mu\ C_{\mu\nu\rho}(x) n^\gamma\ \phi_\gamma{}^\nu(x)\ \phi_{\lambda\beta}(y)]|0> \tag{4.45}$$
$$= \frac{-\alpha}{\kappa\ (2\pi)^{2\omega}} \int d^{2\omega}p\ e^{ip(x-y)} G_{\lambda\beta,\sigma\tau}(p)\ F_{\sigma\tau,\rho}(p)\ ,$$

corresponds to the <u>pincer</u> diagram in Fig. (4.11), and is seen to depend on both n_μ and α. We stress that $F_{\lambda\beta,\rho}$ does not vanish for $\alpha = 0$, i.e.

$$\lim_{\alpha\to 0} F_{\lambda\beta,\rho}(p) \neq 0\ ,$$

explaining, so to speak, the non-transversality of $\Pi_{\mu\nu,\rho\sigma}(p,\alpha=0)$. For the complete expression of $F_{\lambda\beta,\rho}$ and further discussions, see Capper and Leibbrandt (1982b, especially eq. (3.11)).

The structure of counterterms has been analyzed in gravity by Matsuki (1985), and in Yang-Mills theory by Gaigg <u>et al</u>. (1986), while Delbourgo (1979), Baker (1981), West (1982), Sørensen (1983) and others have studied the infrared behaviour of the gluon propagator. Kalashnikov and Casado* (1984), on the other hand, considered the infrared limit of the three-gluon vertex. The axial gauge has also been examined in the context of the BPHZL subtraction scheme (Kreuzer <u>et al</u>., 1986), and in supersymmetry (Capper <u>et al</u>., 1986).

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V. The Planar Gauge

A. Theory

1. Introduction

Although the axial gauge ($\alpha = 0$) possesses a number of significant advantages over covariant gauges, its application to quantum chromodynamics and other non-Abelian theories has been hampered by the complicated structure of the gauge field propagator. The main culprit is the last term in eq. (4.12), proportional to $q_\mu q_\nu\, n^2/(q\cdot n)^2$, which aggravates considerably the analysis of perturbative calculations in quantum chromodynamics (QCD) and elsewhere. It was this unwieldy nature of the gluon propagator that encouraged theorists to search for other ghost-free gauges having simpler propagators.

In their analyses of hard processes in QCD, Kummer (1976) and Dokshitzer et al. (1980) discovered the planar gauge which is intimately related to the axial gauge but possesses a more attractive gluon propagator (Lipatov, 1975; Dokshitzer, 1977). In massless Yang-Mills theory, the general planar gauge is defined by

$$n^\mu A^a_\mu(x) = B^a(x), \quad n^2\neq 0, \quad \alpha\neq 0, \tag{5.1a}$$

$$L_{fix} = -\frac{1}{2\alpha n^2}\, n\cdot A^a\, \partial^2\, n\cdot A^a , \tag{5.1b}$$

leading to the bare gluon propagator

$$G^{ab}_{\mu\nu}(q,\alpha) = \frac{-i\ \delta^{ab}}{(2\pi)^{2\omega}(q^2+i\epsilon)}\left[g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q\cdot n} + \frac{q_\mu q_\nu (1-\alpha) n^2}{(q\cdot n)^2}\right], \quad \epsilon > 0. \tag{5.2a}$$

As $\alpha \to 1$, we obtain the propagator in the planar gauge,

$$G^{ab}_{\mu\nu}(q,\alpha = 1) = \frac{-i\ \delta^{ab}}{(2\pi)^{2\omega}(q^2+i\epsilon)}\left[g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q\cdot n}\right], \epsilon > 0, \tag{5.2b}$$

which is certainly simpler and easier to employ than the axial-gauge version, eq. (4.12). The spurious poles of $(q\cdot n)^{-1}$ in eq. (5.2b) can be treated by the principal-value prescription (4.18), just as in the case of the axial gauge. In fact, the entire procedure of Section IV. B.1 applies also to Feynman integrals in the planar gauge. In short, planar-gauge integrals are the same as axial-gauge integrals. (See Appendix A.) The three-gluon and four-gluon vertices are also the same as in the axial gauge (eqs. (4.14), (4.15)).

The planar gauge is blessed with other attractive features. Apart from being ghost-free and possessing a relatively simple propagator, the gauge is devoid of Gribov gauge copies, like the axial gauge (Gribov, 1977, 1978; Sciuto, 1979; Bassetto et al., 1983; Weisberger, 1983), massless Yang-Mills theory is renormalizable (Andrasi and Taylor, 1981; Mil'shtein and Fadin, 1981), and collinear divergences appear only in self-energy components (Andrasi and Taylor, 1981). The planar gauge has been employed primarily in perturbative QCD, especially in the study of hard processes (Dokshitzer et al., 1980; Humpert and van Neerven, 1981a, 1981b; Bassetto et al., 1983; Bassetto et al., 1984). The gauge has also been used in the renormalization of the twist-four operator and of composite operators in gauge theories (Andrasi and Taylor, 1983a, 1983b). The implementation of the planar gauge in the canonical formalism was carried out by Bassetto et al. (1984).

The key difference between the planar gauge and the axial gauge occurs in the respective self-energies and Ward identities. Not only is the planar-gauge Ward identity (cf. eq. (5.17)) more intricate than in the pure axial gauge ($\alpha = 0$), but so is the one-loop gluon self-energy which turns out to be both nontransverse and n_μ-dependent (cf. eq. (5.19)). These intricacies necessarily complicate the renormalization program (Andrasi and Taylor, 1981; Mil'shtein and Fadin, 1981). Before discussing them, we shall demonstrate the decoupling of ghosts in the planar gauge.

2. Decoupling of ghosts

In order to illustrate the decoupling of ghosts in the planar gauge, we follow Dokshitzer et al. (1980), expressing the generating functional

$$Z[J_\nu^b] = N \int D(A)\ \det(M_F)\ \exp i \int d^4x\ [\ L_{inv}(x) - \frac{1}{2\alpha n^2}\ n\cdot A^a\ \partial^2\ n\cdot A^a + J_\mu^a\ A^{a\mu}] \tag{5.3}$$

as

$$Z[J_\nu^b] = N \int D(A)\ \det(M_F)\ f[A]\ \exp i \int d^4x\ [\ L_{inv}(x) + J_\mu^a\ A^{a\mu}\ ]\ , \tag{5.4}$$

where

$$F^a\ [A_\mu^b] = n\cdot A^a(x) - B^a(x),$$

$$f[A] = \exp\ [\ -\ i\ (2\alpha n^2)^{-1} \int d^4x\ n\cdot A^a\ \partial^2\ n\cdot A^a\ ]\ . \tag{5.5}$$

Implementation of the planar gauge amounts to using the weight function

$$f[B] = \exp\ [\ -\ i\ (2\alpha n^2)^{-1} \int d^4x\ B^a\ \partial^2\ B^a\ ]\ ,$$

with $\int D(B)\ f[B]$ = constant. According to Faddeev and Popov (and replacing g(x) in eq. (3.12) by $\Omega(x)$),

$$\det\ (M) \int D(\Omega)\ \delta\ (B^a - n\cdot{}^{\Omega}A^a) = \text{constant}, \tag{5.6}$$

where

$${}^{\Omega}A_\mu^a = A_\mu^a + \partial_\mu\ \omega^a + gf^{abc}\ \omega^b\ A_\mu^c\ .$$

Ω is given by $\Omega = \Omega_0 + \omega$ ($\omega = \omega^a\ t^a$, t^a are generators) and represents an infinitesimal gauge transformation, Ω_0 being the identity transformation. Inserting $\int D(B)\ f[B]$ = constant into (5.6), we obtain

$$\det\ (M) \int\int D(B)\ D(\Omega)\ \delta\ (B^a - n\cdot{}^{\Omega}A^a)\ \exp\ [\ -\ i\ (2\alpha n^2)^{-1} \int d^4x\ B^a\ \partial^2\ B^a] = \text{constant}, \tag{5.7}$$

with

$$D(\Omega)\ \delta\ (B^a - n\cdot{}^{\Omega}A^a) = D(\Omega)\ \delta\ [(B^a - n\cdot{}^{\Omega_0}A^a) - n\cdot\partial\omega^a - gf^{abc}\ \omega^b\ n\cdot A^c]\ .$$

But in the vicinity Ω_0,

$$B^a - n\cdot{}^{\Omega}A^a \doteq B^a - n\cdot{}^{\Omega_0}A^a = B^a - n\cdot A^a = 0, \tag{5.8}$$

so that

$$D(\Omega)\ \delta(B^a - n\cdot{}^{\Omega}A^a) = D(\omega)\ \delta\ (-n\cdot\partial\omega^a - gf^{abc}\ \omega^b\ n\cdot A^c). \tag{5.9}$$

Substitution of (5.9) into (5.7) yields

$$\det\ (M) = \left\{ \int\int D(B)\ D(\omega)\ \delta\ (\ -\ n\cdot\partial\ \omega^a - gf^{abc}\ \omega^b\ n\cdot A^c)\ \exp\ [\ -\ i\ (2\alpha n^2)^{-1} \int d^4x\ B^a\ \partial^2\ B^a] \right\}^{-1}\ . \tag{5.10}$$

Since $n\cdot A^c = B^c$ from eq. (5.8), and $B^a(x)$ is integrated out, the right-hand-side of (5.10) is indeed independent of the gauge field A_μ^a,

$$\det\ (M) \neq \text{function of } A^a\ , \tag{5.11}$$

which is tantamount to saying that the ghost fields have effectively decoupled from A_μ^a. Accordingly it is legitimate to absorb the Faddeev-Popov determinant, det (M), in eq. (5.3) into the normalization factor N.

This third decoupling argument - the first two were discussed in Section IV. A.2 - seems to apply only to closed ghost loops, but is valid both in the axial gauge and the planar gauge (Taylor, 1986).

3. Ward identity and Yang-Mills self-energy

1. **The Ward identity**

The Ward identity in the general planar gauge may be derived from the complete generating functional for Green functions (cf. eq. (3.19)),

$$Z[J^b_\nu] = N \int D(A) \exp i \int d^4z \left[-\frac{1}{4} (F^a_{\mu\nu})^2 \right.$$
$$\left. - (2\alpha n^2)^{-1}\, n\cdot A^a\, \partial^2\, n\cdot A^a + J^a\cdot A^a \right], \quad n\cdot A = n^\mu A^a_\mu , \tag{5.12}$$

where the ghost Lagrangian density, L_{ghost}, has purposely been omitted, since ghost fields were just shown to decouple from the gauge field A^a_μ. Performing the gauge transformation (4.25) on Z $[J^b_\nu]$ and applying the procedure discussed between eqs. (4.25) and (4.28), we obtain in momentum space (Capper and Leibbrandt, 1982a)

$$-\frac{q\cdot n\, q^2}{\alpha n^2} n^\mu D^{ce}_{\mu\beta}(q) - \frac{i\, g\, f^{abc}}{\alpha n^2} W^{eba}_\beta(q) + \frac{i\, \delta^{ec}}{(2\pi)^{2\omega}} q_\beta + \frac{g\, f^{ebc}}{(2\pi)^{2\omega}} B^b_\beta(q) = 0, \tag{5.13}$$

where $D^{ce}_{\mu\beta}(q)$ and $W^{eba}_\beta(q)$ are, respectively, defined by

$$\langle 0|T\left[A^c_\mu(x)\; A^e_\beta(y)\right]|0\rangle = \int d^{2\omega}q\; e^{i\, q\cdot(x-y)}\; D^{ce}_{\mu\beta}(q) , \tag{5.14a}$$

$$\langle 0|T\left[A^e_\beta(x)\; n\cdot A^b(y)\; \partial^2\; n\cdot A^a(y)\right]|0\rangle = \frac{1}{n^2}\int d^{2\omega}q\; e^{iq\cdot(x-y)}\; W^{eba}_\beta(q). \tag{5.14b}$$

T denotes the conventional time-ordering operator. The last term in eq. (5.13) corresponds to a massless tadpole diagram and can be omitted. Moreover, since the second term in eq. (5.13) does not contribute to lowest order (no loops), $D^{ce}_{\mu\beta}(q)$ reduces to the bare propagator $G^{ce}_{\mu\beta}(q,\alpha)$ given in eq. (5.2a). Hence eq. (5.13) becomes

$$n^\mu G^{ce}_{\mu\beta}(q,\alpha) = \frac{i\; \alpha\; n^2\; \delta^{ce}}{(2\pi)^{2\omega}\; q\cdot n\; q^2}\, q_\beta . \tag{5.15}$$

To obtain the one-loop contribution to the Ward identity (5.13), we multiply the latter by the bare inverse propagator $(G^{ab}_{\mu\nu})^{-1}$ giving

$$q^\beta\, \Pi^{cf}_{\beta\gamma}(q,\alpha) = \frac{-\, g\, f^{abc}(2\pi)^{2\omega}}{\alpha n^2} F^{fba}_\gamma(q,\alpha); \tag{5.16}$$

$F^{fba}_\gamma(q,\alpha)$ is the amputated one-loop contribution to $W^{eba}_\gamma(q,\alpha)$, shown in the "pincer" diagram of Fig. (5.1):

$$W^{eba}_\beta(q,\alpha) = G^{\gamma ef}_\beta(q,\alpha)\; F^{fba}_\gamma(q,\alpha) . \tag{5.17}$$

We note that the function $F^{fba}_\gamma(q,\alpha)$ vanishes identically in the axial gauge. The Ward identity (5.16) may be represented diagrammatically by Fig. (5.2),

<u>Fig. (5.2)</u>

where the two bars on the left leg imply contraction with q^β. Here $\Pi^{ab}_{\mu\nu}$ is the one-loop gluon self-energy, eq. (5.19), while the divergent component of F^{fba}_γ is given by (Capper and Leibbrandt, 1981)

$$F^{fba}_\gamma(q,\alpha) = 2i\pi^2\, g\, \alpha^2\, q\cdot n\, q^2\, f^{bfa}\, \Gamma(2-\omega)\, (2\pi)^{-2\omega}\, (n_\gamma - \frac{q\cdot n}{q^2} q_\gamma). \tag{5.18}$$

For nonvanishing values of α the right-hand side of (5.17) is clearly different from zero, suggesting an n_μ-dependent and <u>nontransverse</u> gluon self-energy, contrary to the claim in Dokshitzer <u>et al</u>. (1980). The properties of

$\Pi^{ab}_{\mu\nu}$ have been confirmed by an explicit calculation, as summarized in the subsequent Section.

2. Gluon self-energy

Use of the propagator (5.2b) and of the three-gluon vertex (4.14), together with the integrals in Appendix A, give the following structure for the gluon self-energy (Fig. 5.3) to one-loop order (Andrasi and Taylor, 1981; Mil'shtein and Fadin, 1981; Capper and Leibbrandt, 1981, 1982a).

$$\Pi^{ab}_{\mu\nu}(q,\alpha) = -g^2 C_{YM}\,\delta^{ab}\left\{\frac{11}{3}(q_\mu q_\nu - g_{\mu\nu}q^2) + 2\alpha\,(q_\mu q_\nu - g_{\mu\nu}q^2) + \frac{2\alpha}{n^2}\left[2n_\mu n_\nu q^2 - q\cdot n\,(q_\mu n_\nu + q_\nu n_\mu)\right]\right\}\bar{I}\,, \qquad (5.19)$$

where $\bar{I} = i\,\pi^2\,\Gamma(2-\omega)$, and $f^{acd}f^{bcd} = \delta^{ab}C_{YM}$. It is readily checked that this self-energy expression obeys the Ward identity (5.17). But since $\Pi^{ab}_{\mu\nu}$ is both gauge-dependent and nontransverse for $\alpha = 1$, the pleasant feature of <u>multiplicative</u> renormalization, characteristic of the pure axial gauge (α=0), is lost in the planar gauge. To illustrate this for the bare gluon propagator, for example, we follow Konetschny (1982) who writes the corrected propagator $G'_{\mu\nu}$,

$$G'_{\mu\nu} = G_{\mu\nu} + G_{\mu\lambda}\,\Pi_{\lambda\rho}\,G_{\rho\nu}\,,\quad G'^{ab}_{\mu\nu} = \delta^{ab}G'_{\mu\nu}\,,$$

as

$$G'_{\mu\nu}(q,\alpha) \simeq \frac{-i}{(2\pi)^{2\omega}(q^2+i\epsilon)}\left\{(1-\Pi_1)g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q\cdot n}\left[1 - \Pi_1 - \frac{\alpha\, n^2\Pi_3}{(q\cdot n)^2}\right] + q_\mu q_\nu\frac{n^2}{(q\cdot n)^2}\left[1 - \Pi_1 - \alpha - \frac{2\alpha n^2}{(q\cdot n)^2}\Pi_3\right] + 0\left(\frac{q^2n^2}{(q\cdot n)^2}\right)\right\}, \qquad (5.20)$$

$$\epsilon > 0,$$

where the scalar functions $\Pi_i(q^2, 2(q\cdot n)^2/q^2) \equiv \Pi_i$, $i = 1,2,3$, are defined through the relation (Konetschny, 1982)

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu}q^2)\,\Pi_1 + \left(q_\mu - n_\mu\frac{q^2}{q\cdot n}\right)\left(q_\nu - n_\nu\frac{q^2}{q\cdot n}\right)\Pi_2 + \frac{1}{q\cdot n}\left[n_\mu\left(q_\nu - n_\nu\frac{q^2}{q\cdot n}\right) + n_\nu\left(q_\mu - n_\mu\frac{q^2}{q\cdot n}\right)\right]\Pi_3\,. \qquad (5.21)$$

For $\alpha = 1$, the corrected propagator is approximately equal to

$$G'_{\mu\nu}(q,\alpha=1) \simeq \frac{-i}{(2\pi)^{2\omega}(q^2+i\epsilon)}\left\{(1-\Pi_1)g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q\cdot n}\left(1 - \Pi_1 - \frac{n^2\,\Pi_3}{(q\cdot n)^2}\right) - q_\mu q_\nu\frac{n^2}{(q\cdot n)^2}\left(\Pi_1 + \frac{2n^2}{(q\cdot n)^2}\Pi_3\right) + 0\,(q^2n^2/(q\cdot n)^2)\right\}, \qquad (5.22)$$

which is certainly not multiplicatively renormalizable. The general conclusion is that massless Yang-Mills theory <u>is</u> renormalizable in the planar gauge, but <u>not multiplicatively</u> renormalizable.

C. Importance of ghosts

As shown in Section (V. B.1), one can derive the proper Ward identity by omitting L_{ghost} from the generating functional for Green functions, eq. (5.12). While the absence of L_{ghost} might seem perfectly logical in view of the ghost-free nature of the planar gauge, it is incorrect to assert, nonetheless, that fictitious fields may also be discarded in other theoretical contexts involving the planar gauge. Ghosts are not only "...helpful in proving the finiteness of the renormalized Green functions", according to Mil'shtein and Fadin (1981), but they are actually necessary in the framework of Becchi-Rouet-Stora invariance (Becchi et al., 1974, 1975), as emphasized by Andrasi and Taylor (1981).

Since ghosts play an equally important role in the light-cone gauge, we thought it might be instructive to reproduce here the essential arguments of Andrasi and Taylor (1981). These authors work with the Yang-Mills Lagrangian density

$$L' = L + L_{fix} ,$$
$$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + L_{ghost} , \qquad (5.23)$$
$$L_{fix} = -(2\alpha n^2)^{-1} n\cdot A^a \partial^2 n\cdot A^a ,$$

where the ghost term reads

$$L_{ghost} = \bar{\eta}^a (n_\mu \partial^\mu \eta^a + g f^{abc} n_\mu A^{b\mu} \eta^c) + J^{a\mu} (\partial_\mu \eta^a + g f^{abc} A^b_\mu \eta^c) - \frac{1}{2} g f^{abc} K^a \eta^b \eta^c .$$

Here η_a, $\bar{\eta}_a$ are ghost fields, and J^μ_a, K_a external sources; the quantities J^μ_a, η_a, $\bar{\eta}_a$ are anti-commuting. L' obeys the Becchi-Rouet-Stora (BRS) identity

$$\int d^4x \left[\frac{\delta L'}{\delta A^a_\mu} \frac{\delta L'}{\delta J^{a\mu}} + \frac{\delta L'}{\delta \eta^a} \frac{\delta L'}{\delta K^a} + (\alpha n^2)^{-1} \partial^2 (n\cdot A^a) \frac{\delta L'}{\delta \bar{\eta}^a} \right] = 0 , \qquad (5.24a)$$

and the ghost equation

$$\frac{\delta L'}{\delta \bar{\eta}^a} - n^\mu \frac{\delta L'}{\delta J^{a\mu}} = 0 . \qquad (5.24b)$$

It is advantageous (Lee, 1976) to work with the generating functional Γ for one-particle-irreducible Green functions, with the gauge-fixing term omitted, in which case eqs. (5.24) become, respectively,

$$\int d^4x \left(\frac{\delta \Gamma}{\delta A^a_\mu} \frac{\delta \Gamma}{\delta J^{a\mu}} + \frac{\delta \Gamma}{\delta \eta^a} \frac{\delta \Gamma}{\delta K^a} \right) = 0 , \qquad (5.25)$$

and

$$\frac{\delta \Gamma}{\delta \bar{\eta}^a} - n^\mu \frac{\delta \Gamma}{\delta J^{a\mu}} = 0 . \qquad (5.26)$$

The divergent parts D of the one-particle irreducible Green functions then satisfy the BRS identity (Andrasi and Taylor, 1981)

$$\sigma D = \int d^4x \left[\frac{\delta L}{\delta A^a_\mu} \frac{\delta}{\delta J^{a\mu}} + \frac{\delta L}{\delta J^{a\mu}} \frac{\delta}{\delta A^a_\mu} + \frac{\delta L}{\delta \eta^a} \frac{\delta}{\delta K^a} + \frac{\delta L}{\delta K^a} \frac{\delta}{\delta \eta^a} \right] D = 0 , \qquad (5.27)$$

where σ is a nilpotent operator, $\sigma^2 = 0$. Employing the ansatz

$$G = a_3 A^a_\mu (J^{a\mu} + n^\mu \bar{\eta}^a) + a_4 n^\mu A^a_\mu (n^\lambda J^a_\lambda + n^2 \bar{\eta}^a) + a_5 \eta^a K^a , \qquad (5.28)$$

Andrasi and Taylor proceed to express the general solution for D as

$$D = -\frac{1}{2} a_1 F^a{}_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} a_2 n_\mu n_\nu F^{a\mu\lambda} F^{a\nu}{}_\lambda + \sigma G , \qquad (5.29)$$

and then derive the following values for the divergent constants a_i, $i = 1,\ldots,5$:

$$a_1 = \frac{11 g^2 C_{YM}}{48\pi^2 \epsilon} , \quad a_2 = 0, \quad \epsilon = 4 - 2\omega ,$$

$$a_3 = - a_5 = - \frac{\alpha g^2 C_{YM}}{8\pi^2 \epsilon} , \quad a_4 = \frac{\alpha g^2 C_{YM}}{4n^2\pi^2 \epsilon} . \qquad (5.30)$$

The coefficient a_1 corresponds to coupling constant renormalization, while a_3 and a_4 represent field renormalizations. Notice, in particular, the nonzero value of the a_5-term corresponding to ghost renormalization.

Moreover, we remind the reader that the Feynman graphs entering the BRS analysis are different, in general, from those needed for the Ward identities. Whereas Ward identities involve, for example, pincer diagrams, the BRS approach requires ghost diagrams instead. (See in this connection the work by Capper and MacLean (1982).) Both approaches lead of course to the same conclusion, namely that the one-loop Yang-Mills self-energy is nontransverse, and that massless Yang-Mills theory is not multiplicatively renormalizable. We shall see in Section VI that the above BRS approach, with its explicit use of ghost fields, also works admirably in the more intricate light-cone gauge, albeit with a modified ansatz for the functional G in eq. (5.28).

VI. The Light-Cone Gauge. Part I

A. Introduction

1. Preliminaries

The history of the light-cone gauge is as colourful and fascinating as that of any noncovariant gauge. Originally the light-cone gauge was a gauge "to fortune and to fame unknown". It was regarded as an odd, if not freakish, member of the family of axial-type gauges which existed more by accident than inventive planning.

But before we delve into the light-cone gauge, we ought to say a few words about the related, but not identical, subject of light-cone coordinates. These were first introduced for any four-vector $x^\mu = (x^0,x^1,x^2,x^3)$ by Dirac (1949) in the form $x^\pm = (1/\sqrt{2})(x^0 \pm x^3)$, $\vec{x} = (x^1,x^2)$, where x^+ is traditionally called the light-cone time and x^-,x^1,x^2 are the light-cone positions. In his article "Forms of relativistic dynamics", Dirac describes various structures of relativistic dynamical systems, among them the so-called front form*. He defines a front, i.e. a plane-wave front, as a three-dimensional surface in space-time which moves with the velocity of light. The front is associated with a subgroup of the Poincaré group that leaves the front invariant. One of the advantages of the front form is the absence of a square root in the Hamiltonian, avoiding thereby particles of negative energy. What is equally important is that Dirac's formulation in terms of light-cone coordinates is, to quote Brodsky and Ji (1986), "...frame-independent, the momentum is always finite". For this reason it is regrettable that the expression "light-cone frame" is sometimes in the

*Root (1973) and other authors employ the name "light front" form.

literature confused with or replaced by the phrase "infinite-momentum frame".

The infinite-momentum frame was originally discussed by Fubini and Furlan (1965) in the context of current algebra "as the limit of a reference frame moving with almost the speed of light" (Kogut and Soper, 1970). It was subsequently studied by Weinberg (1966), Bardakci and Halpern (1968), and others. In 1970, Kogut and Soper developed a canonical formalism for quantum electrodynamics in the infinite-momentum frame which was later extended to massive quantum electrodynamics by Soper (1971). Despite the pioneering work of Kogut, Soper, Bjorken *et al*. (1971) and Tomboulis (1973) on the quantization of the electromagnetic and Yang-Mills fields in the light-cone frame, interest in the light-cone gauge* during the period 1973-1976 remained sparse and was confined to a few researchers (Cornwall, 1974; Chakrabarti and Darzens, 1974; Gross and Wilczek, 1974; Kaku, 1975; Hagen and Yee, 1976, 1977; Scherk and Schwarz, 1975).

Towards the late seventies, however, articles by Konetschny and Kummer (1975, 1976, 1977), Kummer (1976), Beven and Delbourgo (1978), and Konetschny

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*Some authors prefer to use the phrase "null-plane gauge", or "light-front gauge" instead of "light-cone gauge". The reason, according to Aragone** and Gambini (1973), is that "...cones are not characteristic hypersurfaces at their vertex (Hörmander, 1963)", whereas "...the null hypersurfaces x^+ = constant, or x^- = constant, are good simple characteristics at any of their points, they do not present singular points...".

**The author is grateful to Professor C. Aragone for clarifying remarks on this matter and for bringing Hörmander's reference to his attention.

(1978) on the renormalizability of Yang-Mills theory in the axial gauge and on the unitarity of the scattering matrix had managed to dispel some of the ingrained skepticism about noncovariant gauges, and researchers were willing to take a fresh look at the pros and cons of the light-cone gauge. By the end of 1982, several positive features had emerged from studies in perturbative QCD (Kalinowski *et al*., 1981; Furmanski and Petronzio, 1980; Curci *et al*., 1980; Konishi, 1981; Floratos *et al*., 1981). For instance, in deep inelastic processes, only planar diagrams are needed to evaluate the dominant contributions in the leading logarithmic approximation (Pritchard and Stirling, 1980). The major technical problem seemed to be lack of a consistent prescription for the unphysical poles of $(q \cdot n)^{-1}$. The principal-value prescription violated power counting, as well as other basic criteria, and was therefore unsatisfactory for the light-cone gauge.

Early in 1982, Mandelstam (1982) suggested a new prescription for the light-cone gauge and used it to demonstrate the ultraviolet finiteness of the $N = 4$ supersymmetric Yang-Mills model. Later that year an equivalent prescription was discovered independently by the author and implemented for the first time in the context of dimensional regularization (Leibbrandt, 1982, 1984a). Together with Brink, Lindgren and Nilsson (1983), and Bengtsson (1983), Mandelstam (1983) was one of the first to emphasize the computational advantages of the light-cone gauge in supersymmetric theories. The reputation of the gauge was further enhanced in 1984 by the fact that the sophisticated superstring models based on the semi-simple Lie groups Spin $32/Z_2$ and $E_8 \times E_8$ were originally tractable only in the light-cone gauge. Since that time the gauge has found numerous other applications, for example in studies on stochastic quantization (Parisi and Wu, 1981; Zwanziger, 1981;

Egorian and Kalitzin, 1983; Hüffel and Rumpf, 1984), Nicolai maps (Nicolai, 1980, 1982), and stochastic identities (de Alfaro, Fubini and Furlan, 1985).

The aim of this Section is to describe the prominent features of the light-cone gauge and to illustrate its tremendous range of applicability by several examples.

2. Definitions and Ward identity

We begin with some definitions from Yang-Mills theory and then discuss an important Ward identity.

For a massless gauge field A^a_μ, with coupling constant g, the Lagrangian density reads

$$\begin{aligned} L_{YM} &= L_{inv} + L_{fix} + L_{ext} + L_{ghost} , \\ L_{inv} &= -\frac{1}{4}(F^a_{\mu\nu})^2 , \\ L_{fix} &= -\frac{1}{2\alpha}(n^\mu A^a_\mu)^2 , \\ L_{ext} &= J^a_\mu A^\mu_a = J^a \cdot A^a , \\ L_{ghost} &= \bar{\eta}^a n^\mu D^{ab}_\mu \eta^b , \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu , \end{aligned} \tag{6.1}$$

where the various symbols have the same meaning as in eq. (4.1). The light-cone gauge is a noncovariant physical gauge which is defined by

$$n^\mu A^a_\mu(x) = B^a(x) \quad , \; n^2 = 0 \quad , \tag{6.2}$$

with $n_\mu = (n_0, \vec{n})$, and where B^a may or may not be zero. If $B^a = 0$, condition (6.2) is to be understood as the limit $\alpha \to 0$ (in the notation of (6.1)). Condition (6.2) does <u>not</u> specify the light-cone gauge uniquely, because $n \cdot A^a = 0$ remains invariant under gauge transformations that do not involve one of the coordinates, say x^-, where $x^\pm = (1/\sqrt{2})(x^0 \pm x^3)$ (cf. Mandelstam, 1983). This freedom in the choice of the x^- coordinate implies an ambiguity in the $i\epsilon$-prescription for the factor $(q \cdot n)^{-1}$, which will be studied in Section (VI.B). Moreover, the light-cone gauge destroys manifest Lorentz invariance by breaking the group SO(1,3) to the subgroup SO(1,1) x SO(2)* (Namazie <u>et al</u>., 1983).

From (6.1), the bare light-cone gauge propagator reads ($\alpha \neq 0$)

$$G^{ab}_{\mu\nu}(q,\alpha) = \frac{-i\,\delta^{ab}}{(2\pi)^{2\omega}(q^2+i\epsilon)} \left[g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q \cdot n} - \frac{\alpha q^2 q_\mu q_\nu}{(q \cdot n)^2} \right] , \; \epsilon > 0 , \tag{6.3}$$

and for $\alpha = 0$,

$$G^{ab}_{\mu\nu}(q,\alpha{=}0) = \frac{-i\,\delta^{ab}}{(2\pi)^{2\omega}(q^2+i\epsilon)} \left[g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q \cdot n} \right] , \; \epsilon > 0 , \tag{6.4}$$

while the three-gluon vertex has the same form as in the axial gauge, eq.(4.14):

$$\begin{aligned} V^{abc}_{\mu\nu\rho}(p,q,r) = g f^{abc} (2\pi)^{2\omega} \delta^{2\omega}(p+q+r) [& g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-r)_\mu \\ & + g_{\rho\mu}(r-p)_\nu] . \end{aligned} \tag{6.5}$$

The derivation of the Ward identity for the gluon self-energy is similar to that in the axial gauge, Section IV, and will be omitted in favour of a few short remarks.

*In the remainder of this Section we work with four components. The two-component formalism of the light-cone gauge is used in Section VII. C.

Since ghosts decouple in <u>any</u> Feynman diagram whether the ghost lines are open or closed (see Sect. IV.A), the term L_{ghost} may be dropped from the generating functional for complete Green functions,

$$Z[J^b_\mu] = N\int D(A)\tilde{Z} \ , \qquad (6.6)$$

$$\tilde{Z} = \exp i\int d^4z[-\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2\alpha}(n\cdot A^a)^2 + J^a\cdot A^a] \ .$$

Using the invariance of (6.6) under the gauge transformation

$$\delta A^a_\mu = (\delta^{ac}\partial_\mu - g\, f^{abc}A^b_\mu)\, \omega^c(x) \ ,$$

and proceeding as in Section IV between eqs. (4.21) and (4.26), we obtain the Ward identity

$$\frac{1}{\alpha}\, q\cdot n\, n^\mu\, G^{ab}_{\mu\nu}(q,\alpha) - \frac{i\,\delta^{ab}}{(2\pi)^{2\omega}}\, q_\nu + \frac{g\, f^{abc}}{(2\pi)^{2\omega}}\, B^c_\nu(q) = 0, \qquad (6.7)$$

where $G^{ab}_{\mu\nu}(q,\alpha)$ is the bare propagator to one-loop order and $B^c_\nu(q)$ denotes the Fourier-transform vacuum expectation value of $A^c_\nu(x)$. Since the tadpole term $B^c_\nu(q)$ vanishes in dimensional regularization, eq. (6.7) reduces to the Ward identity

$$q^\mu\, \Pi^{ab}_{\mu\nu}(q) = 0 \ , \qquad (6.8)$$

which also follows from BRS invariance (Taylor, 1982). The one-loop Yang-Mills self-energy $\Pi^{ab}_{\mu\nu}$ in the light-cone gauge is shown in Fig.(6.2) and given in eq. (6.34).

The fact that the Ward identity (6.7) can be derived without ghosts might leave the erroneous impression that consideration of ghost fields in (6.1) is completely superfluous. This is not the case. There are situations, where the powerful consequences of BRS invariance provide a welcome tool for analyzing the renormalization structure.

B. Evaluation of light-cone gauge integrals

1. Prescription for unphysical poles

Until about 1982 the spurious singularities of $(q\cdot n)^{-1}$ in light-cone gauge integrals such as

$$\int \frac{d^{2\omega}q\ f(q^2, q_\mu, n_\mu, q\cdot p)}{(q^2+i\epsilon)[(q-p)^2 + i\epsilon]\ q\cdot n} \ ,$$

were invariably treated by the principal-value prescription

$$\frac{1}{q\cdot n} \Rightarrow PV\frac{1}{q\cdot n} = \frac{1}{2}\lim_{\mu\to 0}\left(\frac{1}{q\cdot n + i\mu} + \frac{1}{q\cdot n - i\mu}\right), \ \mu > 0. \qquad (6.9)$$

While this prescription seems to work reasonably well for the axial and planar gauges, it is unsuitable for the light-cone gauge, where it violates power counting and leads to one-loop integrals whose divergent components are either nonlocal or contain double poles. Prescription (6.9) may even fail to satisfy the appropriate Ward identities. The question one has to answer is whether these difficulties are symptomatic of the light-cone gauge <u>per se</u>, or whether they are caused by the prescription itself. It is possible, after all, that (6.9) is mathematically ill defined for $n^2 = 0$.

To see that this is indeed the case, we observe that for $n_0 \neq 0$ (as required by $n^2 = 0$), the poles of $(q\cdot n \pm i\mu)^{-1}$, namely $q_0^{(\pm)} = (\vec{q}\cdot\vec{n} \pm i\mu)/n_0$, lie on a line parallel to the imaginary q_0-axis, i.e. they appear in the first and fourth quadrants of the complex q_0-plane. We assume $n_0 > 0$ and $\vec{q}\cdot\vec{n} > 0$. (Fig (6.1)). The location of $q_0^{(+)}$ and $q_0^{(-)}$ prevents us from making a Wick rotation to Euclidean momenta, <u>without</u> encircling one of these poles (Leibbrandt, 1984a). By comparison, the poles of a typical Feynman propagator like $(q^2 + i\epsilon)^{-1}$ lie in the second and fourth quadrant.

Nor does the principal-value prescription work for momentum integrals like $\int d^{2\omega}q[q^2(q-p)^2 q\cdot n]^{-1}$ which are already defined over Euclidean space. Since $q\cdot n = q_4 n_4 + \vec{q}\cdot\vec{n}$, and since $n^2 = n_4{}^2 + \vec{n}^2 = 0$ implies $n_4 = \pm i|\vec{n}|$, prescription (6.9) gives (we use $n_4 = + i|\vec{n}|$)

$$PV \frac{1}{q\cdot n} = \frac{1}{2} \lim_{\mu\to 0} \left(\frac{1}{iq_4|\vec{n}| + \vec{q}\cdot\vec{n} + i\mu} + \frac{1}{iq_4|\vec{n}| + \vec{q}\cdot\vec{n} - i\mu}\right), \quad \mu > 0,$$

$$= \lim_{\mu\to 0} \left[\frac{\vec{q}\cdot\vec{n} + i\, q_4|\vec{n}|}{(\vec{q}\cdot\vec{n} + i\, q_4|\vec{n}|)^2 + \mu^2} \right]. \tag{6.10}$$

This result is unacceptable, however, because the complex denominator often leads to poorly-defined parameter integrals of the form

$$\int_0^1 d\beta\, \beta^{\omega-1}(1-\beta)^{-\omega-1} = \Gamma(\omega)\Gamma(-\omega)/\Gamma(0).$$

In summary, application of the principal-value prescription to the light-cone gauge creates more problems than it solves and ought to be avoided at any cost.

At the beginning of 1982, Mandelstam (1982) proposed a new light-cone gauge prescription for $(q\cdot n)^{-1}$, which is not of the principal-value type, and used it to demonstrate the ultraviolet finiteness of $N = 4$ supersymmetric Yang-Mills theory (Mandelstam, 1983). Later in 1982, the author independently discovered the following equivalent prescription and implemented it in the framework of dimensional regularization (Leibbrandt, 1982, 1983b, 1984a):

$$\frac{1}{q\cdot n} = \lim_{\epsilon\to 0^+} \frac{q\cdot n^*}{q\cdot n\; q\cdot n^* + i\epsilon}, \quad \epsilon > 0, \tag{6.11}$$

with poles in the second and fourth quadrant of the complex q_0-plane, and where $n_\mu{}^*$ is an arbitrary four-vector, satisfying $(n^*)^2 = 0$, $n\cdot n^* = 1$.# A

#The author is grateful to Professor J.C. Taylor (1986) for providing him with this definition of $n_\mu{}^*$.

convenient choice for $n_\mu{}^*$ is $n_\mu{}^* = (n_0, -\vec{n})$. In terms of n_μ and $n_\mu{}^*$, Mandelstam's prescription reads (Mandelstam, 1983)

$$\frac{1}{q\cdot n} = \lim_{\epsilon\to 0^+} \frac{1}{q\cdot n + i\epsilon\; q\cdot n^*}, \quad \epsilon > 0. \tag{6.12}$$

The two prescriptions (6.11) - (6.12) give identical results (Lee and Milgram, 1986a), at least to one-loop order, and avoid the difficulties created by procedure (6.9). Prescription (6.11) was subsequently recovered by Bassetto et al. (1985) in the context of canonical quantization.

The light-cone gauge prescription (6.11) possesses important properties:

(i) it permits a Wick rotation;

(ii) it satisfies power counting;

(iii) all basic one-loop integrals are local;

(iv) the divergent parts of basic one-loop integrals are at most proportional to simple poles;

(v) the prescription leads in general to Lorentz-noninvariant integrals.

In a basic integral there is merely a single factor $(q\cdot n)^{-\gamma}$, $\gamma = 1,2,3,\ldots,N$. Note that the first four properties (i) - (iv) are the same as for covariant gauges, and that the light-cone gauge shares property (v) with the axial and planar gauge.

2. Light-cone gauge integrals

We illustrate prescription (6.11) by evaluating the basic integral

$$I = \int d^{2\omega}q\, [((q-p)^2 + i\epsilon)\, q\cdot n]^{-1}, \tag{6.13}$$

first in Minkowski, then Euclidean space.

a. Minkowski space: Substituting

$$\left(\frac{1}{q\cdot n}\right)_{\rm Mink} = \lim_{\epsilon\to 0^+}\left[\frac{q_0 n_0 + \vec{q}\cdot\vec{n}}{q_0^2 n_0^2 - (\vec{q}\cdot\vec{n})^2 + i\epsilon}\right] \tag{6.14}$$

into I and observing that the resulting integrand is <u>not</u> Lorentz invariant, we first write

$$d^{2\omega}q = d^{2\omega-1}\vec{q}\, dq_0 ,$$

then integrate over q_0 and $\vec{q}$ separately:

$$I = \lim_{\epsilon\to 0^+}\int \frac{d^{2\omega}q(q_0 n_0 + \vec{q}\cdot\vec{n})}{[(q-p)^2 + i\epsilon][q_0^2 n_0^2 - (\vec{q}\cdot\vec{n})^2 + i\epsilon]} \tag{6.15a}$$

$$= \lim_{\epsilon\to 0^+}\int_0^1 dx\, A^{-2}\int d^{2\omega-1}\vec{q}\int_{-\infty}^{+\infty} dq_0\, (q_0 n_0 + \vec{q}\cdot\vec{n})$$

$$[(q_0 - B/A)^2 - (d/A^2) + i\epsilon/A]^{-2} ,\quad d = B^2 - AC , \tag{6.15b}$$

$$A = x + (1-x)n_0^2 ,\quad B = xp_0 ,$$

$$C = xp_0^2 - x(\vec{q} - \vec{p})^2 - (1-x)(\vec{q}\cdot\vec{n})^2 ,$$

where the denominator in (6.15a) has been combined according to Feynman. The various integrations lead to (Leibbrandt, 1984a):

$$\int d^{2\omega}q\, [(q-p)^2\, q\cdot n]^{-1} = \frac{2p\cdot n^*}{n\cdot n^*}\, i\pi^\omega\Gamma(2-\omega)\left[-p\cdot n\; p\cdot n^*\right]^{\omega-2}$$

$$= \frac{2p\cdot n^*}{n\cdot n^*}\,\bar{I} ,\quad \omega\to 2^+ \tag{6.16}$$

which is local, but Lorentz-noncovariant, and agrees with power counting. Here $n\cdot n^* = 2\,\vec{n}^2 = 2n_0^2$, while $\bar{I} = i\pi^2\,\Gamma(2-\omega)$ has already been defined in eq. (4.23). Comparison of (6.16) with the result in the axial gauge, namely $2\,p\cdot n\,\bar{I}/n^2$, shows that the light-cone gauge is <u>not</u> a limiting case of the

axial gauge when $n^2 \to 0$.

b. **Euclidean space:** The transition from Minkowski to Euclidean space is effected by the transformation

$$q_0 = i\, q_4 ,\quad \vec{q} = \vec{q} ,$$

$$n_0 = n_4 ,\quad \vec{n} = \vec{n} , \tag{6.17}$$

so that prescription (6.11) reads

$$\left(\frac{1}{q\cdot n}\right)_{\rm Eucl.} = \lim_{\epsilon\to 0^+}\left(\frac{\vec{q}\cdot\vec{n} + iq_4 n_4}{-q_4^2 n_4^2 - (\vec{q}\cdot\vec{n})^2 + i\epsilon}\right) \tag{6.18}$$

and I becomes

$$I = i\int d^{2\omega-1}\vec{q}\int_{-\infty}^{+\infty} dq_4\, \frac{(\vec{q}\cdot\vec{n} + iq_4 n_4)}{[(q_4-p_4)^2 + (\vec{q}-\vec{p})^2][q_4^2 n_4^2 + (\vec{q}\cdot\vec{n})^2]} , \tag{6.19}$$

where the $i\epsilon$ has been dropped and $d^{2\omega}q$ replaced by $i\, d^{2\omega-1}\vec{q}\, dq_4$. Note that n_0 is <u>not</u> rotated in (6.17)! Rather than combine the propagators according to Feynman, as was done in Minkowski space, it is more efficient in Euclidean space to apply the exponential parametrization

$$\frac{1}{A^N} = \frac{1}{\Gamma(N)}\int_0^\infty d\alpha\, \alpha^{N-1}\exp(-\alpha A),\quad A > 0,\ N = 1,2,3,\ldots ,$$

giving

$$I = \int_0^\infty d\alpha d\beta\, e^{-\beta p^2}\int_{-\infty}^{+\infty} d^{2\omega-1}\vec{q}\int_{-\infty}^{+\infty} dq_4(\vec{q}\cdot\vec{n} + i\, q_4 n_4)\, e^{-E} ,$$

$$E = \beta\vec{q}^2 - 2\beta\,\vec{q}\cdot\vec{p} + \alpha\,(\vec{q}\cdot\vec{n})^2 + (\beta + \alpha n_4^2)q_4^2 - 2\beta\, q_4 p_4 . \tag{6.20}$$

Integration over q_4 and $\vec{q}$ yields (see Appendix C)

$$I = i\pi^\omega\, p\cdot n^*\int_0^\infty d\alpha\, d\beta\, \beta^{2-\omega}(\beta + \alpha\, n_4^2)^{-2}$$

$$\exp\,[-\beta p^2 + \beta\vec{p}^2 + (-\alpha\beta(\vec{p}\cdot\vec{n})^2 + \beta^2 p_4^2)/(\beta + \alpha n_4^2)] .$$

Rescaling α to α/n_4^2, and introducing the new parameters (λ,ξ) via

$$\alpha = \lambda(1-\xi)\ ,\ \beta = \lambda\xi,\ \int_0^\infty d\alpha\, d\beta \ \Rightarrow\ \frac{1}{\vec{n}^2}\int_0^1 d\xi \int_0^\infty \lambda\, d\lambda\ , \tag{6.21}$$

we eventually get

$$I = \frac{2i\ p\cdot n^*}{n\cdot n^*}\ \bar{I}\ [p_4{}^2 n_4{}^2 + (\vec{p}\cdot\vec{n})^2]^{\omega-2},\quad n_4{}^2 = \frac{1}{2}\, n\cdot n^*, \tag{6.22}$$

where $\bar{I}$ denotes the divergent part of

$$\int \frac{d^{2\omega}q}{q^2[(q-p)^2]} = \frac{\pi^\omega\ \Gamma(2-\omega)(p^2)^{\omega-2}[\Gamma(\omega-1)]^2}{\Gamma(2\omega-2)}\ ,\ p^2 = p_4{}^2 + \vec{p}^2\ ,$$

$$= \pi^2\ \Gamma(2-\omega)\ ,\quad \omega \to 2^+\ .$$

We see from (6.22) that the integral I is <u>local</u> in the external momentum p_μ for $\omega > 2$, but Lorentz-noncovariant.

3. Other technical aspects

a. Tensor method

Prescription (6.11) - (6.12) permits evaluation of any light-cone gauge integral by the conventional Feynman parameter technique, either in Minkowski space or Euclidean space. For some integrals we may replace this safe, but often onerous, Feynman approach by the shorter <u>tensor</u> <u>method</u> which exploits the Lorentz invariance and symmetry of integrals like

$$\int \frac{dq\ F(q_\mu, q_\mu q_\nu, \ldots)}{G(q^2, (q-p)^2)}\ , \tag{6.23}$$

and is known to give satisfactory results for both covariant-gauge integrals and axial-gauge integrals (Kainz <u>et</u> <u>al</u>., 1974; Capper, 1979; Capper and Leibbrandt, 1982b; Jones and Leveille, 1982; Tkachov, 1981). If certain scalar integrals have already been computed, the tensor method allows us to

evaluate integrals like (6.23) efficiently and without further integration.

Take, for instance, the Euclidean-space integral

$$I_\mu = \int d^{2\omega}q\ q_\mu [q^2 (q-p)^2\ q\cdot n]^{-1}\ , \tag{6.24}$$

p_μ, n_μ being free parameters. In the axial gauge ($n^2 \neq 0$), the tensor method involves the ansatz

$$I_\mu^{\text{axial}} = a\ p_\mu + b\ n_\mu\ , \tag{6.25}$$

multiplying (6.25) by n_μ and p_μ, respectively, and solving for the <u>divergent</u> parts of a and b: $a|_{div} = 0$, $b|_{div} = \bar{I}/n^2$.

Thus,

$$I_\mu^{\text{axial}}\Big|_{div} = \int d^{2\omega}q\ q_\mu [q^2\ (q-p)^2\ q\cdot n]^{-1} = n_\mu\ \bar{I}/n^2,\ n^2 \neq 0. \tag{6.26}$$

However, in the light-cone (ℓ.c.) gauge where $n^2 = 0$, the correct ansatz for (6.24) reads

$$I_\mu^{\ell.c.} = A\ p_\mu + B\ n_\mu + C\ n_\mu{}^*\ , \tag{6.27}$$

with the divergent parts of the coefficients A, B and C to be determined. The unusual structure of (6.27) can be justified in the framework of the elegant Newman-Penrose tetrad scheme (Newman and Penrose, 1962, 1963), where any four-dimensional vector is expressible in terms of four <u>null</u> <u>vectors</u>, as explained in Leibbrandt (1984b). Exploiting, moreover, the assumptions of locality and power counting, we obtain $A|_{div} = B|_{div} = 0$, $C|_{div} = \bar{I}/n\cdot n^*$, so that

$$I_\mu^{\ell.c.} = n_\mu{}^* \bar{I}/n\cdot n^*\ ,\ n^2 = 0. \tag{6.28}$$

Notice that the result (6.28) <u>conserves</u> $n_\mu{}^*$, a property characteristic of all light-cone gauge integrals treated with the prescription (6.11) - (6.12). The appearance of the term $C\ n_\mu{}^*$ in (6.27) is related to the fact that the light-cone vector n_μ has linearly dependent components.

b. The operator $\partial/\partial n_\mu$:

Application of the operator $\partial/\partial n_\mu$ to a known integral generates new light-cone gauge integrals, provided n_μ^* is kept fixed and the final indices are symmetrized (Andrasi et al., 1986). We illustrate the procedure by evaluating the integral

$$I_{\mu\nu} = \int \frac{dq\ q_\mu q_\nu}{q^2(q-p)^2(q\cdot n)^2} \qquad (6.29)$$

from the expression

$$\int \frac{dq\ q_\mu}{q^2(q-p)^2 q\cdot n} = n_\mu^*\ \bar{I}/n\cdot n^* . \qquad (6.30)$$

The appropriate ansatz for (6.30) is

$$\int \frac{dq\ q_\mu}{q^2(q-p)^2 q\cdot n} = n_\mu^*(\bar{I}/n\cdot n^*) + n^2\ h(n,n^*,p) , \qquad (6.31)$$

where the function h must be chosen so that the answer for $I_{\mu\nu}$ is symmetric in μ,ν. Moreover, h should conserve n_μ^*, be local in p_μ and of dimension $[n^{-3}]$. Differentiation of (6.31) with respect to n_ν gives, holding n_ν^* fixed,

$$\int \frac{dq\ q_\mu q_\mu}{q^2(q-p)^2(q\cdot n)^2} = \frac{n_\mu^* n_\nu^*}{(n\cdot n^*)^2}\ \bar{I} - n^2 \frac{\partial h}{\partial n_\nu} - 2n_\nu\ h. \qquad (6.32)$$

Setting n^2 equal to zero in (6.32) and observing that the first term on the right-hand side is already symmetric in the indices, we may choose h = 0 to get the divergent part of the integral,

$$\int \frac{dq\ q_\mu\ q_\nu}{q^2(q-p)^2(q\cdot n)^2} = n_\mu^* n_\nu^*\ \bar{I}/(n\cdot n^*)^2. \qquad (6.33)$$

Similar examples are studied in Andrasi et al. (1986).

C. Application to Yang-Mills fields

We illustrate the light-cone prescription (6.11) in the case of quantum chromodynamics, first, by reviewing the Yang-Mills self-energy and then by analyzing the three-gluon vertex function to one-loop order. The three-gluon vertex is studied in some detail in order to display the importance of ghosts in the derivation of nonlocal BRS counterterms.

1. Yang-Mills self-energy to one loop

The relevant Lagrangian density for this calculation is given in eq.(6.1), but with ($L_{ext} + L_{ghost}$) omitted. Application of the Feynman rules (6.4)-(6.5) and of prescription (6.11) leads to the following expression for the Yang-Mills self-energy in Fig.(6.2) (Leibbrandt, 1984a):

$$\text{div}\ \Pi^{ab}_{\mu\nu}(p) = i\pi^2\Gamma(2-\omega)C_{YM}\ g^2\ \delta^{ab}\Big\{\frac{11}{3}(p^2 g_{\mu\nu} - p_\mu p_\nu) + \frac{2p\cdot n}{n\cdot n^*}(p_\mu n_\nu^* + p_\nu n_\mu^*)$$
$$+ \frac{2p\cdot n^*}{p\cdot n\ n\cdot n^*}[2p^2 n_\mu n_\nu - p\cdot n(p_\mu n_\nu + p_\nu n_\mu)]$$
$$- \frac{2p^2}{n\cdot n^*}(n_\mu n_\nu^* + n_\nu n_\mu^*)\Big\} , \qquad (6.34)$$

where $f^{acd} f^{bcd} = \delta^{ab} C_{YM}$. For a more recent study of the gluon self-energy, see Dalbosco (1986).

Apart from the traditional factor, $(11/3)(p^2 g_{\mu\nu} - p_\mu p_\nu)$, the self-energy in the light-cone gauge differs drastically from that in the axial and planar gauges, eqs. (4.34) and (5.19), respectively. Not only is (6.34) gauge-dependent and Lorentz-noninvariant, but it is also nonlocal in the external momentum p_μ, the nonlocality arising from use of the decomposition formula

$$\frac{d^{2\omega}q}{q\cdot n(q-p)\cdot n} = \frac{d^{2\omega}q}{p\cdot n}\left(-\frac{1}{q\cdot n} + \frac{1}{(q-p)\cdot n} \right), \quad p_\mu \neq 0 , \tag{6.35}$$

in integrals like $\int d^{2\omega}q[(q-p)^2\, q\cdot n\, (q-p)\cdot n]^{-1}$. Despite the presence of n^*-terms, $\Pi^{ab}_{\mu\nu}$ obeys the simple Ward identity $p^\mu\, \Pi^{ab}_{\mu\nu}(p) = 0$, derived previously in eq.(6.8). Obviously the nonlocal term $4n_\mu n_\nu\, p\cdot n^* p^2/(p\cdot n\, n\cdot n^*)$ would be damaging for the renormalization program if such a term would turn out to be necessary in order to obtain finite S-matrix elements. In fact, we shall see in the next Section that the nonlocal factors in the self-energy and vertex functions can be matched by a suitable BRS ansatz for the counterterms.

Before leaving this Section, we note that the light-cone gauge formalism has also been applied to the one-loop quark self-energy (Fig. 6.3) and the quark-quark-gluon vertex function (Fig. 6.4) which were shown to respect the Ward identity (Leibbrandt and Nyeo, 1984). Other recent publications include Natroshvili et al. (1985), Gaigg et al. (1987), Ho-Kim et al. (1986), Mann and Tarasov (1986).

2. The three-gluon vertex and nonlocal BRS counterterms

One may gain further insight into the overall structure of nonlocal terms and the importance of ghosts by examining the vertex $\Gamma^{abc}_{\mu\nu\sigma}(p,q,r)$ in Fig. (6.5). The presence of nonlocal terms in the reduced vertex function $\Gamma^{abc}_{\mu\nu\sigma}(p,0,-p)$ was first established by Andrasi et al. (1986), and Lee and Milgram (1986a), and in the general three-gluon vertex $\Gamma^{abc}_{\mu\nu\sigma}(p,q,r)$ by Dalbosco (1985). The results of Dalbosco were later verified by Lee and Milgram (1986b), and Leibbrandt and Nyeo (1986d). In Dalbosco's elegant notation (Dalbosco, 1985), the divergent part of $\Gamma^{abc}_{\mu\nu\sigma}(p,q,r)$ reads, to one-

loop order:

$$\text{div}\, \Gamma^{abc}_{\mu\nu\sigma}(p,q,r) = 2\kappa f^{abc}[(\tfrac{11}{6} A - B^{(1)} - C - D^{(2)}) + (E^{(1)} + E^{(2)} - 2B^{(2)} - H)]_{\mu\nu\sigma} , \tag{6.36}$$

where the first four terms are local,

$$A_{\mu\nu\sigma} = g_{\nu\sigma}(q-r)_\mu + g_{\sigma\mu}(r-p)_\nu + g_{\mu\nu}(p-q)_\sigma = \{g_{\nu\sigma}(q-r)_\mu\}^{(s)} ,$$

$$B^{(1)}_{\mu\nu\sigma} = \{g_{\nu\sigma}n_\mu\, n^*\cdot(q-r)/n\cdot n^*\}^{(s)} ,$$

$$C_{\mu\nu\sigma} = \{g_{\nu\sigma}n_\mu^*\, n\cdot(q-r)/n\cdot n^*\}^{(s)} ,$$

$$D^{(2)}_{\mu\nu\sigma} = \{(q-r)_\mu(n_\nu n_\sigma^* + n_\nu^* n_\sigma)/n\cdot n^*\}^{(s)} ,$$

while the remaining ones are nonlocal:

$$B^{(2)}_{\mu\nu\sigma} = \{g_{\nu\sigma}n_\mu(n\cdot q\, n^*\cdot q - n\cdot r\, n^*\cdot r)/(n\cdot n^*\, n\cdot p)\}^{(s)} ,$$

$$E^{(1)}_{\mu\nu\sigma} = \{p_\mu n_\nu n_\sigma\, (n^*\cdot q/n\cdot q - n^*\cdot r/n\cdot r)/n\cdot n^*\}^{(s)} ,$$

$$E^{(2)}_{\mu\nu\sigma} = \{(q-r)_\mu\, n_\nu n_\sigma\, (n^*\cdot q/n\cdot q + n^*\cdot r/n\cdot r)/n\cdot n^*\}^{(s)} ,$$

$$H_{\mu\nu\sigma} = n_\mu n_\nu n_\sigma\, \{(q^2 n^*\cdot r - r^2 n^*\cdot q)/(n\cdot n^*\, n\cdot q\, n\cdot r)\}^{(s)} .$$

Here p_μ, q_μ, r_μ are incoming momenta with $(p+q+r)_\mu = 0$, $\kappa = g^2 C_{YM}\, \Gamma(2-\omega)/(4\pi)^2$, g is the strong coupling constant and the symbol $\{\ldots\}^{(s)}$ denotes cyclic permutation of the indices (μ,ν,σ) and of the momenta (p,q,r).

The next challenge is to construct a BRS-invariant counterterm-Lagrangian that will match the nonlocal parts in the self-energy and vertex functions. Encouraged by the fact that $\Gamma^{abc}_{\mu\nu\sigma}$ respects the Ward identity (Leibbrandt and Nyeo, 1986d)

$$q_\nu \Gamma^{abc}_{\mu\nu\sigma}(p,q,r) = igf^{abc}(\Pi_{\mu\sigma}(r) - \Pi_{\mu\sigma}(p)) \ , \tag{6.37}$$

where $\Pi_{\mu\sigma}$ is given in eq. (6.34), we shall use the Slavnov-Taylor identities (Slavnov, 1972; Taylor, 1971) to postulate a suitable counterterm-Lagrangian.

Following the procedure between eqs. (5.25) and (5.27), we find that the divergent part D of the one-particle-irreducible Green functions is given by

$$D = Y + \sigma X \tag{6.38}$$

$$Y = -\frac{1}{2} F^a_{\ \mu\nu} F^{\mu\nu a} - \frac{1}{2} a_2 n_\mu n_\nu^* F^{\mu\lambda a} F^\nu{}_\lambda{}^a \ , \tag{6.39}$$

$$\sigma = \frac{\delta L}{\delta A^a_\mu}\frac{\delta}{\delta J^{\mu a}} + \frac{\delta L}{\delta J^a_\mu}\frac{\delta}{\delta A^{\mu a}} + \frac{\delta L}{\delta \eta^a}\frac{\delta}{\delta K^a} + \frac{\delta L}{\delta K^a}\frac{\delta}{\delta \eta^a} \ , \ \sigma^2 = 0. \tag{6.40}$$

where Y is gauge-invariant and σX gauge-noninvariant. J^a, K^a are sources and η_a, $\bar{\eta}_a$ ghost fields. The coefficient a_2 in Y may be dropped on dimensional grounds. (See in this connection, Gaigg et al. (1986).) The functional X is basically arbitrary, but should conserve ghost number N_g (Lee, 1976; Itzykson and Zuber, 1980) and have the proper dimension of mass.

We return to eq. (6.38). Since $\Pi^{ab}_{\mu\nu}$ and $\Gamma^{abc}_{\mu\nu\sigma}$ contain both local and nonlocal terms, it seems reasonable to endow the functional X with similar characteristics. Accordingly, we shall assume the ansatz

$$X = X_{local} + X_{nonlocal} \ , \tag{6.41}$$

$$X_{local} = a_3 A^a_\mu (J^{\mu a} + \bar{\eta}^a n^\mu) + a_4 n_\sigma^* A^{\sigma a} n^\mu (J^a_\mu + \bar{\eta}^a n_\mu) + a_8 \eta^a K^a,$$

$$X_{nonlocal} = a_5 (n^*_\sigma \partial^\sigma / n_\mu \partial^\mu) n_\tau A^{\tau a} n^\lambda (J_\lambda{}^a + \bar{\eta}^a n_\lambda) + a_7 g f^{abc} [(n_\sigma^* \partial^\sigma / n_\mu \partial^\mu) n_\tau A^\tau_a n^\lambda A_\lambda{}^b] \tag{6.42}$$

$$[(n^\tau \partial_\tau)^{-1} (J^c_\rho + \bar{\eta}^c n_\rho) n^\rho]. \tag{6.43}$$

Hence the counterterm Lagrangian is given by

$$L_{count} = -\frac{1}{2} a_1 F^a_{\ \mu\nu} F^{\mu\nu a} - \sigma X. \tag{6.44}$$

As shown by Leibbrandt and Nyeo (1986d), the counterterm for the three-gluon vertex has the form

$$gf^{abc}[-2a_1 A - 3a_3 A + n\cdot n^*(a_5 B^{(1)} - a_4 C - a_4 D^{(2)}) + n\cdot n^* (2a_5 B^{(2)} - a_5 E^{(2)} + a_7 E^{(1)} - a_7 H)]_{\mu\nu\sigma} \ , \tag{6.45}$$

where $A_{\mu\nu\sigma}, \ldots, H_{\mu\nu\sigma}$ are given in eq. (6.36). By explicit computation all ghost diagrams vanish (see Figs. (6.6) and (6.7)), the chief reason being that $n^\mu G^{ab}_{\mu\nu} = 0$. Comparison of (6.45) with (6.36) leads to the coefficients:

$$a_1 = +\frac{11}{6}\kappa, \ a_3 = 0 \ , \qquad a_4 = -a_5 = a_7 = -2\kappa/n\cdot n^* \ . \tag{6.46}$$

The coefficient a_8 vanishes from a study of the ghost diagrams. Notice that the last four terms in (6.45) correspond to the nonlocal terms in (6.36).

Due to the presence of nonlocal counterterms, and despite considerable effort in recent years, there remain a number of unresolved questions about the renormalization structure of Yang-Mills theory in the light-cone gauge which has been studied by various groups, including Bassetto et al., (1985, 1986), Bassetto (1985, 1986), Lee and Milgram (1985c, 1986a), Andrasi et al., (1986), Nyeo (1986a, 1986c), and Leibbrandt and Nyeo (1986c, 1986d). The general hope is that the nonlocal divergent terms can be controlled in a systematic way by working, for example, in an extended BRS formalism, or by proving that nonlocal terms eventually cancel in all "observable" quantities. This second and, from a historical point of view, more appealing approach has been studied with some success by Bassetto et al. (1987).

VII. The Light-Cone Gauge. Part II

A. Supersymmetric Yang-Mills theory

1. Introduction

The purpose of this Section is to apply the light-cone gauge to a study of the finiteness properties of supersymmetric Yang-Mills theories. Gell-Mann and Schwarz (unpublished) had suggested some time ago that the N=4 supersymmetric Yang-Mills model might be ultraviolet convergent. The problem was subsequently analyzed by two distinct methods, the Lorentz covariant method and the noncovariant light-cone gauge technique. Despite the loss of manifest Lorentz covariance, the light-cone technique seemed superior. It was easier to apply and permitted implementation of the full N=4 supersymmetry. With the proof of the ultraviolet finiteness of the N=4 Yang-Mills model by Mandelstam (1983), and the pioneering work of Brink et al. (1983a, 1983b), the reputation of the light-cone gauge as an effective and viable gauge was finally established. There soon appeared other articles on the N=4 model (Namazie et al., 1983; Ögren, 1984; Capper et al., 1984; Brink and Tollstén, 1985; Leibbrandt and Matsuki, 1985), as well as on the N=2 model (Smith, 1985a, 1985b) and on N=1 (Capper and Jones, 1985a, 1985b).

Our plan is to summarize the principal steps leading to the light-cone gauge superfield formulation of the N=4 model as given by Brink et al. (1983b). These steps include elimination of the unphysical field components, embedding of the remaining physical modes in a complex scalar superfield ϕ and, finally, rewriting of the Lagrangian as a function of the light-cone superfield ϕ. We base our review on Section II of Namazie, Salam and Strathdee (1983), highlighting those features characteristic of the light-cone formalism.

2. N=4 supersymmetric Yang-Mills theory

The Lagrangian density for the N = 4 supersymmetric Yang-Mills model (Gliozzi et al., 1976, 1977) can be written as (Namazie et al., 1983)

$$L = -\frac{1}{4}(F_{\mu\nu})^2 - i\,\bar{\Psi}^\alpha \nabla \Psi_\alpha + \frac{1}{4}\nabla_\mu \bar{H}^{\alpha\beta}\nabla^\mu H_{\alpha\beta}$$
$$-\frac{g}{\sqrt{2}}(\bar{H}^{\alpha\beta}\cdot \Psi_\alpha{}^T \times C^{-1}\Psi_\beta + \text{herm. conj.})$$
$$-\frac{g^2}{16}\bar{H}^{\alpha\beta}\times \bar{H}^{\gamma\delta}\cdot H_{\alpha\beta}\times H_{\gamma\delta}, \qquad (7.1)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,A_\mu \times A_\nu, \quad \mu,\nu = 0,1,2,3,$$

where A_μ is a Yang-Mills field, Ψ_α a chiral spinor, $H_{\alpha\beta}$ a scalar, and α, β = 1,2,3,4 are SU(4) indices. Gauge indices are suppressed in this Section and all fields are in the adjoint representation of the gauge group. C is the charge conjugation matrix. This N = 4 model possesses the following symmetries:

1) a local symmetry (any semi-simple gauge group, with all fields in the adjoint representation);

2) a global supersymmetry and a global SU(4) symmetry, under which the supersymmetry charge transforms as a 4. This implies that there is only one independent coupling constant g. Moreover, $\Psi_\alpha \sim 4$, $\bar{\Psi}_\alpha \sim \bar{4}$, and $H_{\alpha\beta} \sim 6$ of this SU(4), with the "reality" condition $H_{\alpha\beta} = (1/2)\,\epsilon_{\alpha\beta\gamma\delta}\,\bar{H}^{\gamma\delta}$ imposed, and $\bar{H}^{\gamma\delta} = (H_{\gamma\delta})^*$.

Star (*) means complex conjugation and the superscript T in (7.1) indicates the transpose. $\nabla = \gamma^\mu\nabla_\mu$, where ∇_μ is the gauge derivative

$$\nabla_\mu = \partial_\mu + g\,A_\mu \times . \qquad (7.2)$$

The Lagrangian (7.1) may be rewritten in light-cone form by defining, in

4-dimensional space-time,

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \quad x^\mu = (x^0,x^1,x^2,x^3),$$
$$x_T = \frac{1}{\sqrt{2}}(x^1 + ix^2), \quad \bar{x}_T = \frac{1}{\sqrt{2}}(x^1 - ix^2),$$
$$x^\mu x_\mu = 2\,(x^+x^- - x_T\,\bar{x}_T); \qquad (7.3)$$

and

$$A_\pm = \frac{1}{\sqrt{2}}(A_0 \pm A_3), \quad A_\mu = (A_0,A_1,A_2,A_3),$$
$$A_T = \frac{1}{\sqrt{2}}(A_1 - i\,A_2), \quad \bar{A}_T = \frac{1}{\sqrt{2}}(A_1 + i\,A_2), \qquad (7.4)$$

where the subscript T labels the transverse components. It is customary to call x^+ the light-cone time, or evolution parameter, and x^-, x_T, $\bar{x}_T$ the spatial light-cone coordinates. Equations of motion containing $\partial_+ = \partial/\partial x^+$ are, therefore, dynamical equations, whereas those not involving ∂_+ are treated as constraint equations. Imposition of the light-cone gauge condition $n^\mu A_\mu = 0$, or

$$A_- = 0 \qquad (7.5)$$

for the special choice $n_\mu = (1,0,0,1)$, allows one to use the constraint equations to eliminate some components of the various fields in favour of the remaining physical degrees of freedom. In the light-cone gauge (7.5), the equations of motion for A_μ,

$$J_\nu = \nabla^\mu F_{\mu\nu}, \quad \mu,\nu = (+,-,T,\bar{T})$$

can be solved for A_+:

$$A_+ = (\partial_-)^{-2}\,(\nabla_T\partial_- A_T + \bar{\nabla}_T\,\partial_-\bar{A}_T - J_-), \qquad (7.6)$$

where $\nabla_T = \frac{1}{\sqrt{2}}(\nabla_1 - i\nabla_2)$, $\bar{\nabla}_T = \frac{1}{\sqrt{2}}(\nabla_1 + i\nabla_2)$. We recall that the nonlocal operator $1/\partial_-$ represents an indefinite integral and will, therefore, be burdened by the usual integration constant ambiguity. In momentum space the ambiguity asserts itself when $p^- = 0$. For loop integrations it is important to handle the unphysical singularities of $(p\cdot n)^{-1} = (p^-)^{-1}$ by a meaningful prescription which must satisfy power counting. Such a prescription has been given by Mandelstam (1983) and, independently, by Leibbrandt (1982, 1984a).

It remains to express J_- in terms of propagating modes. Following Namazie et al. (1983), we write the chiral fermion field Ψ_α as

$$\Psi_\alpha = 2^{1/4} \begin{pmatrix} \zeta_\alpha \\ \chi_\alpha \\ 0 \\ 0 \end{pmatrix} , \tag{7.7}$$

and then use the Dirac equation to solve for the unphysical component ζ_α,

$$\zeta_\alpha = \frac{1}{i\,\partial_-} (i\,\nabla_T \chi_\alpha + g\, H_{\alpha\beta} \times \bar{\chi}^\beta) , \quad \alpha, \beta = 1,2,3,4. \tag{7.8}$$

Hence

$$J_- = -2ig\,\bar{\chi}^\alpha \times \chi_\alpha - \frac{g}{2} \bar{H}^{\alpha\beta} \times \partial_- H_{\alpha\beta} , \tag{7.9}$$

so that A_+ becomes

$$A_+ = (\partial_-)^{-2} [\nabla_T \partial_- A_T + \bar{\nabla}_T \partial_- \bar{A}_T + 2ig\, \bar{\chi}_\alpha \times \chi^\alpha + \frac{g}{2} \bar{H}^{\alpha\beta} \times \partial_- H_{\alpha\beta}] . \tag{7.10}$$

By eliminating A_-, ζ_α, and A_+ by means of eqs. (7.5), (7.8) and (7.10), Brink et al. (1983a) managed to rewrite the Lagrangian (7.1) in terms of the set $(A_T, \chi_\alpha, H_{\alpha\beta})$ and its complex conjugate, and then proceeded to embed these physical components in a scalar light-cone superfield $\phi\,(x^\mu, \theta_\alpha, \bar{\theta}^\alpha)$ defined on N = 4 extended superspace $(x^\mu, \theta_\alpha, \bar{\theta}^\alpha)$ (Brink et al., 1983a; Mandelstam, 1983). The coordinate θ_α, and its complex conjugate $\bar{\theta}^\alpha$, $\alpha = 1,\ldots,4$, are Grassman parameters, transforming under SU(4) as a 4 and $\bar{4}$, respectively. In

the light-cone gauge, an SO(1,1) x SO(2) subgroup of the Lorentz group survives intact; under this subgroup,

$$\theta_\alpha \to e^{(\lambda - i\sigma)/2}\, \theta_\alpha , \quad \bar{\theta}^\alpha \to e^{(\lambda + i\sigma)/2}\, \bar{\theta}^\alpha, \tag{7.11}$$

while the coordinate vector $x^\mu = (x^+, x^-, x^T, \bar{x}^T)$ changes according to

$$x^\pm \to e^{\pm\lambda} x^\pm , \quad -\infty < \lambda < +\infty ,$$
$$x_T \to e^{+i\,v} x_T , \quad \bar{x}_T \to e^{-i\,v} \bar{x}_T , \quad 0 \le v \le 2\pi . \tag{7.12}$$

Under light-cone supertranslations these variations transform as

$$\theta_\alpha \to \theta_\alpha + \epsilon_\alpha , \quad \bar{\theta}^\alpha \to \bar{\theta}^\alpha + \bar{\epsilon}^\alpha ,$$
$$x^+ \to x^+ , \quad x^- \to x^- + \frac{i}{2} (\bar{\theta}\epsilon + \bar{\epsilon}\theta) + \frac{i}{2} \bar{\epsilon}\, \epsilon ,$$
$$x_T \to x_T , \quad \bar{x}_T \to \bar{x}_T, \tag{7.13}$$

with ϵ_α an infinitesimal anti-commuting parameter.

As usual, it is possible to define spinor covariant derivatives (Salam and Strathdee, 1978) on the extended superspace

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} + \frac{i}{2} \theta_\alpha \partial_- , \quad \bar{D}^\alpha = \frac{\partial}{\partial\theta_\alpha} - \frac{i}{2} \bar{\theta}^\alpha \partial_- , \tag{7.14}$$

which satisfy the following anti-commutation relations:

$$\{D_\alpha , D_\beta\} = 0 , \quad \{\bar{D}^\alpha , \bar{D}^\beta\} = 0, \quad \{D_\alpha , \bar{D}^\beta\} = -i\, \delta_\alpha{}^\beta\, \partial_- ; \tag{7.15}$$

$\partial/\partial\theta_\alpha$ and $\partial/\partial\bar{\theta}^\alpha$ act as right and left derivatives, respectively. The scalar superfield ϕ is chiral in the sense that

$$D_\alpha\, \phi = 0 , \tag{7.16}$$

and since the N = 4 multiplet is CPT self-conjugate, ϕ obeys the "reality" condition:

$$D_1 D_2 D_3 D_4\, \bar{\phi} = (\partial_-)^2\, \phi \quad ; \tag{7.17}$$

equation (7.17) implies that the superfields ϕ and $\bar{\phi}$ are linearly dependent. Explicitly,

$$\phi = e^{-(1/2)\bar{\theta}\theta\partial_-} \left\{ \frac{1}{i\partial_-} A_T(x) + \frac{1}{i\partial_-} \theta_\alpha \bar{\chi}^\alpha(x) + \frac{i}{2} \theta_\alpha \theta_\beta \bar{H}^{\alpha\beta}(x) \right.$$

$$\left. - \frac{1}{3!} \epsilon^{\alpha\beta\gamma\delta} \theta_\alpha\theta_\beta\theta_\gamma \chi_\delta(x) + \frac{i}{4!} \epsilon^{\alpha\beta\gamma\delta} \theta_\alpha\theta_\beta\theta_\gamma\theta_\delta \partial_- \bar{A}_T(x) \right\} . \quad (7.18)$$

It then follows from the SO(1,1) x SO(2) light-cone symmetry that

$$\phi \to e^{\lambda - i\sigma} \phi \ , \ \bar{\phi} \to e^{\lambda + i\sigma} \bar{\phi} \ ,$$

$$d^4\theta \to e^{2(\lambda - i\sigma)} d^4\theta \ , \ d^4\bar{\theta} \to e^{2(\lambda + i\sigma)} d^4\bar{\theta} \ ,$$

$$\partial_\pm \to e^{\mp\lambda} \partial_\pm \ , \ \partial_T \to e^{-i\sigma} \partial_T \ , \ \bar{\partial}_T \to e^{+i\sigma} \bar{\partial}_T . \quad (7.19)$$

3. **Superfield representation**

In terms of the superfield ϕ , eq. (7.18), the Lagrangian (7.1) now assumes the light-cone gauge form (see eq. (4.10) of Brink <u>et al</u>. (1983a))

$$L = \int d^4\theta \, d^4\bar{\theta} \left\{ \frac{1}{2} \phi \cdot \frac{\partial^2}{(\partial_-)^2} \phi \ + \right.$$

$$+ \left[\frac{2}{3} g \left(\bar{\phi} \cdot \frac{1}{i\,\partial_-} \phi \times \bar{\partial}_T \phi \right) + \text{herm. conj.} \right]$$

$$\left. - \frac{g^2}{2} \left[\frac{1}{\partial_-} \phi \times \partial_- \phi \cdot \frac{1}{\partial_-} \bar{\phi} \times \partial_- \bar{\phi} + \frac{1}{2} \phi \times \bar{\phi} \cdot \phi \times \bar{\phi} \right] \right\} , \quad (7.20)$$

which is manifestly invariant under the global symmetry expressed by the transformations (7.11)-(7.13).

The light-cone gauge superfield formulation of the Lagrangian (7.20) provides a convenient starting point for proving the ultraviolet finiteness of the N=4 supersymmetric Yang-Mills model. By examining the detailed structure of each vertex in an arbitrary amplitude of the theory, Mandelstam (1983) showed that sufficiently many powers of momentum are associated with each <u>external</u> line so as to render the corresponding Green function "power-

counting ultraviolet finite". A crucial ingredient in the proof is his light-cone prescription which allows the amplitude to be Wick-rotated to Euclidean space so that naive power-counting is indeed valid.

Further details may be found in Namazie <u>et al</u>. (1983) who proceed to demonstrate, among other things, that the off-shell finiteness of the N=4 Yang-Mills theory remains intact even when supersymmetry is broken explicitly by the addition of mass terms for the scalars and spinors of the model. In addition, the local symmetry can be spontaneously broken. For an SU(2) gauge group, for instance, the resulting theory has a spectrum that is entirely massive and, hence, both infrared and ultraviolet finite.

This completes our brief review of the basic light-cone gauge nomenclature in supersymmetric Yang-Mills theory.

B. Applications in gravity

1. Pure gravity

Scherk and Schwarz (1975) and Kaku (1975) were among the first to study pure gravity in the light-cone gauge (Root, 1973). By eliminating the redundant degrees of freedom of the metric tensor $g_{\mu\nu}$ and expressing the latter in terms of two physical transverse modes, they were able to simplify the Einstein-Hilbert Lagrangian density L_{Ein} considerably. Simplification of the theory and ease of computation are the major advantages of the light-cone formalism, both for Einstein gravity and other sophisticated theories, like supergravity or supersymmetric string theories. As in Section VII. A, the purpose of this part is to acquaint the reader with those features characteristic of the light-cone gauge. With this in mind we shall summarize the main steps in the elimination of the redundant modes of $g_{\mu\nu}$, following closely the approach of Scherk and Schwarz (1975).

Consider the Lagrangian density

$$L = L_{Ein} + L_{fix} \quad , \tag{7.21a}$$

$$L_{Ein} = \frac{2}{\kappa^2} \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu} \ , \tag{7.21b}$$

$$L_{fix} = (2\alpha)^{-1} (n^\mu g_{\mu\nu})^2 \ , \quad n^\mu n_\mu = 0 \ , \tag{7.21c}$$

where the nomenclature is the same as in eq. (4.37). $R_{\mu\nu}$ is the Ricci tensor, $g = \det(g_{\mu\nu})$ and $g^{\mu\sigma} g_{\nu\sigma} = \delta^\mu_\nu$. L_{Ein} describes massless, helicity-two gravitons: it is invariant under general coordinate transformations and possesses, therefore, gravitational gauge symmetry. The light-cone gauge condition

$$n^\mu g_{\mu\nu} = 0 \quad , \quad n^\mu n_\mu = 0 \quad , \quad \mu,\nu = 0,1,2,3, \tag{7.22}$$

is implemented by letting the gauge parameter α in eq. (7.21c) approach zero.

In the absence of matter, Einstein's equations for the gravitational field in empty space read (cf. Capper and Leibbrandt, 1982b)

$$R_{\mu\nu} = \partial_\nu \Gamma^\rho{}_{\mu\rho} - \partial_\rho \Gamma^\rho{}_{\mu\nu} - \Gamma^\sigma{}_{\mu\nu} \Gamma^\rho{}_{\sigma\rho} + \Gamma^\rho{}_{\sigma\nu} \Gamma^\sigma{}_{\mu\rho} = 0 \ ,$$

$$\Gamma^\sigma{}_{\beta\gamma} = \frac{1}{2} g^{\sigma\omega} (\partial_\gamma g_{\beta\omega} + \partial_\beta g_{\omega\gamma} - \partial_\omega g_{\beta\gamma}) . \tag{7.23}$$

In order to reduce L_{Ein} to two-component form, it is advantageous to employ light-cone coordinates defined in 4-dimensional space-time by

$$x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^3) \ , \ \vec{x} = (x^1, x^2) \ ,$$

$$x \cdot x = 2(x^+ x^- - \vec{x}\cdot\vec{x}) \quad , \tag{7.24a}$$

$$x \cdot y = (x^+ y^- + x^- y^+ - 2\vec{x}\cdot\vec{y}) \ ,$$

and, in d-dimensional space-time, by

$$x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^{d-1}) \ ,$$

$$\vec{x} = (x^1, \ldots, x^{d-2}) = (x^j), \ j = 1, \ldots, d-2, \tag{7.24b}$$

and to treat x^+ as the light-cone time, and $x^-, x^1, \ldots, x^{d-2}$ as the spatial light-cone coordinates or light-cone positions. Note that the definition (7.24) differs slightly from (7.3) for supersymmetric Yang-Mills theory. For ease of back-checking with the original literature and rather than run the risk of introducing errors by standardizing the notation, we decided to maintain as much as possible the notation of the original references.

Our primary task is to express L_{Ein} in terms of physical propagating modes only. We recall that the symmetric tensor $g_{\mu\nu}$ has originally ten components. The four <u>gauge conditions</u>

$$n^\mu g_{\mu\nu} = 0, \quad n^\mu n_\mu = 0, \quad \mu,\nu = +,-,1,2, \tag{7.25}$$

reduce that number from ten to six, while gravitational gauge invariance eliminates a further four redundant modes. In four dimensions the graviton field possesses, therefore, two physical degrees of freedom, whereas in d dimensions the number of propagating modes equals $\frac{1}{2}$ (d-2)(d-1) -1 (Goroff and Schwarz, 1983).

To demonstrate the elimination of the redundant $g_{\mu\nu}$ modes, we follow Scherk and Schwarz (1975) who define new variables Ψ, γ_{ij} and ϕ by writing, respectively,

$$g_{ij} = e^{\Psi}\,\gamma_{ij} \quad , \qquad i,j = 1,2, \tag{7.26}$$

$$\det(\gamma_{ij}) = 1 \quad , \tag{7.27}$$

$$g_{+-} = e^{\phi} \, , \tag{7.28}$$

where the symmetric, unimodular matrix γ_{ij} is characterized by two independent variables, ρ and θ:

$$\gamma_{ij} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{\rho} & 0 \\ 0 & e^{-\rho} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} , \; \gamma^{ik}\gamma_{jk} = \delta^i_j \, . \tag{7.29}$$

The next step is to replace the gauge constraints (7.25) by the more suitable set ($n^+ = \sqrt{2}$, $n^- = 0$)

$$g_{++} = g_{+1} = g_{+2} = 0 \, , \tag{7.30a}$$

$$\phi = \frac{1}{2}\Psi \, , \tag{7.30b}$$

and then to rewrite Ψ, g_{--}, and g_{-i} in terms of γ_{ij}.

Consider first Ψ. Substitution of eqs. (7.26), (7.27), (7.28), (7.30a) into eq. (7.23) yields

$$R_{++} = 2(\partial_+\phi)(\partial_+\Psi) - 2(\partial_+)^2\Psi - (\partial_+\Psi)^2 + \frac{1}{2}(\partial_+\gamma^{ij}\,\partial_+\,\gamma_{ij}) = 0, \tag{7.31}$$

leading to the solution ($\phi = \frac{1}{2}\Psi$)

$$\Psi = \frac{1}{4}(\partial_+)^{-2}(\partial_+\gamma^{ij}\,\partial_+\,\gamma_{ij}) \, , \tag{7.32}$$

where the nonlocal operator $(\partial_+)^{-1}$ is equivalent to the operator $(\partial_-)^{-1}$ used in Sect. VII.A.

The components g_{-i} and g_{--}, on the other hand, follow from $g^{\mu+}g_{\mu i} = \delta^+_i$ and $g^{\mu+}g_{\mu-} = \delta^+_-$, respectively:

$$g_{-i} = -\,e^{3\Psi/2}\,\gamma_{ij}g^{+j} \, , \qquad i,j = 1,2, \tag{7.33a}$$

$$g_{--} = e^{-\Psi}\,\gamma^{ij}\,g_{-i}g_{-j} - e^{\Psi}g^{++} \, , \tag{7.33b}$$

where $g_{+i} = \delta^+_i = \delta^+_- = 0$; g^{+j} may be deduced from $R_{+j} = 0$, and g^{++} from $R_{+-} = 0$. Eqs. (7.32)-(7.33) permit us to write all three variables Ψ, g_{-i} and g_{--} as functions of γ_{ij}, so that L_{Ein} finally becomes (see eq. (3.17) of Scherk and Schwarz, 1975)

$$\begin{aligned} L_{Ein} \propto e^{\Psi/2} \Big[& \gamma^{ij}\partial_i\partial_j\Psi - \frac{3}{4}\gamma^{ij}\partial_i\Psi\,\partial_j\Psi + \\ & + \gamma^{ik}\,\partial_i\,\gamma^{jm}\,\partial_j\gamma_{km} - \frac{1}{2}\gamma^{ij}\,\partial_-\,\gamma^{km}\,\partial_j\,\gamma_{km} \Big] \\ & + e^{\Psi}\Big[4\partial_+\partial_-\,\Psi - \partial_+\gamma^{ij}\partial_-\,\gamma_{ij}\Big] \\ & + e^{-3\Psi/2}\,\gamma^{ij}\left(\frac{1}{\partial_+}R_i\right)\left(\frac{1}{\partial_+}R_j\right) , \quad i,j,k,m = 1,2, \end{aligned} \tag{7.34}$$

with

$$R_i = \frac{1}{2}e^{\Psi}\Big[\partial_+\gamma^{jk}\partial_i\,\gamma_{jk} + \partial_i\Psi\,\partial_+\Psi - 3\,\partial_+\partial_i\Psi\Big] + \partial_k\left(e^{\Psi}\,\gamma^{jk}\,\partial_+\,\gamma_{ij}\right) .$$

We have succeeded in rewriting Einstein's Lagrangian entirely in terms of the physical degrees of freedom of the graviton. The relative simplicity of the two-component version (7.34) is partially offset by its lack of locality and manifest Lorentz covariance, two features reminiscent of most light-cone gauge formulations.

This completes our review, according to Scherk and Schwarz (1975), of the elimination of the nonpropagating modes in pure gravity, and of the derivation of L_{Ein} in light-cone form.

The structure of 4-dimensional gravity was also examined by Kaku (1975) who eliminated the eight redundant components by integrating functionally over g^{++}, g^{+-}, g^{+i}, g^{--}, g^{-i} and $\det(g_{ij})$. Goroff and Schwarz (1983), on the other hand, studied pure gravity in d dimensions. They showed that the theory possesses an SL (d-2,R) symmetry, so that the graviton may be identified with the coset SL(d-2,R)/SO(d-2), SO(d-2) being the helicity group. Recently, Ögren (1986) evaluated the one-loop self-energy in light-front gravity.

2. Supergravity

Since the early 1980's the light-cone formalism has also been applied to different models of supergravity, but with a lower success rate than in Yang-Mills theory because of the nonpolynomial structure and greater complexity of the gravitational interaction. Nevertheless, the presence of only physical modes in the light-cone gauge and the absence of FP-ghosts have led to simpler and more attractive supergravity theories in which the transformations obey a global super-Poincaré algebra.

Among the earliest contributors to the field was Bengtsson (1983) who studied the linear structure of N=1 supergravity in four dimensions, constructing the dynamical supersymmetry transformations and deriving the

Hamiltonian to first order in the gravitational coupling constant κ. Bengtsson et al. (1983a, 1983b) constructed cubic interaction terms for massless fields of arbitrary helicity and for all maximally extended supermultiplets. Extended supergravity in ten dimensions was investigated by Green and Schwarz (1983). In 1984, Randjbar-Daemi et al. computed the vacuum energy in 11-dimensional supergravity and a year later Randjbar-Daemi and Sarmadi (1985) analyzed the graviton induced compactification of a (4+N)-dimensional space-time into the group $(\text{Minkowski})_4 \times S^N$.

C. Strings and Superstrings

1. Introduction

In Sections VI, VII.A. and VII.B. we illustrated the practical side of the light-cone gauge in the case of ordinary Yang-Mills theory, supersymmetric Yang-Mills theory and Einstein gravity, respectively. The purpose of the present discussion is to introduce the notational framework of the light-cone gauge formalism also for supersymmetric string theories, which burst onto the scene in the Orwellian year and have since captured the imagination of a diverse spectrum of particle physicists. While the basic idea of the light-cone gauge, namely elimination of the nonpropagating fields and reformulation of the theory in terms of physical modes only, is the same for all models, the reader will have noticed a gradual increase in the complexity of the light-cone formalism, coupled with minor but annoying changes in notation. This trend continues for the various supersymmetric string theories, or superstring theories for short, including those based on the gauge groups Spin $32/Z_2$ and $E_8 \times E_8$. Since the original superstring model of Green and Schwarz (1982) and numerous subsequent calculations are formulated in the light-cone gauge, we decided to include here a few remarks on the implementation of this exceedingly versatile gauge. We stress that the present discussion is in no way meant to replace the excellent review articles available in the literature (Mandelstam, 1974; Scherk, 1975; Schwarz, 1982; Green, 1982).

Superstring theories emerged from dual string models (Veneziano, 1968) which were developed between 1968 and 1975 as a theory of hadrons, but were later abandoned, because they were unable to provide a satisfactory physical description of the hadronic world (see, for instance, Alessandrini et al.,

1971; Frampton, 1974; Veneziano, 1974; Jacob, 1974). Among the defects of the "old" string theory were the appearance of massless states in the hadronic spectrum and a lack of awareness for the need of a critical dimension of space-time, a dimension which differs from four and in which these theories were meaningful. Today we know that the critical dimension for bosonic string theories is 26, for fermionic string theories 10. With the discovery of supersymmetry, however (Gol'fand and Likhtman, 1971; Volkov and Akulov, 1973; Wess and Zumino, 1974), and supergravity (Ferrara et al., 1976; Deser and Zumino, 1976; van Nieuwenhuizen, 1981), and the development of various grand unified models, the conceptual framework of quantum field theory changed dramatically and led to a revival of the ideas of Kaluza (1921) and Klein (1926). (For a review, see Duff et al. (1986).) In essence, particle physicists became "conditioned" to working with higher dimensions and were, therefore, quite willing to consider the implications of the radically new theory of superstrings.

Superstring models appear to be good candidates for a unified theory of the known interactions, offering for the first time a realistic opportunity for combining quantum mechanics with general relativity. The models based on the semi-simple gauge groups $E_8 \times E_8$, and Spin $32/Z_2$, are particularly attractive, since they are free of tachyons, ultraviolet finite and anomaly-free to one-loop order. Nevertheless, superstring theories are not easy to work with. They demand a different mode of thinking and the application of unconventional mathematics. For this reason, many of the calculations have been and still are being carried out in the light-cone gauge which, as we know, breaks Lorentz covariance. The first-quantized light-cone gauge string action is supersymmetric, Lorentz-noncovariant and possesses the following two symmetries. It is invariant under Weyl rescaling and reparametrization of the two-dimensional world sheet coordinates. Since we wish to implement the light-cone gauge, we need to concentrate on the second invariance, because fixing the gauge is equivalent to choosing a specific parametrization.

There are closed and open strings. Closed strings can be classified as being either of type I or type II. Type I theories contain only states that are symmetric under the interchange of the oscillators $\alpha_n^i \leftrightarrow \beta_n^i$ (cf. eqs. (7.48), (7.50)), while type II theories contain states which can be either symmetric or anti-symmetric under the interchange $\alpha_n^i \leftrightarrow \beta_n^i$. In addition, there exist planar, orientable and non-orientable strings, but for a thorough discussion of these and related properties we refer the reader to the literature.

The remainder of this chapter is organized as follows. In Section C.2 we review ordinary (bosonic) strings and in Section C.3, superstrings. As indicated, there are both open and closed strings, but we shall not distinguish between these two categories except in situations where the distinction is essential, as in the case of boundary conditions, for example.

2. Strings

Consider a string spanning a two-dimensional surface in space-time, a world-sheet which is parametrized by the variables σ and τ (Scherk, 1975). This world sheet is described by the coordinates $X^\mu(\sigma,\tau)$, where σ is the spatial coordinate that labels points along the string, $0 \le \sigma \le \pi$, and where τ may be identified with the time parameter. In other words, the zero-component of $X^\mu(\sigma,\tau)$, namely X^0, may be chosen proportional to the time τ,

$X^0 \propto \tau$, as emphasized by Schwarz (1982). The metric associated with the two-dimensional world-sheet is denoted by $g_{\alpha\beta}(\sigma,\tau)$, $\alpha,\beta = 1,2$; the space-time metric is labeled by $\eta_{\mu\nu}$ and taken to be the flat Minkowski metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,\ldots,1). \tag{7.35}$$

Following the work of Nambu (1970) and Goto (1971), we can write the string action as (Schwarz, 1982)

$$S = \frac{-1}{4\pi\alpha'}\int_0^\pi d\sigma \int_{\tau_i}^{\tau_f} d\tau\, \eta_{\mu\nu}\sqrt{-g}\, g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\ , \quad \alpha,\beta = 1,2, \tag{7.36}$$

where $g = \det g_{\alpha\beta}$ and $\mu,\nu = (0,1,\ldots,D-1)$. D is the dimension of space-time and equal to D = 26 for the bosonic string; the parameter α' denotes the Regge slope and has dimension $[\alpha'] = (\text{length})^2$, while $\hbar = c = 1$.

As mentioned earlier, the action (7.36) is invariant under Weyl rescaling of the world sheet metric, $g_{\alpha\beta} \to e^{\lambda(\sigma,\tau)} g_{\alpha\beta}$, and under reparametrization of the coordinates σ,τ: $\sigma \to \sigma'(\sigma,\tau)$, $\tau \to \tau'(\sigma,\tau)$ (Goddard et al., 1973). Let us take a closer look at the last symmetry. Since invariance under reparametrization implies a certain gauge freedom, we are at liberty to work in whatever gauge we choose. A particularly convenient gauge is the orthonormal gauge (Douglas, 1939; Goddard et al., 1973; Scherk, 1975; Schwarz, 1982) which may be defined as

$$(\partial_\sigma X_\mu)(\partial_\tau X^\mu) = 0\ , \qquad \partial_\sigma = \partial/\partial\sigma,\ \text{etc.} \tag{7.37a}$$

$$(\partial_\sigma X^\mu)^2 + (\partial_\tau X^\mu)^2 = 0\ . \tag{7.37b}$$

Unfortunately, the constraints (7.37a) and (7.37b) do not completely specify the coordinate system for the string, since the associated two-dimensional world-sheet admits infinitely many orthogonal systems. In order to remove the remaining gauge degrees of freedom, we select a specific axis# n_μ in space by constructing $n_\mu X^\mu$, with n_μ an <u>arbitrary</u> D-dimensional vector. We then choose $n_\mu X^\mu$ proportional to the evolution parameter τ:

$$n_\mu X^\mu(\sigma,\tau) = n_\mu x^\mu(\tau) = n_\mu x^\mu + 2\alpha' n_\mu p^\mu \tau, \tag{7.37c}$$

where p^μ is the total D-momentum of the string (Scherk, 1975), x^μ an integration constant, and where the $x^\mu(\sigma,\tau)$ are "centre-of-mass" coordinates given by

$$x^\mu(\tau) = \frac{1}{\pi} \int_0^\pi X^\mu(\sigma,\tau)\, d\sigma . \tag{7.38}$$

Eqs. (7.37) form a unique orthonormal system which may be still further simplified by choosing n_μ light-like, $n^2 = 0$ (the resulting gauge constraint is also referred to as the <u>transverse</u> gauge) and by working in D-dimensional light-cone coordinates:

$$X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^{D-1}) , \tag{7.39}$$

$$\vec{X} = (X^1,\ldots,X^{D-2}) = (X^i),\ i = 1,2,\ldots,D-2; \tag{7.40}$$

$$X\cdot Y = \eta_{\mu\nu} X^\mu Y^\nu = - X^+ Y^- - X^- Y^+ + X^i Y^i. \tag{7.41}$$

Notice the difference in the overall sign between eq. (7.41) and eq. (7.24a) which is due to the metric (7.35). With $n_\mu = (1,0,\ldots,0,1)$, and in view of eq. (7.37c), the light-cone condition reads

$$X^+(\sigma,\tau) = x^+(\tau) = x^+ + 2\alpha' p^+ \tau , \tag{7.42}$$

while the string action (7.36) reduces to (Schwarz, 1982)

#Note that the choice of such an axis breaks manifest Lorentz covariance.

$$S^{l.c.} = \frac{-1}{4\pi\alpha'} \int_{\sigma=0}^{\sigma=\pi} d\sigma \int_{\tau_i}^{\tau_f} d\tau\, \partial_\alpha X^i \partial^\alpha X^i , \quad \alpha = 1,2,\ i = 1,\ldots,D-2. \tag{7.43}$$

Since it is possible to express X^- in terms of X^i - see the detailed discussion in Scherk (1975) - the dynamical content of the theory is completely determined by the <u>transverse</u> coordinates $X^i(\sigma,\tau)$. The latter satisfy the free wave equation

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^i (\sigma,\tau) = 0 , \quad i = 1,2,\ldots,D-2, \tag{7.44}$$

subject to the following boundary conditions for open and closed strings, respectively:

$$\frac{\partial}{\partial\sigma} X^i_{open}(\sigma,\tau)\Big|_{\sigma=0} = \frac{\partial}{\partial\sigma} X^i_{open} (\sigma,\tau)\Big|_{\sigma=\pi} = 0 , \tag{7.44a}$$

$$X^i_{closed}(0,\tau) = X^i_{closed} (\pi,\tau). \tag{7.44b}$$

To solve system (7.44) - (7.44a) for open strings, one simply expands X^i in terms of normal modes so that

$$X^i_{open}(\sigma,\tau) = x^i + 2\alpha' \tau p^i + 2i\alpha' \sum_{n=1}^{\infty} \frac{1}{n} \alpha^i_n \cos(n\sigma) e^{-in\tau} , \tag{7.45}$$

while quantization of the string gives

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} , \quad \mu,\nu = 0,1,\ldots,D-1, \tag{7.46}$$

$$[\alpha^i_m, \alpha^j_n] = m\, \delta_{m+n,0}\, \delta^{ij} \ ; \quad i,j = 1,2,\ldots,D-2;\ m,n = 1,2,\ldots . \tag{7.47}$$

At this stage it is customary to introduce the lowering and raising operators a^i_n and (a^i_n) , respectively,

$$a_n^i = \frac{1}{\sqrt{n}}\alpha_n^i, \quad (a_n^i)^{\dagger} = \frac{1}{\sqrt{n}}\alpha_{-n}^i, \quad n = 1,2,\ldots, \tag{7.48}$$

which describe infinitely many harmonic oscillators. Ghost states are absent, because we are working in a physical gauge, the light-cone gauge.

The solution of system (7.44)-(7.44b) for closed strings is similar to the open case and leads to (Moffat, 1986; Schwarz, 1982; Dine, 1986)

$$X^i_{closed}(\sigma,\tau) = x^i + i\alpha' \sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_n^i\, e^{-2in(\tau-\sigma)} + \beta_n^i\, e^{-2in(\tau+\sigma)}\right), \tag{7.49}$$

where the first term in the sum represents waves moving along the string to the right, or "right movers", while the second term represents waves moving to the left, or "left movers". Quantization of this string system yields

$$[x^\mu, p^\nu] = i\,\eta^{\mu\nu}, \tag{7.50a}$$
$$[\alpha_m^i, \alpha_n^j] = m\,\delta_{m+n,0}\,\delta^{ij}, \tag{7.50b}$$
$$[\beta_m^i, \beta_n^j] = m\,\delta_{m+n,0}\,\delta^{ij}, \tag{7.50c}$$
$$[\alpha_m^i, \beta_n^j] = 0, \tag{7.50d}$$

where the indices have the same range as in eqs. (7.46) - (7.47).

In summary, the first-quantized bosonic string theory in the light-cone gauge has a critical dimension of $D = 26$ and yields $(D-2)$ massless states in the open case, and $(D-2)^2$ states in the closed case. Its major defects are the appearance of tachyons and the absence of fermions. Although we have purposely restricted our discussion to the light-cone gauge, we should mention that progress during the past couple of years has also been made on the covariant formulation of bosonic strings.

3. Superstrings

The supersymmetric string has bosonic as well as fermionic degrees of

freedom, it is free of tachyons, ultraviolet-finite - at least to one-loop order (Dine, 1986) - and its critical dimension is $D = 10$. The superstring represents a decisive improvement over the bosonic string, especially since anomalies can be shown to cancel at the one-loop level provided the gauge group is Spin $32/Z_2$ (Green and Schwarz, 1984).

The original first-quantized version of superstrings by Green and Schwarz (1982) was formulated in the light-cone gauge, because it was not clear in 1982 how to construct a superstring action that was both Lorentz covariant and supersymmetric. Since that time considerable progress has been made in constructing covariant models, and in establishing equivalence between the light-cone gauge fomulation and the covariant formalism.

Apart from work in the first-quantized version, much effort has gone into deriving a second-quantized, field theoretic formulation of interacting superstrings (Kaku and Kikkawa, 1974; Kaku, 1985; Banks and Peskin, 1986; Witten, 1986; Siegel and Zwiebach, 1986; Hata et al., 1986; Neveu and West, 1985, 1986; Samuel, 1986). In this framework the string field is represented by a scalar functional of the light-cone string coordinate (Green, 1986) and there exist now creation and destruction operators for strings. This functional formulation (Hsue et al., 1970; Gervais and Sakita, 1971; Polyakov, 1981a, 1981b) is being pursued both in the light-cone gauge (Mandelstam, 1985; Restuccia and Taylor, 1985), and in a covariant setting (Green, 1986; Ohta, 1986; see also West (1986)). Here we shall take a brief look only at the first-quantized superstring formalism in the light-cone gauge.

As remarked earlier, superstrings contain both bosonic and fermionic degrees of freedom, i.e. a superstring is characterized by the coordinates

$\{X^\mu, \theta^{Aa}\}$ which define a superspace. $X^\mu(\sigma,\tau)$, $\mu = 0,1,\ldots,D-1$, are the usual bosonic space-time coordinates in D dimensions, while $\theta^{Aa}(\sigma,\tau)$ are Grassmann coordinates expressing the fermionic degrees of freedom. The two-component object θ^{Aa}, $A = 1,2$, $a = 1,2,\ldots,2^{D/2}$, transforms like a spinor in D-dimensional space-time, thereby connecting bosons and fermions (Schwarz, 1982; Green, 1986). Since the critical dimension for superstrings is $D = 10$, there are exactly $D-2 = 8$ physical modes (matching the number of transverse components $X^i(\sigma,\tau)$) and eight physical spinor modes. The variable θ^{Aa} is assumed to be self-conjugate (Majorana) and obeys the chirality (Weyl) condition (Green, 1986; Schwarz, 1982)

$$\frac{1}{2}\,(1 + \eta^A \gamma_{11})^{ab}\, \theta^{Ab}(\sigma,\tau) = 0, \quad A = 1,2, \quad a,b = 1,\ldots,32, \tag{7.51}$$

where $(\gamma^\mu)^{ab}$ are space-time Dirac matrices in a Majorana representation, $\eta^A = \pm 1$, $\gamma_{11} = \gamma^0\gamma^1\ldots\gamma^9$ and $\{\gamma^\mu,\gamma^\nu\} = -\eta^{\mu\nu}$, where the last negative sign is due to the particular choice of Minkowski metric, $\eta^{\mu\nu} = \mathrm{diag}(-1,1,\ldots,1)$.

The next task is to give the form of the light-cone constraints on the superspace coordinates $\{X^\mu, \theta^{Aa}\}$. For the bosonic coordinates X^μ, the constraint is of course the same as in eq. (7.42),

$$X^+(\sigma,\tau) = x^+ + 2\alpha'\, p^+ \tau \quad , \tag{7.52}$$

but for the spinor θ^{Aa} the light-cone condition assumes quite a different form (Schwarz, 1982):

$$(\gamma^+)^{ab}\, \theta^{Ab}\,(\sigma,\tau) = 0, \quad A = 1,2;\ a,b = 1,2,\ldots,32, \tag{7.53}$$

with $\gamma^+ = \frac{1}{\sqrt{2}}\,(\gamma^0 + \gamma^9)$.

To obtain the symmetric string action, one just adds to eq. (7.43) the Dirac action for spinors, $i\,(4\pi)^{-1} \int d\sigma \int d\tau\ \bar\theta\, \gamma^- \rho^\alpha \partial_\alpha \theta$, so that the total action is given by (Schwarz, 1982)

$$S^{\ell.c.} = \frac{1}{4\pi}\int_0^\pi d\sigma \int_{\tau_i}^{\tau_f} d\tau \left[\frac{-1}{\alpha'}\, \partial_\alpha X_i\, \partial^\alpha X^i + i\, \bar\theta^a\, \gamma^- \rho^\alpha\, \partial_\alpha\, \theta^a\right] , \tag{7.54}$$

$$i = 1,2,\ldots,8;\ \alpha = 1,2.$$

Here

$$(\bar\theta)^{Aa} = (\theta^+)^{Bb}\,(\gamma^0)^{ba}\,(\rho^0)^{BA}\ , \qquad A,B = 1,2;\quad a,b = 1,2,\ldots,32,$$

and $(\rho^\alpha)^{AB}$ are two-dimensional world-sheet Dirac matrices with $\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. The light-cone action (7.54), first proposed by Green and Schwarz (1982), is invariant under the supersymmetry transformations

$$\delta X^i = (p^+)^{-1/2}\ \bar\epsilon\, \gamma^i\, \theta, \tag{7.55}$$

$$\delta\theta = i(p^+)^{-1/2}\ \gamma_-\, \gamma_\mu (\rho\cdot\partial\, X^\mu)\epsilon\ , \quad \mu = 0,\ldots,9, \tag{7.56}$$

ϵ^{Aa} being Majorana-Weyl spinors in ten dimensions.

The equations of motion for X^i and θ^{Aa} follow readily from the string action (7.54). For open strings, $X^i(\sigma,\tau)$ satisfies

$$\partial_-\, X^i(\sigma,\tau) = 0\ , \quad \partial_\pm = \frac{1}{\sqrt{2}}\,(\partial_\tau \pm \partial_\sigma)\ , \tag{7.57}$$

with boundary conditions

$$\frac{\partial}{\partial\sigma}\, X^i_{\mathrm{open}}(\sigma,\tau)\Big|_{\sigma=0} = \frac{\partial}{\partial\sigma}\, X^i_{\mathrm{open}}(\sigma,\tau)\Big|_{\sigma=\pi}, \tag{7.58}$$

while θ^{Aa} obeys

$$\theta^{1a} = 0\ , \quad \partial_-\, \theta^{2a} = 0\ , \tag{7.59}$$

with boundary conditions

$$\theta^{1a}(0,\tau) = \theta^{2a}(0,\tau),\ \theta^{1a}(\pi,\tau) = \theta^{2a}(\pi,\tau). \tag{7.60}$$

Systems (7.57) - (7.58) and (7.59) - (7.60) lead to the following open string solutions:

$$X^i(\sigma,\tau) = x^i + 2\alpha' p^i\tau + 2i\alpha' \sum_{n=1}^{\infty} \frac{1}{n} \left\{\alpha_n^i e^{-in\tau} - \alpha_{-n}^i e^{+in\tau}\right\}, \quad (7.61)$$

and

$$\theta^{1a}(\sigma,\tau) = \sum_{n=-\infty}^{\infty} \theta_n^a e^{-in(\tau-\sigma)}, \quad (7.62a)$$

$$\theta^{2a}(\sigma,\tau) = \sum_{n=-\infty}^{\infty} \theta_n^a e^{-in(\tau+\sigma)}, \quad (7.62b)$$

with $\alpha_{-n}^i = \alpha_n^{i*}$, $\theta_{-n}^a = \theta_n^{a*}$.

This concludes our brief introduction to superstrings in the light-cone gauge, but we would be remiss if we did not at least mention a second type of superstring, the heterotic string of Gross, Harvey, Martinec and Rohm (1985). The heterotic string is a hybrid theory, combining the D = 10 fermionic string with the D = 26 bosonic string. Gross *et al*. have demonstrated that the heterotic string, with N = 1 supersymmetry, is likewise finite, free of tachyons and anomaly-free, provided the gauge group is $E_8 \times E_8$ or Spin $32/Z_2$. For a detailed discussion of the heterotic string, which has only been constructed in the light-cone gauge, we refer the reader to the references (Moffat, 1986; Gross, 1986; Dine, 1986).

VIII. The Temporal Gauge

A. Introduction

The temporal gauge is the last of the four axial-type gauges to be surveyed in this review. The temporal gauge is almost as old as quantum mechanics itself having been used half a century ago by Weyl (1931), Heisenberg and Pauli (1930) and others in the quantization of the Maxwell-Dirac field. In quantum electrodynamics the temporal gauge is given by $A_0(x) = 0$ and in Yang-Mills theory by $A_0^a(x) = 0$, with the constant four-vector n_μ taken time-like: $n^2 = n_0^2 - \vec{n}^2 > 0$.

In recent years, the problems related to the quantization of gauge theories in the temporal gauge have been studied both in the context of canonical quantization and within Feynman's path-integral formalism. At the same time, practical calculations have received about equal attention (Goldstone and Jackiw, 1978; Polyakov, 1978; Baluni and Grossman, 1978; Frenkel, 1979; Rossi and Testa, 1980a, 1980b, 1984a; Leroy *et al*., 1984; Müller and Rühl, 1981). The temporal gauge has been applied to the vacuum tunneling by instantons (Rossi and Testa, 1984b) and to the computation of mass singularities from planar graphs. It has also appeared in connection with one-loop thermodynamic potentials (Actor, 1986), Nicolai maps (Claudson and Halpern, 1985; Bern and Chan, 1985) and lattice gauge formulations (Curci *et al*., 1984).

The difficulties encountered in the quantization of gauge theories in the temporal gauge may be traced back to the condition $A_0^a(x) = 0$ which does not fix the gauge uniquely#. The point is that time-independent gauge

#Leroy *et al*. (1986) have considered fixing the gauge completely by adding an extra gauge constraint (Curci and Menotti, 1984; Girotti and Rothe, 1985).

transformations are still a symmetry of the action. This residual invariance manifests itself as an unphysical pole in the longitudinal part of the gauge field propagator. To solve this delicate problem, the following schemes have been proposed.

(1) The canonical quantization scheme. In this approach one tries to eliminate explicitly the unwanted degrees of freedom associated with time-independent gauge transformations (Goldstone and Jackiw, 1978; Bjorken, 1980; Christ and Lee, 1980; Haller, 1986). The procedure can become complicated, especially in non-Abelian models, and does not lead to a practicable set of "Feynman rules".

(2) The pragmatic approach. Its basic idea is to remove the ambiguities arising in integrals like $\int dq\, [(q-p)^2\, q\cdot n]^{-1}$, $\int dq\, [(q-p)^2\, q^2\, (q\cdot n)^2]^{-1}$, etc., by finding a suitable prescription for $(q\cdot n)^{-\alpha}$, $\alpha = 1,2,\ldots$. This strategy, pursued by Caracciolo et al. (1982), Curci and Menotti (1982), Landshoff (1986a), Steiner (1986) and others, has led to several concrete results and some much needed insight into the technical subtleties. It is too early to say how successful this approach will turn out to be, since none of the consistency checks have been carried out beyond the one-loop level.

(3) The path-integral approach. In this scheme, Rossi and Testa (1980a, 1980b; 1984a, 1984b), Leroy et al. (1984a, 1984b) and Chan (1986) achieve quantization by invoking the Faddeev-Popov prescription. Working with a finite-time propagation kernel (Feynman and Hibbs, 1965), they are able to (i) identify the physical states, (ii) derive a set of consistent Feynman

rules, and (iii) prove equivalence between the temporal gauge and Coulomb-gauge formulations. Though unconventional, this method appears to be the only consistent one available to date. Starting from first principles, we are led to a functional representation for the Feynman propagation kernel which then allows us to derive a perturbative expansion. There are no spurious singularities in the gauge field propagator and hence no ambiguities in the loop integrals. Practical problems, related to the complexity of the perturbative expansion, have been solved to some extent by Chan (1986).

B. Path-integral approach

Let us apply the finite-time path-integral method to the temporal gauge. The main ingredient in this approach is the Feynman propagation kernel $K(\vec{A}_2, T/2; \vec{A}_1, -T/2)$ which represents the amplitude for finding the field in the configuration $\vec{A}_2(\vec{x})$ at time $t = -T/2$, if it was in the configuration $\vec{A}_1(\vec{x})$ at time $t = -T/2$.* In Euclidean space, the kernel is given by the functional integral (Rossi and Testa, 1980a, 1980b)

$$K(\vec{A}_2,T/2;\vec{A}_1,-T/2) = \int \prod_{-T/2 \le t \le T/2} Dg(\vec{x},t) \int_{\substack{\vec{A}(\vec{x},T/2) = \vec{A}_2(\vec{x}) \\ \vec{A}(\vec{x},-T/2) = \vec{A}_1(\vec{x})}} \delta A_\mu(x)\, e^{-S}\, \delta({}^{U(g)}A_0)\ , \tag{8.1}$$

where we have used the identity

$$1 = \Delta \int \prod_{-T/2 \le t \le T/2} Dg(\vec{x},t)\, \delta({}^{U(g)}A_0)\ ; \tag{8.2}$$

g(x) is a generic element of the local gauge group G, and U(g) an N x N

*In this Section, three-vectors are frequently denoted by an arrow: $\vec{E} = (E_i)$, $i = 1,2,3$.

unitary matrix, for SU(N). Δ is the familiar Faddeev-Popov factor and Dg the invariant Haar measure over the group of all gauge transformations. This measure is an infinite product of invariant measures, taken at each time $t \in [-T/2, T/2]$ and at each point in space. Changing variables,

$$A'_\mu = {}^{U(g)} A_\mu \ , \qquad (8.3)$$

we can employ the delta-function in (8.1) to integrate over A'_0. Notice, however, that this change affects the boundaries of the functional integral, namely $\vec{A}_1$ and $\vec{A}_2$. Since $\delta(A_0 = 0)$ is invariant under time-independent gauge transformations and since Δ is a field-independent (infinite) constant and may be dropped, we obtain from (8.1):

$$K(\vec{A}_2, T/2; \vec{A}_1, -T/2) = K(\vec{A}_2, \vec{A}_1; T) \qquad (8.4)$$

$$= \int_{G_0} Dg(\vec{x})\, \tilde{K}\,({}^{U(g)} \vec{A}_2, \vec{A}_1; T) \ , \qquad (8.5)$$

$$\tilde{K}(\vec{A}_2, \vec{A}_1; T) = \int_{\substack{\vec{A}(\vec{x},-T/2) = \vec{A}_1(\vec{x}) \\ \vec{A}(\vec{x},T/2) = \vec{A}_2(\vec{x})}} \delta \vec{A}(x)\, e^{-S(A_0 = 0)} \ , \qquad (8.6)$$

$$S(A_0 = 0) = \int_{-T/2}^{T/2} dt \int d\vec{x}\ L(A_0 = 0) \ , \qquad (8.7)$$

$$L(A_0 = 0) = \frac{1}{2} \dot{A}^a_i \dot{A}^a_i + \frac{1}{4} F^a_{ij} F^a_{ij} \ , \qquad (8.8)$$

$$\dot{A}^a_i = \partial A^a_i / \partial t \ , \quad a = 1,2,\dots,N^2-1; \quad i,j = 1,2,3 \ , \qquad (8.9)$$

$$F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + g f^{abc} A^b_i A^c_j \ . \qquad (8.10)$$

In (8.5), G_0 is the group of time-independent gauge transformations which

tend to the unit operator as $|\vec{x}| \to \infty$. To motivate the gauge integration in eqs. (8.5) - (8.6) we recall that $\tilde{K}$ is just the matrix element of the (Euclidean) operator e^{-HT} in the coordinate representation, namely in the representation in which the field variables are diagonal at time $t = t_0$:

$$\tilde{K}(\vec{A}_2, \vec{A}_1; T) = \langle \vec{A}_2 | e^{-HT} | \vec{A}_1 \rangle \ , \qquad (8.11)$$

$$\vec{A}(\vec{x}, t_0) | \vec{A} \rangle = \vec{A}(\vec{x}) | \vec{A} \rangle \ . \qquad (8.12)$$

The gauge integration in eq. (8.5) effectively leaves in K only those eigenstates of H which are invariant under G_0 (i.e. the physical states). These states are annihilated by the Gauss operator which, as is well known, is the generator of the time-independent gauge transformations. Finally, we note that the conjugate momentum in this representation is given by

$$\vec{\Pi}(\vec{x}, t_0) = \dot{\vec{A}}(\vec{x}, t_0) \to \frac{1}{i} \frac{\delta}{\delta \vec{A}(\vec{x})} \ , \qquad (8.13)$$

so that the Hamiltonian reads

$$H = \int d\vec{x} \left[-\frac{1}{2} \frac{\delta^2}{\delta \vec{A}^a(\vec{x})\, \delta \vec{A}^a(\vec{x})} + \frac{1}{4} F^a_{ij}(\vec{x})\, F^a_{ij}(\vec{x}) \right] \ . \qquad (8.14)$$

Further details, especially on the implementation of Gauss' law, can be found in the cited literature.

C. Canonical approach

1. The Abelian case

We begin our review of canonical quantization with a discussion of the Abelian case in Minkowski space. Consider the classical Lagrangian density

$$L_{EM}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \ \mu,\nu = 0,1,2,3, \tag{8.15}$$

$$L_{EM}(x) = \frac{1}{2} (\vec{B}\cdot\vec{B} - \vec{E}\cdot\vec{E}) \ ,$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \ , \ \vec{E} = -\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}A_0 \ , \ x^0 = t = \text{time},$$

where $A_\mu(x)$ is the four-vector potential and $F_{\mu\nu}(x)$ the field strength. For the Hamiltonian formulation it is essential to identify the canonical coordinates and canonical momenta. Choosing A_μ to be the field variables and $F_{\mu 0}$ the corresponding canonical momenta, we observe that not all the components of A_μ can be independent, since L_{EM} in (8.15) does not contain F_{00}. Hence there is no momentum which is conjugate to A_0. Accordingly one defines $A_i(x)$, $i = 1,2,3$, as the independent canonical <u>coordinates</u> and F_{i0} as the corresponding conjugate <u>momenta</u>, F_{i0} being the electric field:

$$F_{i0} = E_i = -\partial_0 A_i \quad , \quad \partial_0 = \partial/\partial t. \tag{8.16}$$

Since Maxwell's theory is gauge invariant (Weyl, 1931; Heisenberg and Pauli, 1930; Feynman, 1977), we may set $A_0(x)$ equal to zero, leading to Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = J_0(x) \ , \tag{8.17}$$

$$\partial_0 E_j = \partial_i F_{ij} \ , \ \partial_i = \partial/\partial x^i \ , \tag{8.18}$$

where J_μ is a conserved current. (The current component in (8.17) has been added "by hand" for later convenience.) The total Hamiltonian reads

$$H = \frac{1}{2} \int d^3x \left[\vec{E}\cdot\vec{E} + (\vec{\nabla} \times \vec{A})^2 \right] , \tag{8.19}$$

and the theory is quantized by imposing the equal-time commutation relations (Itzykson and Zuber, 1980),

$$[A_i(x), E_j(y)]_{x^0 = y^0} = i\,\delta_{ij}\,\delta^3(\vec{x} - \vec{y}) \ , \tag{8.20a}$$

$$[A_i(x), A_j(y)]_{x^0 = y^0} = 0 \ , \tag{8.20b}$$

$$[E_i(x), E_j(y)]_{x^0 = y^0} = 0 \ . \tag{8.20c}$$

We observe that (8.17) appears to be <u>inconsistent</u> with (8.20a) and that it is <u>not</u> a dynamical equation, but rather a constraint equation, known as <u>Gauss' law</u> (Willemsen, 1978; Jackiw, 1980). Implementation of Gauss' law (8.17), and of the temporal gauge constraint $A_0(x) = 0$, eliminates all unphysical degrees of freedom from the theory.

We shall now take a closer look at the role played by Gauss' law operator G,

$$G(x) = \vec{\nabla}\cdot\vec{E} - J_0(x) \ , \tag{8.21}$$

in removing the unphysical modes from a gauge-invariant theory such as QED. What is crucial here is to note that imposition of the constraint $A_0(x) = 0$ removes <u>some</u> degrees of freedom, but by no means all. The question is where do the remaining, i.e. residual, degrees of freedom come from and how are they to be eliminated? (In quantum mechanics, the residual degrees correspond to the center-of-mass degrees of freedom (Bialynicki-Birula and Kurzepa, 1984).) As emphasized in Section B, residual gauge invariance is due to local, time-independent gauge transformations which are generated precisely by Gauss' law operator G(x), eq. (8.21). Since the Hamiltonian H is independent of these residual gauge degrees of freedom, it must commute with G,

$$[H,G] = 0 , \tag{8.22}$$

so that G is, in fact, a constant of the motion. In order to remove the inconsistency between eqs. (8.17) and (8.20a), it is customary to define the Hamiltonian system by eqs. (8.15), (8.19), (8.20a), and <u>subject to the condition</u> that the physical states of the theory obey (Partovi, 1984; Willemsen, 1978)

$$G(x)|P\rangle = 0 , \tag{8.23}$$

where |P> are physical states. The problem of consistency between eq.(8.20a) and eq.(8.23) has been the subject of some debate, both in the Abelian and non-Abelian case (Kakudo <u>et al</u>., 1983; Partovi, 1984; Hatfield, 1984; Rossi and Testa, 1984b). The difficulty can be resolved most readily in the formalism of Rossi and Testa (1984b), discussed in Section VIII.B. In the simple case of the Maxwell field, the <u>physical</u> states are just the transverse fields, while the longitudinal field components are non-dynamical and must be eliminated. With this in mind, one first decomposes $\vec{A}$ and $\vec{E}$ into transverse (T) and longitudinal (L) parts,

$$\vec{A} = \vec{A}_L + \vec{A}_T , \vec{E} = \vec{E}_L + \vec{E}_T , \tag{8.24}$$

so that (8.19) becomes

$$H = \frac{1}{2} \int d^3x \left[\vec{E}_T \cdot \vec{E}_T + \vec{E}_L \cdot \vec{E}_L + (\vec{\nabla} \times \vec{A}_T)^2 \right] , \tag{8.25}$$

and then invokes Gauss' law (8.17) to extract the longitudinal component of the electric field $\vec{E}_L$ (Bjorken, 1980):

$$\vec{E}_L(x) = \vec{\nabla} (\vec{\nabla}^2)^{-1} J_0(x) , \tag{8.26}$$

$$= - \vec{\nabla}_x \int d^3y \, (4\pi|\vec{x}-\vec{y}|)^{-1} J_0 . \tag{8.26a}$$

Substitution of (8.26) into (8.25) effectively removes the nondynamical variable $\vec{E}_L$. Notice that the solution for $\vec{E}_L$ is easy here, because the theory is linear and its Hamiltonian at most quadratic in the potentials A_μ.

2. The non-Abelian case

The purpose of the ensuing discussion is to mimic the procedure of the preceding Section in the non-Abelian case, paying particular attention to the generalized version of Gauss' law operator $G^a(x)$,

$$G^a(x) = \vec{D}^{ab} \cdot \vec{E}^b(x) - J_0^a(x) , \tag{8.27}$$

$$D_j^{ab} = \delta^{ab} \partial_j + g f^{abc} A_j^c , \; j = 1,2,3, \quad a,b,c, = 1,\ldots,8,$$

J_μ^a being a conserved current. Since the construction of physical states in the temporal gauge

$$A_0^a(x) = 0 \quad , \quad a = 1,2,\ldots,8, \tag{8.28}$$

is intimately associated with the operator $G^a(x)$ (Eylon, 1978; Senjanovic, 1978; Hatfield, 1984; Rossi and Testa, 1984; Buchholz, 1986; Yamagishi, 1986), we shall briefly highlight the main steps leading to the formal elimination of the longitudinal degrees of freedom in the Hamiltonian ,eq. (8.45).

Consider the Lagrangian density

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \quad , \quad \mu,\nu = 0,1,2,3, \qquad a = 1,\ldots,8, \tag{8.29}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu ,$$

which is independent of F^a_{00}, so that A^a_0 cannot be considered a dynamical variable. In analogy with QED, the <u>independent</u> canonical coordinates are A^a_i, $i = 1,2,3$, and the corresponding conjugate momenta are F^a_{io},

$$F^a{}_{io} = E^a_i = -\partial_0 A^a_i , \tag{8.30}$$

where E^a_i is the color electric field (Feynman, 1977). In the temporal gauge (8.28), the equation of motion for E^a_i is

$$\partial_0 E^a_i = D^{ab}_j F^b_{ji} , \tag{8.31}$$

Gauss' law constraint reads (cf. eq. (8.27))

$$D^{ab}_j(A)\, E^b_j(x) = J^a_0(x) , \tag{8.32}$$

and the canonical equal-time commutation relations are given by

$$\left[A^a_i(x),\, E^b_j(y)\right]_{x^0 = y^0} = i\, \delta_{ij}\, \delta^{ab}\, \delta^3(\vec{x}-\vec{y}) , \tag{8.33a}$$

$$\left[A^a_i(x),\, A^b_j(y)\right]_{x^0 = y^0} = 0 , \tag{8.33b}$$

$$\left[E^a_i(x),\, E^b_j(y)\right]_{x^0 = y^0} = 0 . \tag{8.33c}$$

The Hamiltonian reads

$$H = \frac{1}{2} \sum_{a=1}^{8} \int d^3x \left[(E^a_i)^2 + (B^a_i)^2\right] , \tag{8.34}$$

with the color magnetic field $\vec{B}^a$ defined by (Feynman, 1977; Huang, 1982)

$$\vec{B}^a = \vec{\nabla} \times \vec{A}^a + \frac{1}{2} g f^{abc} \vec{A}^b \times \vec{A}^c . \tag{8.35}$$

As in Maxwell's case, the equal-time commutation relation (8.33a) is <u>inconsistent</u> with the constraint (8.32). To remedy the situation and, at the same time, incorporate Gauss' law into the Hamiltonian structure, we demand that only those states of the full Hilbert space be acceptable which satisfy the subsidiary condition

$$G^a(x)\, |P\rangle = 0 , \tag{8.36}$$

where $|P\rangle$ are physical states (Bjorken, 1980; Jackiw, 1980). The residual gauge invariance of the theory may again be attributed to Gauss' law operator

$$G^a(x) = D^{ab}_j(A)\, E^b_j - J^a_0(x),$$

generating local, time-independent gauge transformations. Since the Hamiltonian (8.34) does not depend on these residual degrees of freedom, it must commute with $G^a(x)$:

$$[H, G^a] = 0 . \tag{8.37}$$

A more challenging task is to render the longitudinal components of the vector potential and electric field ineffective. Following Bjorken's clear analysis (1980), we split $\vec{A}^a$ and $\vec{E}^a$ into transverse and longitudinal parts,

$$\vec{A}^a = \vec{A}^a_T + \vec{A}^a_L , \tag{8.38}$$

$$\vec{E}^a = \vec{E}^a_T + \vec{E}^a_L , \tag{8.39}$$

define $\vec{E}^a_L$ by

$$\vec{E}^a_L = \vec{\nabla} \phi^a , \tag{8.40}$$

and then exploit Gauss' law (8.32) to solve for the variable $\phi^a(x)$. (See Appendix A of Bjorken (1980).) Substitution of (8.39) and (8.40) into (8.32) gives

$$\vec{\nabla} \cdot \vec{D}^{ab} \phi^b = g f^{abc} \vec{A}^b_T \cdot \vec{E}^c_T + J^a_0 , \tag{8.41}$$

leading to the formal solution

$$\phi^a(x) = \int d^3y \, K^{ab}(x,y;A) \left[g f^{bcd} \vec{A}^c_T(y) \cdot \vec{E}^d_T(y) + J^b_0(y) \right] ; \tag{8.42}$$

the kernel K (Bjorken, 1980),

$$K = [\vec{\nabla}\cdot\vec{D}(A)]^{-1} , \tag{8.43}$$

satisfies

$$(\vec{\nabla}\cdot\vec{D})_{ab} \quad K(x,y;A)_{bc} = \delta^3 (\vec{x}-\vec{y}) , \tag{8.44a}$$

$$= \vec{\nabla}^2 K(x,y;A)_{ac} - g f_{abd} \vec{A}^d_T \cdot \vec{\nabla} K_{bc} . \tag{8.44b}$$

Hence one may formally solve for ϕ^a, compute $\vec{\nabla} \phi^a = \vec{E}^a_L$, and then rewrite the Hamiltonian (8.34) in terms of the variables $\vec{E}^a_T$ and $\vec{A}^a_T$ only:

$$H = \frac{1}{2} \sum_{a=1}^{8} \int d^3x \left[\vec{E}^a_T\cdot\vec{E}^a_T + (\vec{\nabla} \times \vec{A}^a_T)^2 + (\vec{\nabla}\phi^a)^2 \right] . \tag{8.45}$$

Concerning the elimination of $\vec{E}^a_L$, the major difference between QCD and

QED lies in the appearance of the operator $(\vec{\nabla}\cdot\vec{D})^{-1}$ in eq. (8.41), in place of the operator $(\vec{\nabla}^2)^{-1}$ in eq (8.26), since $\vec{\nabla}\cdot\vec{D}$ is now a function of the vector potential A . For weak coupling, the dependence on A in the second term of (8.44b) is small compared with $\vec{\nabla}^2 K_{ac}$, and the situation is similar to QED. For large values of $\vec{A}^a_T$, on the other hand, the explicit solution for ϕ^a is much harder to attain. We shall not pursue this topic further here, but refer the curious reader to the following papers: Gribov (1977, 1978), Jackiw (1978, 1980), Mandelstam (1977), Singer (1978) and Bjorken (1980, Appendix A).

D. Pragmatic approaches

In pure Yang-Mills theory the bare gauge field propagator in the temporal gauge (8.28) is given by (Burnel, 1982)

$$G^{ab}_{\mu\nu}(q) = \frac{-i\,\delta^{ab}}{(2\pi)^{2\omega}(q^2+i\epsilon)} \left[g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q\cdot n} + \frac{n^2 q_\mu q_\nu}{(q\cdot n)^2} \right] ,$$

$$n^2 > 0, \ \epsilon > 0, \tag{8.46}$$

where the last term in (8.46) reflects the residual gauge invariance of the theory. The crucial question again is how to interpret the unphysical singularities arising from $(q\cdot n)^{-\alpha}$, $\alpha = 1,2$. Since the propagator (8.46) has the same structure as in the homogeneous axial gauge one might be inclined to think that the principal-value prescription would also give reasonable results in the temporal gauge. But recent calculations do not seem to support this view.

As noted in Section A, the residual gauge invariance manifests itself as an unphysical pole of $(q\cdot n)^{-2}$ in the <u>longitudinal</u> part of the gluon propagator. Application of the principal-value prescription, for instance, leads to the longitudinal propagator (Lim, 1984)

$$G_{ij}^{abL}(\vec{x}_2,t_2;\vec{x}_1,t_1) = -\frac{i\,\delta^{ab}}{2}|t_2-t_1|\int\frac{d\vec{k}}{(2\pi)^3}\frac{k_ik_j}{\vec{k}^2}e^{i\vec{k}\cdot(\vec{x}_2-\vec{x}_1)}, \quad (8.47)$$

while the prescription of Caracciolo, Curci and Menotti (1982) gives the form (Yamagishi, 1986)

$$G_{ij}^{abL}(\vec{x}_2,t_2;\vec{x}_1,t_1) = \frac{-i\,\delta^{ab}}{2}\left\{|t_2-t_1| \mp (t_2+t_1)+\beta\right\}\int\frac{d\vec{k}}{(2\pi)^3}\frac{k_ik_j}{\vec{k}^2}e^{i\vec{k}\cdot(\vec{x}_2-\vec{x}_1)}, \quad (8.48)$$

β being a constant. The form of the longitudinal gluon propagator has also been scrutinized by Frenkel (1979), Müller and Rühl (1981), Dahmen et al. (1982), and Girotti and Rothe (1986). Since this form depends on the choice of regularization and can be tested by computing the Wilson loop, for example, we have some control over the type of prescription to be chosen for $(q\cdot n)^{-2}$. A good case in point is the calculation of Caracciolo, Curci and Menotti (1982). (See also the article by Landshoff, 1986b.) They found that use of the principal-value prescription at the one-loop level did not lead, as anticipated, to exponentiation of the time dependence of the Wilson-loop operator.

More recently, Landshoff (1986a) has proposed a different prescription which he calls "α-prescription". It consists of replacing (8.46) by the propagator

$$G_{\mu\nu}^{ab}(q) = \frac{i\,\delta^{ab}}{(2\pi)^{2\omega}(q^2+i\epsilon)}\left[g_{\mu\nu} - \frac{(q_\mu n_\nu + q_\nu n_\mu)q\cdot n}{(q\cdot n)^2+\alpha(n^2)^2} + \frac{n^2q_\mu q_\nu - \alpha^2n^2n_\mu n_\nu}{(q\cdot n)^2+\alpha(n^2)^2}\right], \quad \epsilon > 0, \quad (8.49)$$

of performing all calculations with $\alpha \neq 0$, and letting $\alpha \to 0$ only at the end. Note that the structure of (8.49) does not imply a principal-value prescription for $(q\cdot n)^{-1}$, a situation reminiscent of the light-cone gauge (Mandelstam, 1983; Leibbrandt, 1984a).

The propagator (8.49) possesses a number of pleasant features (e.g. translation invariance, also $n^\mu G_{\mu\nu}^{ab}(q) = 0$) that prompted Steiner (1986) to search for an "explanation" of Landshoff's scheme. He suggested that Landshoff's α-prescription may be derived from an improved temporal gauge of the form

$$A_0^a(x) = \alpha\,\phi^a(x), \quad (8.50)$$

where ϕ^a is a functional of $A_i^a(x)$ and the parameter α in (8.50) is the same as Landshoff's α in (8.49). Steiner's soft temporal gauge is then defined by the limit $\alpha \to 0$ (Chan and Halpern, 1985). The ansatz (8.50) gives the bare propagator:

$$G_{\mu\nu}^{ab}(q) = \frac{-i\,\delta^{ab}}{(2\pi)^{2\omega}}\left[G_{\mu\nu}^{(1)}(q) + \alpha^2 G_{\mu\nu}^{(2)}(q) + i\alpha\, G_{\mu\nu}^{(3)}(q) + \frac{1}{\lambda}\frac{q_\mu q_\nu}{(q\cdot n)^2+\alpha^2}\right], \quad \lambda \to \infty, \quad (8.51)$$

where λ is a gauge parameter, and the components $G_{\mu\nu}^{(i)}(q)$, $i = 1,2,3$, can be found in Steiner (1986). Debate on this topic continues.

Concerning the pragmatic approach, the present situation may, therefore, be summarized as follows:

(1) Momentum-space prescriptions, containing $G_{00}^{ab}(q) \neq 0$, have been derived by Steiner (1986) and Cheng and Tsai (1986), but are of little practical value.

(2) Several authors (Caracciolo et al., 1982; Leroy et al., 1984; Girotti and Rothe, 1986) have considered the addition of a non-translation invariant part to the propagator in t-space, such as $(t_1 + t_2)$ in eq. (8.48). This prescription for the temporal gauge has been obtained in different ways, but is again only of limited practical use.

(3) Landshoff's α-prescription (Landshoff, 1986a) has several advantages and is straight forward, but has not been proved.

(4) Steiner's "proof" of Landshoff's α-prescription (Steiner, 1986) remains to be completed.

(5) The difficulties in Minkowski space and Euclidean space should be tractable by the same prescription.

(6) Since the problems in the temporal gauge $A_0 = 0$ ($n^2 > 0$) are related to those in the axial gauge $A_3 = 0$($n^2<0$), it would be helpful to have mechanism which interpolates between these two gauges.

E. Conclusion

As is evident from the general discussion, the temporal gauge is suitable in selected circumstances, but it is certainly not an easy gauge to work with. Apart from the formal difficulties encountered in the strong-coupling limit, nagging problems persist in the computation of one-loop momentum integrals. What is missing here is a simple, unambiguous prescription for the unphysical singularities of $(q \cdot n)^{-\alpha}$, $\alpha = 1,2$, a prescription that obeys power counting, that is equally applicable in Euclidean and Minkowski space, and also satisfies other requirements such as locality.

IX. Related Topics

A. Higher-loop integrals

The feasibility of performing higher-loop calculations in the light-cone gauge has been demonstrated by several groups. Leibbrandt and Nyeo (1986a) have evaluated various Feynman integrals arising in the two-loop Yang-Mills self-energy, while Capper, Jones and Suzuki (1985) have computed the scalar anomalous dimension in a general gauge theory. Working in the context of supersymmetry to two-loop order, Capper and his co-workers concluded that the light-cone gauge is manifestly supersymmetric and free of auxiliary fields. Smith, on the other hand, was the first to tackle the two-loop beta function in N=2 Yang-Mills theory and to compute the counterterm for the four-point function to two-loop order (Smith 1985a, 1985b, 1986). The consensus at this stage is that two-loop integrals can indeed be calculated consistently and unambiguously, but that some of the integrals - for instance, those with overlapping divergences - are certainly more complicated than in the axial gauge. The increased complexity may be attributed to the vector n_μ^*, as will be illustrated now.

Consider the two-loop Yang-Mills self-energy in Fig. (9.1) which gives rise to the double integral (Leibbrandt and Nyeo, 1986a)

$$I_\mu(p) = \iint \frac{dq\, dk\, q_\mu}{q^2k^2(q-p+k)^2 q\cdot n} \quad , \quad n^2 = 0, \quad d^{2\omega}k = dk, \; d^{2\omega}q = dq. \qquad (9.1)$$

Application of the light-cone gauge prescription (6.11) and integration over k_μ, with

$$\int \frac{dk}{k^2[k - (p-q)]^2} = \frac{i(-\pi)^\omega\, \Gamma(2-\omega)[\Gamma(\omega-1)]^2[(p-q)^2]^{\omega-2}}{\Gamma(2\omega-2)} ,$$

leads to the intermediate expression

$$I_\mu(p) = \frac{i(-\pi)^\omega \Gamma(2-\omega)[\Gamma(\omega-1)]^2}{\Gamma(2\omega-2)} \int \frac{dq\, q_\mu}{q^2[(p-q)^2]^{2-\omega} q\cdot n} . \tag{9.2}$$

The remaining q_μ-integration gives (see Appendix C.3)

$$\int \frac{dq\, q_\mu}{q^2[(q-p)^2]^{2-\omega}\, q\cdot n} = -\frac{2i(-\pi)^\omega \Gamma(4-2\omega)\, p\cdot n^*}{(n\cdot n^*)^2\, \Gamma(2-\omega)} \int_0^1\int_0^1 dx\, dy\, y^{\omega-2} H^{2\omega-4},$$

$$\left(\frac{n\cdot n^* H}{2(2\omega-3)\, p\cdot n^*} n_\mu^* - y\, n\cdot n^* p_\mu + xy\, (p\cdot n\, n_\mu^* + p\cdot n^*\, n_\mu)\right) , \tag{9.3}$$

and $H = (1-y)\, p^2 + 2xy\, p\cdot n\, p\cdot n^*/n\cdot n^*$. Substituting (9.3) into (9.2) and noting that

$$\lim_{\omega\to 2^+} (\Gamma(4-2\omega)/\Gamma(2-\omega)) = 1/2,$$

we obtain for the divergent part of I_μ,

$$I_\mu(p) = \frac{(-\pi)^{2\omega}\, \Gamma(2-\omega)}{2\, n\cdot n^*} \left(p^2 n_\mu^* - 2p\cdot n^* p_\mu + \frac{2p\cdot n\, p\cdot n^*}{n\cdot n^*} n_\mu^* + \frac{(p\cdot n^*)^2}{n\cdot n^*} n_\mu \right) , \quad \omega \to 2^+ , \tag{9.4}$$

which is seen to possess only a simple pole.

There are other two-loop integrals in the self-energy, Fig. (9.1), such as

$$\int\int \frac{dq\, dk\, q\cdot k}{q^2\, k^2\, (k-p+q)^2\, q\cdot n\, k\cdot n} , \tag{9.5}$$

which give rise to both single and double poles. The integral (9.5) is particularly challenging since it contains an overlapping divergence (Leibbrandt and Nyeo, 1986a). Other two-loop integrals and clever schemes of

evaluation can be found in Capper et al. (1985) and Smith (1985a, 1985b, 1986).

B. Stochastic quantization

Noncovariant gauges also play a significant role in the area of stochastic quantization (Parisi and Wu, 1981). We shall, therefore, review some of the basic features of the stochastic approach which is based on the celebrated Langevin equation of non-equilibrium statistical mechanics:

$$\frac{\partial\phi}{\partial\tau}(x,\tau) = -\frac{\delta S}{\delta\phi(x,\tau)} + \eta(x,\tau) , \tag{9.6}$$

where S denotes the action of the field theory under study in (d+1)-dimensional Euclidean space (we may, for example, consider a real, self-interacting scalar field ϕ), and where τ is an extra dimension usually called the "fictitious" time. The system evolves with respect to τ, reaching an equilibrium distribution for $\tau \to +\infty$. The random variable $\eta(x,\tau)$ in (9.6) is a Gaussian "white" noise with correlations

$$\langle\eta(x,\tau)\rangle_\eta = 0 ,$$

$$\langle\eta(x,\tau)\eta(x',\tau')\rangle_\eta = 2\, \delta^d(x-x')\, \delta(\tau-\tau') . \tag{9.7}$$

The correlations are defined by performing averages over the noise η with Gaussian distribution. Let us suppose that eq. (9.6) can be solved for some initial conditions and denote the solution by $\phi_\eta(x,\tau)$, indicating explicitly the dependence on η. Correlation functions over ϕ_η are then defined, as in (9.7), by performing Gaussian averages over η. The basic claim in stochastic quantization (Parisi and Wu, 1981; Floratos and Iliopoulos, 1983; Grimus and Hüffel, 1983) is that as the ficitious time $\tau \to +\infty$, the stochastic averages approach quantum Green functions, namely

$$\lim_{\tau \to +\infty} < \phi_\eta(x_1,\tau)\ldots\phi_\eta(x_n,\tau)>_\eta = < \phi(x_1)\ldots\phi(x_n)> . \qquad (9.8)$$

The stochastic formalism is particularly relevant for gauge-invariant theories, since neither ghost particles nor gauge-fixing terms are required (Parisi and Wu, 1981; Namiki et al., 1983; see also Zwanziger, 1981). The absence of ghost fields suggests a possible link between stochastic quantization and quantizaiton in a noncovariant gauge. Such a link has recently been discussed by Hüffel and Landshoff (1985) who showed that it is possible to formulate a stochastic perturbation theory which reproduces conventional theory in the axial gauge (see also Landshoff (1986b) and Chan and Halpern (1986)).

But there is another noncovariant gauge which is even more popular than the axial gauge. This is the light-cone gauge of Sections VI and VII which has proven remarkably effective in studying the relationship betwen supersymmetry and stochastic quantization (de Alfaro et al., 1984; Amati and Veneziano, 1985; Floreanini, 1985; Floreanini et al., 1985). The origin of this intimate relationship between supersymmetry and stochastic processes (Parisi and Sourlas, 1979, 1983; Cecotti and Girardello, 1983) may be traced back to the existence of Nicolai maps (Nicolai, 1980, 1982). The proof, for example, that N = 1 supersymmetric Yang-Mills theory is a four-dimensional field theory with a local Nicolai map (de Alfaro et al., 1984) has to date only been possible in the light-cone gauge (Amati and Veneziano, 1985). The importance of the light-cone gauge is also highlighted in the construction of stochastic identities for supersymmetric Yang-Mills theories (de Alfaro et al., 1985, 1984; de Alfaro, Fubini and Furlan, 1985; Lechtenfeld, 1986).

We are fully aware that our microscopic review of stochastic

quantization does not do justice to this fascinating, provocative topic, but hope that the interested reader will find an opportunity to consult the original literature and a forthcoming review by Damgaard and Hüffel (1987).

X. Concluding Remarks

In this review we have concentrated on four prominent noncovariant gauges: the axial gauge, the planar gauge, the light-cone gauge and the temporal gauge. Our aim has been to acquaint the reader not only with the basic properties of these ghost-free gauges, but also with their advantages and deficiencies, their computational idiosyncracies and different ranges of applicability. As seen from the discussion in the main text, the usefulness of a particular gauge depends ultimately on its effectiveness in eliminating the unwanted gauge degrees of freedom, and on the availability of a reliable prescription for $(q \cdot n)^{-1}$. In this context, the axial gauge and the planar gauge are in good shape, both from a theoretical and technical point of view. The standard prescription for $(q \cdot n)^{-1}$ for these two gauges is the principal-value prescription which provides internally consistent integrals at the one-loop level and leads to satisfactory answers in most, thought not all, practical calculations.

The related, but computationally superior, light-cone gauge is endowed with unusual characteristics, including an unorthodox prescription for $(q \cdot n)^{-1}$. The new prescription, which is <u>not</u> of principal-value form, satisfies locality and naive power counting, and permits an unambiguous evaluation of one- and two-loop integrals. A novel feature of this prescription is the apppearance of <u>nonlocal</u> expressions in the gluon self-energy and three-gluon vertex which require the introduction of nonlocal BRS-invariant counterterms. As a result of these counterterms, and despite progress in this area during the last two years, there remain several unresolved questions about the renormalization structure of Yang-Mills theory in the light-cone gauge.

Further effort and fresh ideas are also needed in order to place the tricky temporal gauge on a par with the other noncovariant gauges. The key problem is that the temporal gauge choice is not sufficiently powerful to eliminate all degrees of freedom. There remains in the theory a residual gauge symmetry which is due to Gauss' law operator generating local, time-independent gauge transformations. While canonical quantization in the temporal gauge is satisfactory for Abelian models, it is problematic in non-Abelian theories, especially in the strong-coupling limit. Uncertainties also prevail in the covariant path-integral formalism, where absence of a reliable prescription for $(q \cdot n)^{-1}$ tends to undermine user confidence. However, given the tenacity and eternal optimism of theorists, it seems only a matter of time before the temporal gauge will be placed on a firm mathematical foundation.

Today's preoccupation with gauges is neither new nor surprising. What is novel perhaps is the guarded enthusiasm with which the search for and study of suitable gauges is being conducted, an enthusiasm that will likely persist as long as there is a demand for non-Abelian models with gauge symmetry. We hope that this article will encourage judicious application, and provide some insight into the character and potential usefulness, of noncovariant gauges.

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Appendix A

Axial Gauge Integrals

We list, in connection with Section IV.B, the <u>divergent parts</u> of some massless one-loop integrals in the axial/planar gauge. Here $n^2 \neq 0$, $d^{2\omega}q = dq$ and $\bar{I}$ is defined by (cf. eq.(4.23))

$$\bar{I} = \text{divergent part of } \int dq[q^2(q-p)^2]^{-1} ,$$

$$= \begin{cases} \pi^2/(2-\omega) : \text{Euclidean space,} \\ i\pi^2/(2-\omega) : \text{Minkowski space.} \end{cases}$$

The integrals listed below have been collected from Capper and Leibbrandt (1982b) and Leibbrandt (1983a).

$$\int \frac{dq}{(q-p)^2 q\cdot n} = \frac{2p\cdot n}{n^2} \bar{I} ;$$

$$\int \frac{dq\, q_\mu}{(q-p)^2 q\cdot n} = \frac{2p\cdot n}{n^2} (p_\mu - n_\mu \frac{p\cdot n}{n^2}) \bar{I} ;$$

$$\int \frac{dq\, q_\mu q_\nu}{(q-p)^2 q\cdot n} = \frac{-2p\cdot n\, p^2}{3n^2} \left[\frac{(p\cdot n)^2}{p^2 n^2} \delta_{\mu\nu} - \frac{3}{p^2} p_\mu p_\nu - \frac{4(p\cdot n)^2}{p^2 n^4} n_\mu n_\nu + \frac{3p\cdot n}{p^2 n^2} (p_\mu n_\nu + p_\nu n_\mu) \right] \bar{I} ;$$

$$\int \frac{dq\, q^2}{(q-p)^2 q\cdot n} = \frac{-2p\cdot n\, p^2}{3n^2} \left[\frac{2(\omega+1)(p\cdot n)^2}{p^2 n^2} - 3 \right]_{\omega=2} \bar{I} ,$$

$$= \frac{2p\cdot n\, p^2}{n^2} (1 - \frac{2(p\cdot n)^2}{p^2 n^2}) \bar{I} ,$$

$$\int \frac{dq}{(q-p)^2 (q\cdot n)^2} = \frac{-2}{n^2} \bar{I} ;$$

$$\int \frac{dq\; q_\mu}{(q-p)^2(q\cdot n)^2} = \frac{-2}{n^2}\left(p_\mu - n_\mu \frac{2p\cdot n}{n^2}\right)\bar{I} \;;$$

$$\int \frac{dq\; q_\mu q_\nu}{(q-p)^2(q\cdot n)^2} = \frac{2(p\cdot n)^2}{n^4}\Big[\delta_{\mu\nu} - \frac{n^2}{(p\cdot n)^2}\, p_\mu p_\nu + \frac{2}{p\cdot n}(p_\mu n_\nu + p_\nu n_\mu) - \frac{4}{n^2} n_\mu n_\nu\Big]\bar{I} \;;$$

$$\int \frac{dq\; q^2}{(q-p)^2(q\cdot n)^2} = \frac{2p^2}{n^2}\left[\frac{2\omega(p\cdot n)^2}{p^2 n^2} - 1\right]_{\omega=2}\bar{I} \;,$$

$$= \frac{2p^2}{n^2}\left(\frac{4(p\cdot n)^2}{p^2n^2} - 1\right)\bar{I} \;;$$

$$\int \frac{dq}{q^2(q-p)^2 q\cdot n} = \text{finite;}$$

$$\int \frac{dq\; q_\mu}{q^2(q-p)^2 q\cdot n} = \frac{1}{n^2} n_\mu \bar{I} \;;$$

$$\int \frac{dq\; q_\mu q_\nu}{q^2(q-p)^2 q\cdot n} = \frac{p\cdot n}{2n^2}\Big[\delta_{\mu\nu} + \frac{1}{p\cdot n}(p_\mu n_\nu + p_\nu n_\mu) - \frac{2}{n^2} n_\mu n_\nu\Big]\bar{I} \;;$$

$$\int \frac{dq}{(q-p)^2(q-k)^2 q\cdot n} = \text{finite} \;;$$

$$\int \frac{dq\; q_\mu}{(q-p)^2(q-k)^2 q\cdot n} = \frac{1}{n^2} n_\mu \bar{I} \;;$$

$$\int \frac{dq\; q_\mu q_\nu}{(q-p)^2(q-k)^2 q\cdot n} = \frac{p\cdot n}{2n^2}\Big[\delta_{\mu\nu} + \frac{1}{p\cdot n}(p_\mu n_\nu + p_\nu n_\mu) - \frac{2}{n^2} n_\mu n_\nu\Big]\bar{I}$$

$$+ \frac{k\cdot n}{2n^2}\Big[\delta_{\mu\nu} + \frac{1}{k\cdot n}(k_\mu n_\nu + k_\nu n_\mu) - \frac{2}{n^2} n_\mu n_\nu\Big]\bar{I} \;;$$

$$\int \frac{dq\; q^2}{(q-p)^2(q-k)^2 q\cdot n} = \left[\frac{\omega(p+k)\cdot n}{n^2}\right]_{\omega=2}\bar{I} \;,$$

$$= \frac{2(p+k)\cdot n}{n^2}\bar{I}.$$

Appendix B

The Tensors $T^i_{\mu\nu,\rho\sigma}$

We list the fourteen independent tensors (Matsuki, 1979) which appear in the text in connection with the graviton propagator, eq.(4.38a), and the nontransverse component of the graviton self-energy, eq.(4.40b). The tensors $T^i_{\nu\mu,\rho\sigma}$, $i = 1,\ldots,14$, are formed from n_μ, p_μ, $\delta_{\mu\nu}$ and satisfy $T^i_{\mu\nu,\rho\sigma} = T^i_{\nu\mu,\rho\sigma} = T^i_{\mu\nu,\sigma\rho} = T^i_{\rho\sigma,\mu\nu}$ (Capper and Leibbrandt, 1982b).

$$T^1_{\mu\nu,\rho\sigma} = 2^{-1}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}) \;,$$

$$T^2_{\mu\nu,\rho\sigma} = \delta_{\mu\nu}\delta_{\rho\sigma} \;,$$

$$T^3_{\mu\nu,\rho\sigma} = (p^2)^{-1}(\delta_{\mu\nu}p_\rho p_\sigma + \delta_{\rho\sigma}p_\mu p_\nu) \;,$$

$$T^4_{\mu\nu,\rho\sigma} = (2p\cdot n)^{-1}(\delta_{\mu\nu}p_\rho n_\sigma + \delta_{\mu\nu}p_\sigma n_\rho + \delta_{\rho\sigma}p_\mu n_\nu + \delta_{\rho\sigma}p_\nu n_\mu) \;,$$

$$T^5_{\mu\nu,\rho\sigma} = (n^2)^{-1}(\delta_{\mu\nu}n_\rho n_\sigma + \delta_{\rho\sigma}n_\mu n_\nu) \;,$$

$$T^6_{\mu\nu,\rho\sigma} = (p^2)^{-1}(\delta_{\mu\rho}p_\nu p_\sigma + \delta_{\mu\sigma}p_\nu p_\rho + \delta_{\nu\rho}p_\mu p_\sigma + \delta_{\nu\sigma}p_\mu p_\rho) \;,$$

$$T^7_{\mu\nu,\rho\sigma} = (2p\cdot n)^{-1}[(\delta_{\mu\rho}p_\nu + \delta_{\nu\rho}p_\mu)n_\sigma + (\delta_{\mu\sigma}p_\nu + \delta_{\nu\sigma}p_\mu)n_\rho$$

$$+ (\delta_{\mu\rho}n_\nu + \delta_{\nu\rho}n_\mu)p_\sigma + (\delta_{\mu\sigma}n_\nu + \delta_{\nu\sigma}n_\mu)p_\rho] \;,$$

$$T^8_{\mu\nu,\rho\sigma} = (n^2)^{-1}(\delta_{\mu\rho}n_\nu n_\sigma + \delta_{\mu\sigma}n_\nu n_\rho + \delta_{\nu\rho}n_\mu n_\sigma + \delta_{\nu\sigma}n_\mu n_\rho) \;,$$

$$T^9_{\mu\nu,\rho\sigma} = (p^2)^{-2}\, p_\mu p_\nu p_\rho p_\sigma \;,$$

$$T^{10}_{\mu\nu,\rho\sigma} = (4p^2 p\cdot n)^{-1}(n_\mu p_\nu p_\rho p_\sigma + n_\nu p_\mu p_\rho p_\sigma + n_\rho p_\mu p_\nu p_\sigma + n_\sigma p_\mu p_\nu p_\rho) \;,$$

$$T^{11}_{\mu\nu,\rho\sigma} = (2p^2n^2)^{-1}(p_\mu p_\nu n_\rho n_\sigma + p_\rho p_\sigma n_\mu n_\nu) \;,$$

$$T^{12}_{\mu\nu,\rho\sigma} = [4(p\cdot n)^2]^{-1}\,(p_\mu n_\nu + p_\nu n_\mu)(p_\rho n_\sigma + p_\sigma n_\rho) \;,$$

$$T^{13}_{\mu\nu,\rho\sigma} = (4p\cdot n\, n^2)^{-1}(p_\mu n_\nu n_\rho n_\sigma + p_\nu n_\mu n_\rho n_\sigma + p_\rho n_\mu n_\sigma n_\nu + p_\sigma n_\mu n_\nu n_\rho),$$

$$T^{14}_{\mu\nu,\rho\sigma} = (n^2)^{-2}\, n_\mu n_\nu n_\rho n_\sigma \;.$$

Appendix C

Light-Cone Gauge Integrals

This appendix contains a partial list of massless and massive one-loop integrals in the light-cone gauge which are relevant for the discussion in Sections VI, VII and IX. A.

1. Gaussian integrals

(a) Gaussian integrals in one dimension:

$V_0 = Aq_4{}^2 - 2Bq_4$; $E_0 = B^2/A$; A, B are arbitrary coefficients.

$$\int_{-\infty}^{+\infty} dq_4\, e^{-V_0} = \frac{\pi^{1/2}}{A^{1/2}}\, e^{E_0} \;,$$

$$\int_{-\infty}^{+\infty} dq_4\, q_4\, e^{-V_0} = \frac{B\,\pi^{1/2}}{A^{3/2}}\, e^{E_0} \;,$$

$$\int_{-\infty}^{+\infty} dq_4\, q_4{}^2\, e^{-V_0} = \pi^{1/2}\left[\frac{1}{2A^{3/2}} + \frac{B^2}{A^{5/2}}\right] e^{E_0} \;,$$

$$\int_{-\infty}^{+\infty} dq_4\, q_4{}^3\, e^{-V_0} = B\,\pi^{1/2}\left[\frac{3}{2A^{5/2}} + \frac{B^2}{A^{7/2}}\right] e^{E_0} \;.$$

(b) Gaussian integrals in $(2\omega-1)$ dimensions:

$$V = \gamma\, \vec{q}^{\,2} - 2\beta\, \vec{q}\cdot\vec{p} + \alpha(\vec{q}\cdot\vec{n})^2 \;,$$

$$E = \frac{\beta^2\, \vec{p}^{\,2}}{\gamma} - \frac{\alpha\beta^2(\vec{p}\cdot\vec{n})^2}{\gamma A} \;, \quad A = \gamma + \alpha\vec{n}^{\,2} \;;$$

α, β, γ are arbitrary coefficients.

$$\int d^{2\omega-1}\vec{q}\; e^{-V} = \frac{\pi^{\omega-1/2}\gamma^{1-\omega}}{A^{1/2}}\, e^{E} \quad ,$$

$$\int d^{2\omega-1}\vec{q}\;\vec{q}\; e^{-V} = \frac{\pi^{\omega-1/2}\gamma^{-\omega}\beta}{A^{1/2}} \left(\vec{p} - \frac{\alpha\vec{n}\;\vec{p}\cdot\vec{n}}{A} \right) e^{E} \; ,$$

$$\int d^{2\omega-1}\vec{q}\;\vec{q}\cdot\vec{n}\; e^{-V} = \frac{\pi^{\omega-1/2}\gamma^{1-\omega}\beta\;\vec{p}\cdot\vec{n}}{A^{3/2}}\, e^{E} \quad ,$$

$$\int d^{2\omega-1}\vec{q}\;\vec{q}^{2}\; e^{-V} = \frac{\pi^{\omega-1/2}\gamma^{-\omega}}{2A^{1/2}} \left\{ 2\omega-1 - \frac{\alpha\,\vec{n}^{2}}{A} + \frac{2\beta^{2}}{\gamma}\left[\vec{p}^{2} - \frac{2\alpha(\vec{p}\cdot\vec{n})^{2}}{A} + \frac{\alpha^{2}\,\vec{n}^{2}(\vec{p}\cdot\vec{n})^{2}}{A^{2}} \right]\right\} e^{E} \quad ,$$

$$\int d^{2\omega-1}\vec{q}\;\vec{q}\;\vec{q}\cdot\vec{n}\; e^{-V} = \frac{\pi^{\omega-1/2}\gamma^{1-\omega}}{2A^{3/2}} \left[\vec{n} + \frac{2\beta^{2}\;\vec{p}\cdot\vec{n}}{\gamma} \left(\vec{p} - \frac{\alpha\,\vec{n}\;\vec{p}\cdot\vec{n}}{A} \right)\right] e^{E} \quad ,$$

$$\int d^{2\omega-1}\vec{q}\;(\vec{q}\cdot\vec{n})\;\vec{q}^{2}\; e^{-V} = \frac{\pi^{\omega-1/2}\gamma^{-\omega}\beta\;\vec{p}\cdot\vec{n}}{A^{3/2}} \left\{ \omega + \frac{1}{2} - \frac{3\alpha\,\vec{n}^{2}}{2A} + \frac{\beta^{2}}{\gamma}\left[\vec{p}^{2} - \frac{2\alpha\,(\vec{p}\cdot\vec{n})^{2}}{A} + \frac{\alpha^{2}\,\vec{n}^{2}(\vec{p}\cdot\vec{n})^{2}}{A^{2}} \right]\right\} e^{E} \; .$$

2. **One-loop massless Feynman integrals in 2ω-space**

All integrals in this Section and the next have been derived by applying the light-cone prescription, eq. (6.11). The variable $\bar{I}$ is defined in eq.(4.23), and $d^{2\omega}q = dq$.

(a) Two propagators:

$$\int \frac{dq}{(q-p)^{2}\; q\cdot n} = \frac{2\; p\cdot n^{*}}{n\cdot n^{*}}\; \bar{I} \quad , \quad n^{2} = 0 \quad ,$$

$$\int \frac{dq\; q_{\mu}}{(q-p)^{2}\; q\cdot n} = \frac{p\cdot n^{*}}{n\cdot n^{*}} \left(2p_{\mu} - \frac{2p\cdot n}{n\cdot n^{*}}\, n_{\mu}^{*} - \frac{p\cdot n^{*}}{n\cdot n^{*}}\, n_{\mu} \right) \bar{I} \; ,$$

$$\int \frac{dq\; q_{\mu}q_{\nu}}{(q-p)^{2}\; q\cdot n} = \frac{p\cdot n^{*}}{n\cdot n^{*}} \left[\frac{-p\cdot n\; p\cdot n^{*}}{n\cdot n^{*}}\, \delta_{\mu\nu} - \frac{2p\cdot n}{n\cdot n^{*}} (p_{\mu}\, n_{\nu}^{*} + p_{\nu}\, n_{\mu}^{*}) - \frac{p\cdot n^{*}}{n\cdot n^{*}} (p_{\mu}\, n_{\nu} + p_{\nu}\, n_{\mu}) + 2p_{\mu}\, p_{\nu} + \frac{2(p\cdot n)^{2}}{(n\cdot n^{*})^{2}}\, n_{\mu}^{*}\, n_{\nu}^{*} + \frac{2p\cdot n\; p\cdot n^{*}}{(n\cdot n^{*})^{2}} (n_{\mu}\, n_{\nu}^{*} + n_{\nu}\, n_{\mu}^{*}) + \frac{2(p\cdot n^{*})^{2}}{3(n\cdot n^{*})^{2}}\, n_{\mu}\, n_{\nu} \right] \bar{I} \; ,$$

$$\int \frac{dq\; q^{2}}{(q-p)^{2}\; q\cdot n} = \frac{2p\cdot n^{*}}{n\cdot n^{*}} \left(p^{2} - \frac{3p\cdot n\; p\cdot n^{*}}{n\cdot n^{*}} \right) \bar{I} \; ,$$

$$\int \frac{dq}{(q-p)^{2}\; (q\cdot n)^{2}} = \text{finite},$$

$$\int \frac{dq\; q_{\mu}}{(q-p)^{2}\; (q\cdot n)^{2}} = \frac{2p\cdot n^{*}}{(n\cdot n^{*})^{2}}\, n_{\mu}^{*} \left(\frac{-2p\cdot n\; p\cdot n^{*}}{n\cdot n^{*}} \right)^{\omega-2} \bar{I} \; ,$$

$$\int \frac{dq\; q_{\mu}q_{\nu}}{(q-p)^{2}\; (q\cdot n)^{2}} = \frac{p\cdot n^{*}}{(n\cdot n^{*})^{2}} \left[p\cdot n^{*}\, \delta_{\mu\nu} + 2(p_{\mu}\, n_{\nu}^{*} + p_{\nu}\, n_{\mu}^{*}) - \frac{4p\cdot n}{n\cdot n^{*}}\, n_{\mu}^{*}\, n_{\nu}^{*} - \frac{2p\cdot n^{*}}{n\cdot n^{*}} (n_{\mu}\, n_{\nu}^{*} + n_{\nu}\, n_{\mu}^{*}) \right] \bar{I} \quad ,$$

$$\int \frac{dq\; q^{2}}{(q-p)^{2}\; (q\cdot n)^{2}} = \left[2\omega \left(\frac{p\cdot n^{*}}{n\cdot n^{*}} \right)^{2} \right]_{\omega=2} \bar{I} \quad .$$

(b) Three propagators:

$$\int \frac{dq}{q^{2}\; (q-p)^{2}\; q\cdot n} = \text{finite},$$

$$\int \frac{dq\; q_{\mu}}{q^{2}\; (q-p)^{2}\; q\cdot n} = \frac{1}{n\cdot n^{*}}\, n_{\mu}^{*}\; \bar{I} \quad ,$$

$$\int \frac{dq\, q_\mu q_\nu}{q^2\,(q\cdot p)^2\; q\cdot n} = \frac{1}{2(n\cdot n^*)^2}\Big[\, p\cdot n^*\; n\cdot n^*\; \delta_{\mu\nu} + n\cdot n^*\,(p_\mu\, n_\nu^* + p_\nu\, n_\mu^*)$$

$$- p\cdot n\; n_\mu^*\, n_\nu^* - p\cdot n^*\,(n_\mu\, n_\nu^* + n_\nu\, n_\mu^*)\Big]\; \bar{I}\;;$$

$$\int \frac{dq}{q^2 (q\cdot p)^2\;(q\cdot n)^2} = \text{finite},$$

$$\int \frac{dq\; q_\mu}{q^2 (q\cdot p)^2\;(q\cdot n)^2} = \text{finite},$$

$$\int \frac{dq\; q_\mu q_\nu}{q^2 (q\cdot p)^2\;(q\cdot n)^2} = (n\cdot n^*)^{-2}\; n_\mu^*\, n_\nu^*\; \bar{I}\;,$$

$$\int \frac{dq\; q_\mu q_\nu q_\rho}{q^2 (q\cdot p)^2\;(q\cdot n)^2} = \frac{1}{2}\,(n\cdot n^*)^{-3}\Big\{ n\cdot n^*\; p\cdot n^*\;(\delta_{\mu\nu}\, n_\rho^* + \delta_{\mu\rho}\, n_\nu^* + \delta_{\nu\rho}\, n_\mu^*)$$

$$+ n\cdot n^*\,(p_\mu\, n_\nu^*\, n_\rho^* + p_\nu\, n_\mu^*\, n_\rho^* + p_\rho\, n_\mu^*\, n_\nu^*)$$

$$- 2p\cdot n^*\,(n_\mu\, n_\nu^*\, n_\rho^* + n_\nu\, n_\mu^*\, n_\rho^* + n_\rho\, n_\mu^*\, n_\nu^*)$$

$$- 2p\cdot n\; n_\mu^*\, n_\nu^*\, n_\rho^*\Big\}\; \bar{I}\;;$$

$$\int \frac{dq}{q^2\;(q\cdot p)^4\; q\cdot n} = \text{finite},$$

$$\int \frac{dq\; q_\mu}{q^2\;(q\cdot p)^4\; q\cdot n} = \text{finite},$$

$$\int \frac{dq\; q_\mu q_\nu}{q^2\;(q\cdot p)^4\; q\cdot n} = \text{finite}.$$

The remaining integrals in this Section and the next have been obtained with the help of the decomposition formulae

$$\frac{1}{(q\cdot p)\cdot n\;\, q\cdot n} = \frac{1}{p\cdot n}\left(\frac{1}{(q\cdot p)\cdot n} - \frac{1}{q\cdot n}\right),$$

$$\frac{1}{(q\cdot p)\cdot n(q\cdot n)^2} = \frac{1}{(p\cdot n)^2 (q\cdot p)\cdot n} - \frac{1}{(p\cdot n)^2\; q\cdot n} - \frac{1}{p\cdot n\;(q\cdot n)^2}\,.$$

$$\int \frac{dq}{q^2\;\, q\cdot n\;(q\cdot p)\cdot n} = \frac{-2p\cdot n^*}{n\cdot n^*\; p\cdot n}\;\bar{I}\;,$$

$$\int \frac{dq\; q_\mu}{q^2\;\, q\cdot n\;(q\cdot p)\cdot n} = \frac{-p\cdot n^*}{(n\cdot n^*)^2\; p\cdot n}\,(p\cdot n^*\, n_\mu + 2p\cdot n\; n_\mu^*)\;\bar{I}\;,$$

$$\int \frac{dq\; q_\mu q_\nu}{q^2\;\, q\cdot n(q\cdot p)\cdot n} = \frac{p\cdot n^*}{(n\cdot n^*)^3\; p\cdot n}\Big[n\cdot n^*\; p\cdot n\; p\cdot n^*\; \delta_{\mu\nu} - \frac{2}{3}\,(p\cdot n^*)^2\, n_\mu\, n_\nu$$

$$- 2p\cdot n\; p\cdot n^*\,(n_\mu\, n_\nu^* + n_\nu\, n_\mu^*) - 2(p\cdot n)^2\, n_\mu^*\, n_\nu^*\Big]\;\bar{I}\;;$$

$$\int \frac{dq}{q^2\;(q\cdot p)\cdot n\;(q\cdot n)^2} = \frac{-2p\cdot n^*}{n\cdot n^*\;(p\cdot n)^2}\;\bar{I}\;,$$

$$\int \frac{dq\; q_\mu}{q^2\;(q\cdot p)\cdot n\;(q\cdot n)^2} = \frac{-p\cdot n^*}{(n\cdot n^*)^2\;(p\cdot n)^2}\,(p\cdot n^*\, n_\mu + 2p\cdot n\; n_\mu^*)\;\bar{I}\;,$$

$$\int \frac{dq\; q_\mu q_\nu}{q^2\;(q\cdot p)\cdot n\;(q\cdot n)^2} = \frac{p\cdot n^*}{(n\cdot n^*)^3\;(p\cdot n)^2}\Big\{ n\cdot n^*\; p\cdot n\; p\cdot n^*\; \delta_{\mu\nu}$$

$$-2\big[p\cdot n\; p\cdot n^*\,(n_\mu\, n_\nu^* + n_\nu\, n_\mu^*) + \frac{1}{3}\,(p\cdot n^*)^2\, n_\mu\, n_\nu\big]$$

$$-2(p\cdot n)^2\, n_\mu^*\, n_\nu^*\Big\}\;\bar{I}\;,$$

$$\int \frac{dq}{(q\cdot p)^2\;(q\cdot p)\cdot n\;(q\cdot n)^2} = \frac{-2p\cdot n^*}{n\cdot n^*\;(p\cdot n)^2}\;\bar{I}\;,$$

$$\int \frac{dq\; q_\mu}{(q\cdot p)^2\;(q\cdot p)\cdot n\;(q\cdot n)^2} = \frac{p\cdot n^*}{(n\cdot n^*)^2\;(p\cdot n)^2}\,(p\cdot n^*\, n_\mu - 2n\cdot n^*\, p_\mu)\;\bar{I}\;,$$

$$\int \frac{dq\, q_\mu q_\nu}{(q\cdot p)^2\, (q\cdot p)\cdot n\, (q\cdot n)^2} = \frac{-p\cdot n^*}{(n\cdot n^*)^3\, (p\cdot n)^2} \Big[2(n\cdot n^*)^2\, p_\mu\, p_\nu$$

$$- n\cdot n^*\, p\cdot n^*\, (p_\mu\, n_\nu + p_\nu\, n_\mu)$$

$$+ \frac{2}{3}\, (p\cdot n^*)^2\, n_\mu\, n_\nu - 2(p\cdot n)^2\, n_\mu^*\, n_\nu^* \Big] \bar{I} \; .$$

(c) Four propagators:

$$\int \frac{dq}{q^2\, (q\cdot p)^2\, q\cdot n\, (q\cdot p)\cdot n} = \text{finite},$$

$$\int \frac{dq\, q_\mu}{q^2\, (q\cdot p)^2\, q\cdot n\, (q\cdot p)\cdot n} = \text{finite},$$

$$\int \frac{dq\, q_\mu q_\nu}{q^2\, (q\cdot p)^2\, q\cdot n\, (q\cdot p)\cdot n} = (n\cdot n^*)^{-2} \Big[- \frac{n\cdot n^*\, p\cdot n^*}{p\cdot n}\, \delta_{\mu\nu} + n_\mu^*\, n_\nu^*$$

$$+ \frac{p\cdot n^*}{p\cdot n}\, (n_\mu\, n_\nu^* + n_\nu\, n_\mu^*) \Big] \bar{I} \; ,$$

$$\int \frac{dq}{q^2\, (q\cdot p)^2\, (q\cdot p)\cdot n\, (q\cdot n)^2} = \text{finite},$$

$$\int \frac{dq\, q_\mu}{q^2\, (q\cdot p)^2\, (q\cdot p)\cdot n\, (q\cdot n)^2} = \text{finite},$$

$$\int \frac{dq\, q_\mu q_\nu}{q^2\, (q\cdot p)^2\, (q\cdot p)\cdot n\, (q\cdot n)^2} = \frac{p\cdot n^*}{(n\cdot n^*)^2\, (p\cdot n)^2}\, (-n\cdot n^*\, \delta_{\mu\nu}$$

$$+ n_\mu\, n_\nu^* + n_\nu\, n_\mu^*)\; \bar{I} \; .$$

3. **Massive light-cone gauge integrals in 2ω-space**

In the following one-loop integrals, m is a mass, $n^2 = 0$ and $d^{2\omega}q = dq$ (Leibbrandt and Nyeo, 1984).

$$\int \frac{dq}{[(q\cdot p)^2 - m^2]\; q\cdot n} = \frac{2p\cdot n^*}{n\cdot n^*}\, \bar{I} + F_1 \; ,$$

$$\int \frac{dq\, q_\mu}{[(q\cdot p)^2 - m^2]\; q\cdot n} = \Big[\frac{m^2}{n\cdot n^*}\, n_\mu^* - \frac{2p\cdot n\; p\cdot n^*}{(n\cdot n^*)^2}\, n_\mu^* + \frac{2p\cdot n^*}{n\cdot n^*}\, p_\mu$$

$$- \frac{(p\cdot n^*)^2}{(n\cdot n^*)^2}\, n_\mu \Big] \bar{I} + F_2 \; ,$$

$$\int \frac{dq}{q^2\, [(q\cdot p)^2 - m^2]\; q\cdot n} = F_3 \; ,$$

$$\int \frac{dq\, q_\mu}{q^2\, [(q\cdot p)^2 - m^2]\; q\cdot n} = \frac{1}{n\cdot n^*}\, n_\mu^*\, \bar{I} + F_4 \; ,$$

$$\int \frac{dq\, q_\mu q_\nu}{q^2\, [(q\cdot p)^2 - m^2]\; q\cdot n} = \frac{1}{2n\cdot n^*} \Big[p_\mu\, n_\nu^* + p_\nu\, n_\mu^* - \frac{p\cdot n}{n\cdot n^*}\, n_\mu^*\, n_\nu^*$$

$$- \frac{p\cdot n^*}{n\cdot n^*}\, (n_\mu\, n_\nu^* + n_\nu\, n_\mu^*) + p\cdot n^*\, \delta_{\mu\nu} \Big] \bar{I} + F_5 \; ,$$

$$\int \frac{dq}{q^2\, [(q\cdot p)^2 - m^2][(q\cdot k)^2 - m^2]\; q\cdot n} = F_6 \; ,$$

$$\int \frac{dq\, q_\mu}{q^2\, [(q\cdot p)^2 - m^2][(q\cdot k)^2 - m^2]\; q\cdot n} = F_7 \; ,$$

$$\int \frac{dq\, q_\mu q_\nu}{q^2\, [(q\cdot p)^2 - m^2][(q\cdot k)^2 - m^2]\; q\cdot n} = F_8 \; ,$$

$$\int \frac{dq\, q_\mu q_\nu q_\sigma}{q^2\, [(q\cdot p)^2 - m^2][(q\cdot k)^2 - m^2]\; q\cdot n} = \frac{1}{4n\cdot n^*} \Big[n_\mu^*\, \delta_{\nu\sigma}$$

$$+ n_\nu^*\, \delta_{\mu\sigma} + n_\sigma^*\, \delta_{\mu\nu} - \frac{1}{n\cdot n^*}\, (n_\mu^*\, n_\nu^*\, n_\sigma$$

$$+ n_\nu^*\, n_\sigma^*\, n_\mu + n_\sigma^*\, n_\mu^*\, n_\nu) \Big] \bar{I} + F_9 \; ,$$

$$\int \frac{dq\, q_\mu q_\nu}{[(q-p)^2 - m^2][(q-k)^2 - m^2]\; q\cdot n} = \frac{1}{2n\cdot n^*}\Big[(p+k)_\mu\, n_\nu^* + (p+k)_\nu\, n_\mu^*$$

$$- \frac{(p+k)\cdot n}{n\cdot n^*}\, n_\mu^*\, n_\nu^*$$

$$- \frac{(p+k)\cdot n^*}{n\cdot n^*}\,(n_\mu^*\, n_\nu + n_\nu^*\, n_\mu)$$

$$+ (p+k)\cdot n^*\, \delta_{\mu\nu}\Big]\, \bar{I} + F_{10}\ ,$$

where $\bar{I} = i\pi^2\ (2/\epsilon)$, $2\omega = 4-\epsilon$, and the F_j's, $j = 1,2,\ldots,10$, are finite expressions which are known exactly.

4. Special integrals ($n^2 = 0$)

The following integrals arise in the computation of two-loop massless Feynman integrals in the light-cone gauge (See Section IX.A, and Leibbrandt and Nyeo (1986a).)

(a) $$\int \frac{dq\, (q^2)^{\omega-1}}{(q-p)^2\, (q\cdot n)^2} = \frac{4i\, (-\pi)^\omega\, \Gamma(4-2\omega)(p\cdot n^*)^2}{\Gamma(1-\omega)\, (n\cdot n^*)^2}$$

$$\int_0^1 du\, dv\, (1-u) v^{-\omega}\, (1-v)^{2\omega-2}$$

$$\left\{ vp^2 + \frac{2(1-u)\, (1-v)\, p\cdot n\, p\cdot n^*}{n\cdot n^*} \right\}^{2\omega-4}\ .$$

(b) $$\int \frac{dq\, (q^2 + t\, q\cdot n\, q\cdot n^*)^{\omega-1}}{(q-p)^2\, (q\cdot n)^2} = \frac{4i(-\pi)^\omega\, \Gamma(4-2\omega)\, (p\cdot n^*)^2}{\Gamma(1-\omega)\, (n\cdot n^*)^2}\ .$$

$$\int_0^1 du\, dv\, (1-u)\, v^{-\omega}\, (1-v)^{2\omega-2}\, (1 + u\, v\, t\, n_0^2)^{-3}$$

$$\left\{ vp^2 + \frac{2(1-v)\, (1-u + u\, v\, t\, n_0^2)\, p\cdot n\, p\cdot n^*}{(1 + u\, v\, t\, n_0^2)\, n\cdot n^*} \right\}^{2\omega-4}\ ,$$

where t is a parameter and $n_0^2 = \vec{n}^2$.

(c) $$\int \frac{dq\, q_\mu}{q^2[(q-p)^2]^\sigma\, q\cdot n} = \frac{i(-\pi)^\omega \Gamma(\sigma+1-\omega) n_\mu^*}{\Gamma(\sigma)\, n\cdot n^*} \int_0^1 dx\, dy\, y^{\omega-2}\, H^{\omega-\sigma-1}$$

$$+ \frac{2i(-\pi)^\omega\, \Gamma(\sigma+2-\omega)\, p\cdot n^*}{\Gamma(\sigma)\, n\cdot n^*}\, p_\mu \int_0^1 dx\, dy\, y^{\omega-1}\, H^{\omega-\sigma-2}$$

$$- \frac{2i(-\pi)^\omega\, \Gamma(\sigma+2-\omega)\, p\cdot n\, p\cdot n^*}{\Gamma(\sigma)\, (n\cdot n^*)^2}\, n_\mu^* \int_0^1 dx\, dy\, x\, y^{\omega-1}\, H^{\omega-\sigma-2}$$

$$- \frac{2i(-\pi)^\omega\, \Gamma(\sigma+2-\omega)\, (p\cdot n^*)^2}{\Gamma(\sigma)\, (n\cdot n^*)^2}\, n_\mu \int_0^1 dx\, dy\, x\, y^{\omega-1}\, H^{\omega-\sigma-2}\ ,$$

where $H = (1-y)p^2 + 2xy\, p\cdot n\, p\cdot n^*/n\cdot n^*$, and σ is a complex number.

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Figure Captions

Fig. (1.1): Location of poles in the complex q_0-plane.

Fig. (4.1): Ghost loops with m external gauge bosons attached to it. Broken lines represent ghost particles, while wavy lines denote external gauge bosons.

Fig. (4.2): Gauge boson propagator.

Fig. (4.3): Ghost propagator.

Fig. (4.4): Three-gluon vertex.

Fig. (4.5): Four-gluon vertex.

Fig. (4.6): Ghost-ghost-gluon vertex.

Fig. (4.7a): One-loop self-energy in the axial gauge. All lines correspond to Yang-Mills fields.

Fig. (4.7b): Massless tadpole diagram.

Fig. (4.8): One-loop diagram for the graviton self-energy in the axial gauge. All dotted-dashed lines denote gravitons.

Fig. (4.9): The three-graviton vertex used in the computation of the graviton self-energy.

Fig. (4.10): Gravitational Ward identity in the axial gauge.

Fig. (4.11): The "pincer" diagram for the one-loop contribution to $F_{\lambda\beta,\rho}$ in the axial gauge.

Fig. (5.1): The "pincer" diagram for the one-loop contribution to F_{γ}^{fba} in the planar gauge. Wavy lines correspond to Yang-Mills fields.

Fig. (5.2): Yang-Mills Ward identity in the planar gauge.

Fig. (5.3): One-loop Yang-Mills self-energy in the planar gauge.

Fig. (6.1): The poles of a typical Feynman propagator such as $(q^2+i\epsilon)^{-1}$, denoted by a cross (x), lie in the second and fourth quadrant, whereas the poles connected with the principal-value prescription (6.9), and denoted by (*), are seen to lie in the first and fourth quadrant of the complex q_0-plane.

Fig. (6.2a): Pure Yang-Mills self-energy diagram in the light-cone gauge.

Fig. (6.2b): Massless tadpole diagram, vanishing in dimensional regularization.

Fig. (6.3): One-loop fermion self-energy diagram. The wavy line corresponds to a gluon field, while the solid lines denote fermions.

Fig. (6.4a): QED-like fermion-fermion-gauge vertex diagram. Wavy and solid lines correspond, respectively, to gluons and fermions.

Fig. (6.4b): Non-Abelian fermion-fermion-gauge vertex diagram.

Fig. (6.5): Three-gluon vertex diagrams.

Fig. (6.6): Ghost-loop diagrams vanishing in noncovariant gauges (cf. Fig. (4.1)). Broken lines represent ghost fields, wavy lines gluon fields.

Fig. (6.7): (a) J-ω ghost diagram. (Broken lines represent ghost fields.)

(b) J-A-ω vertex diagram.

(c) J-A-ω vertex diagram.

(d) K-ω vertex diagram.

Fig. (9.1): A two-loop Yang-Mills self-energy diagram in the light-cone gauge.

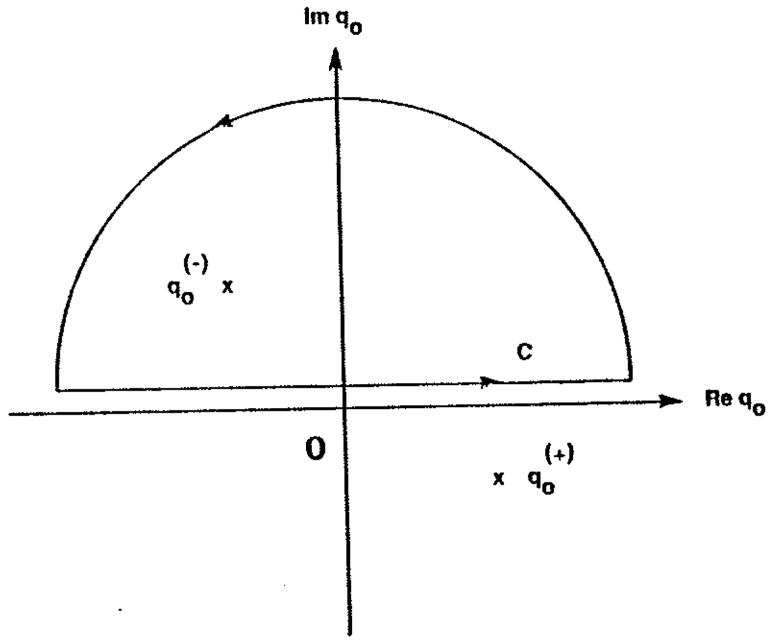


Fig. (1.1)

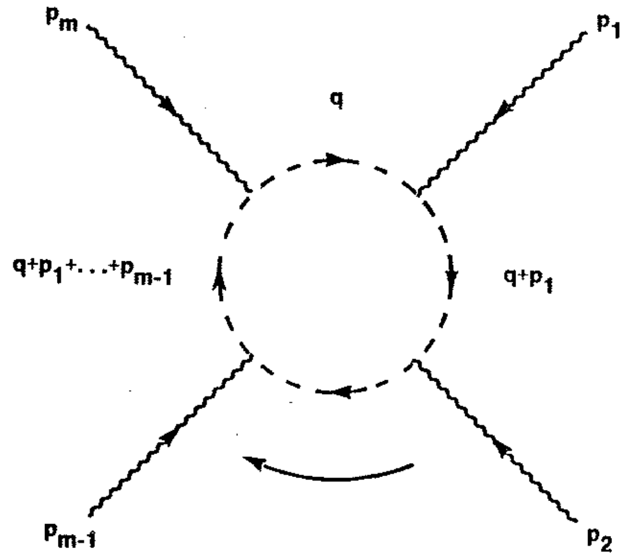
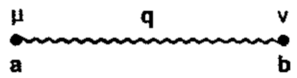


Fig. (4.1)

Fig. (4.2)



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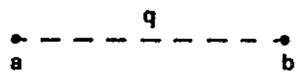

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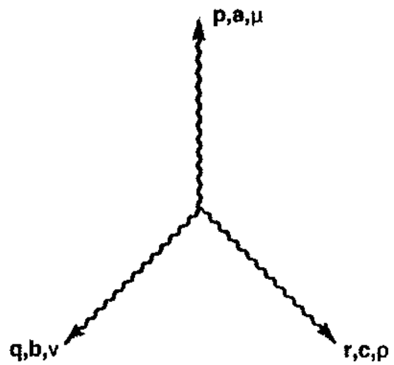

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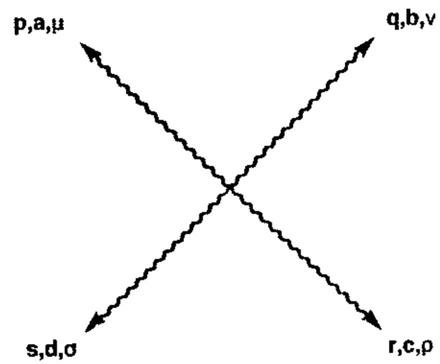

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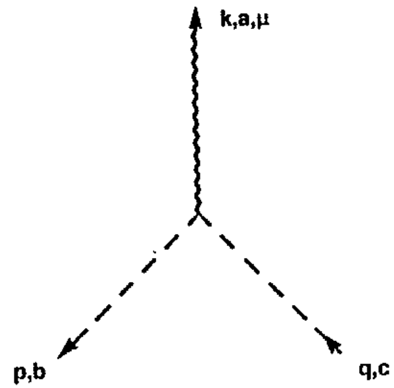

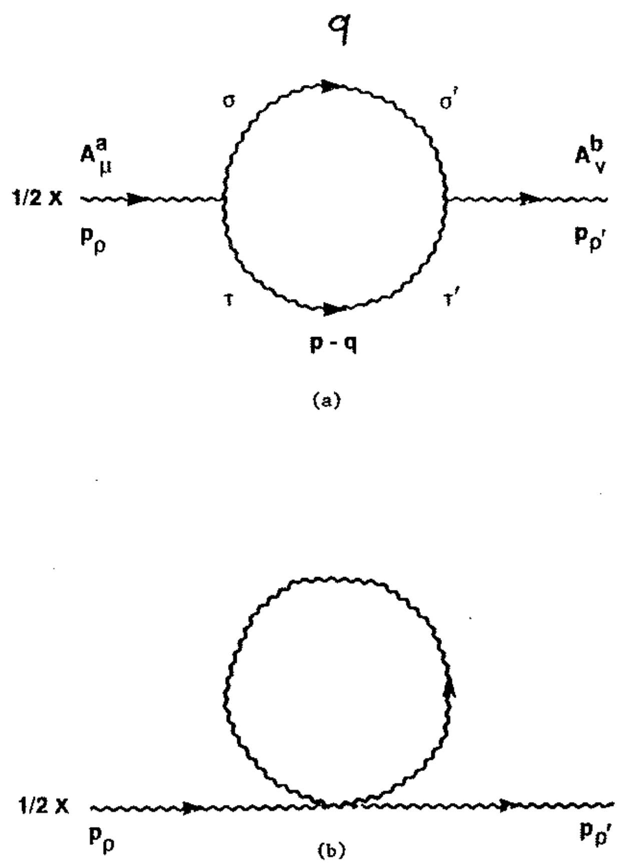

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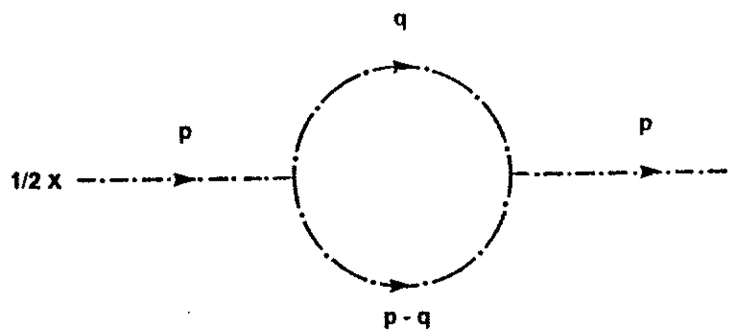

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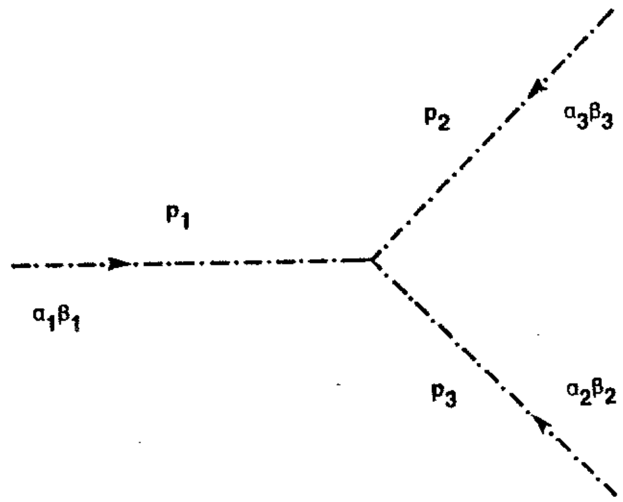

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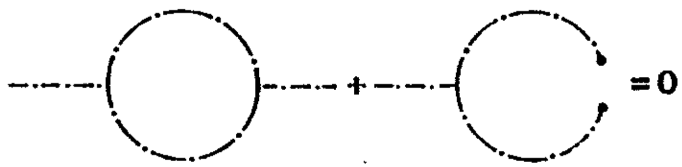

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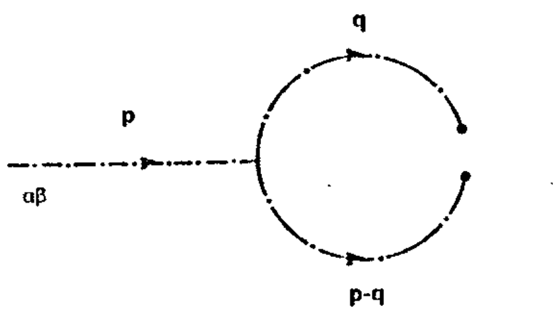

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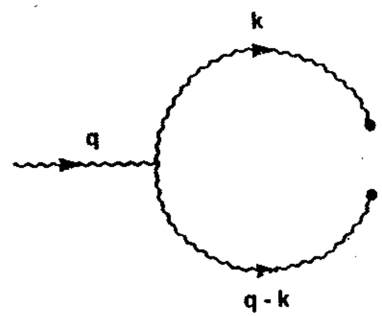

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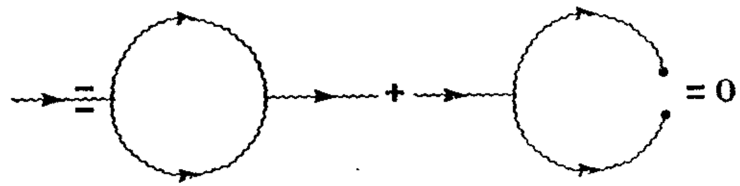

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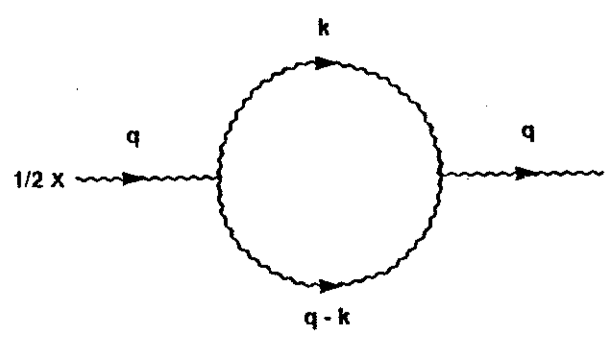

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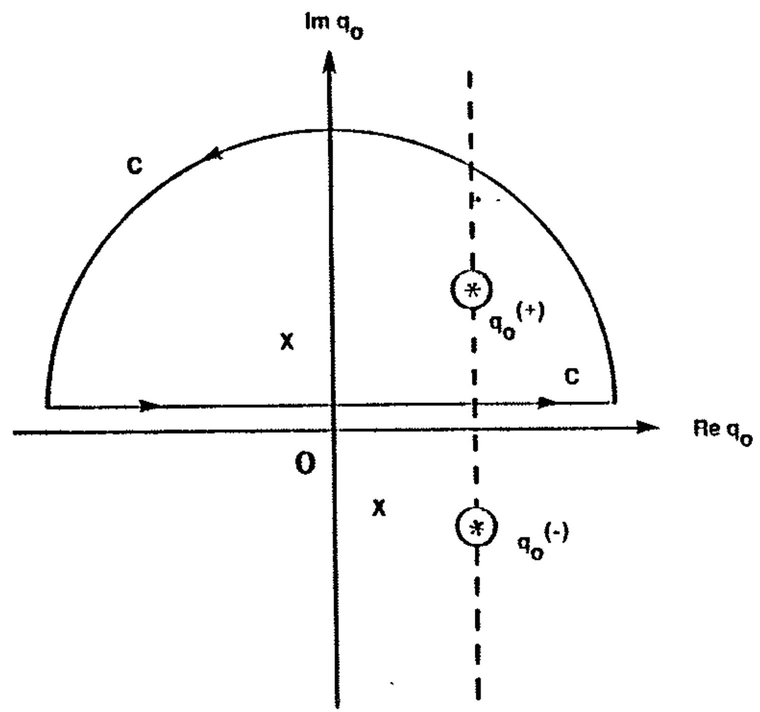

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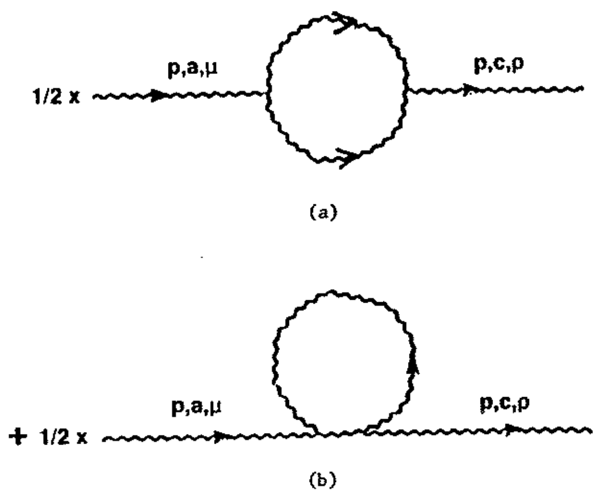

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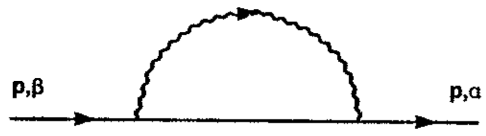

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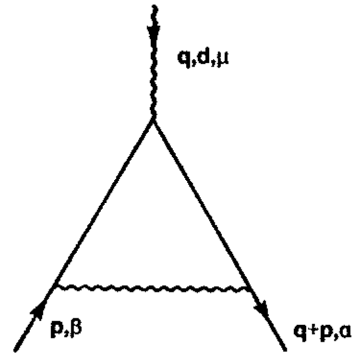

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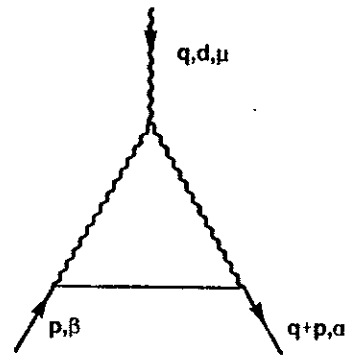

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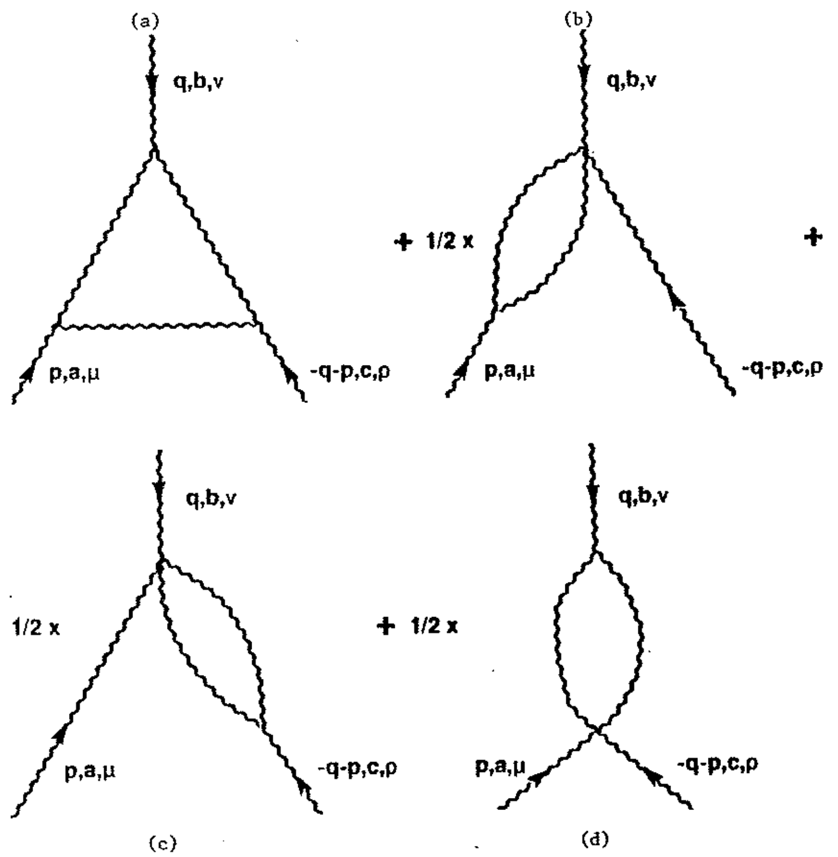
(a)
q,b,ν
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(b)
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+
1/2 x
q,b,ν
p,a,μ
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(d)

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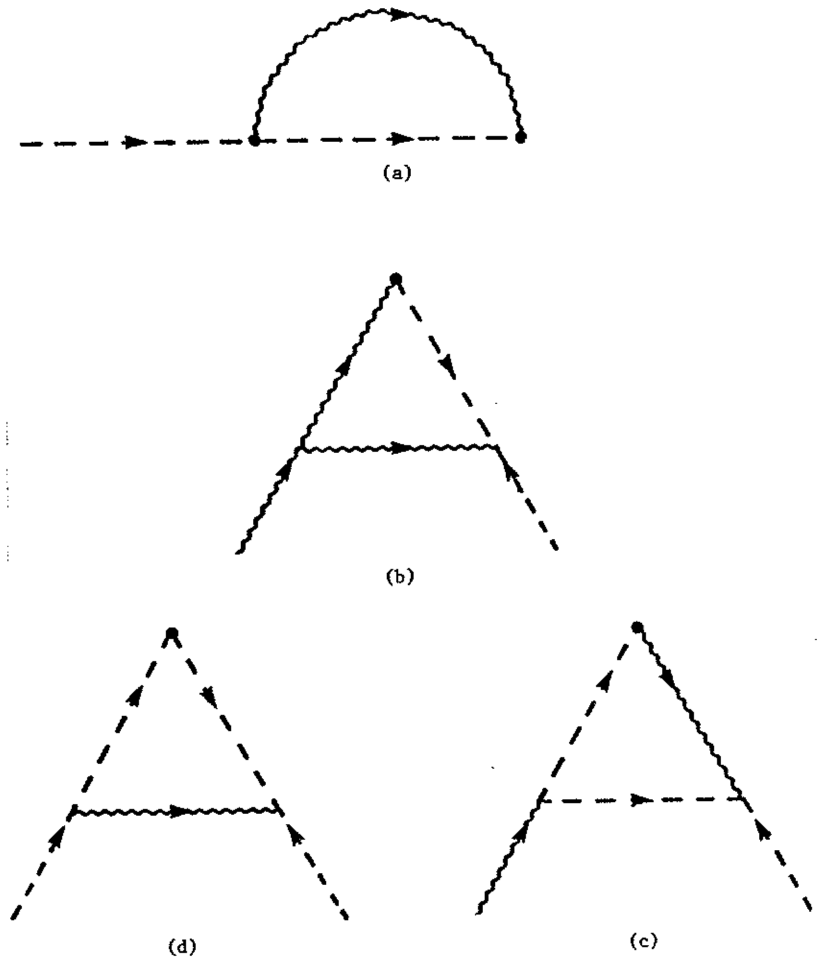
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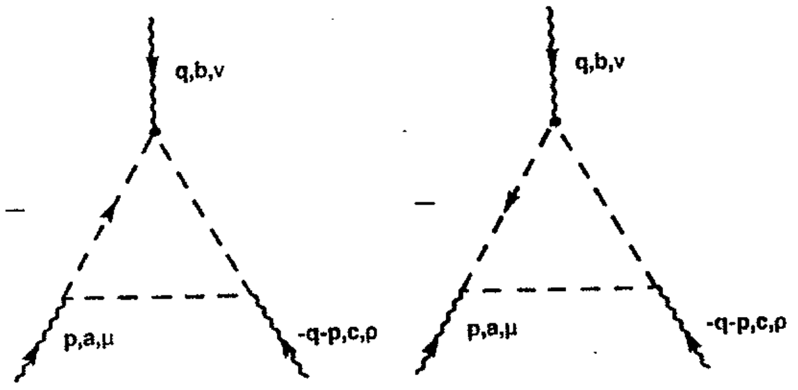

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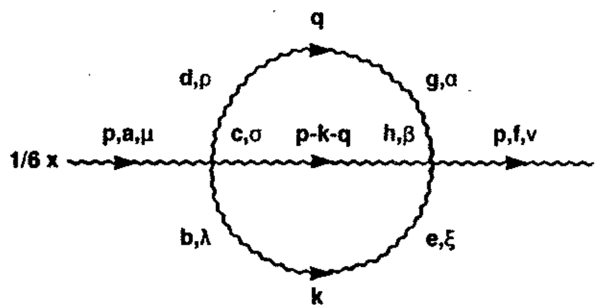

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