LHC note 8

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HANDY FORMULAE FOR BUNCHED P-P AND P-P̄ COLLIDERS

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1. Introduction

This report consists of two parts. The first part contains handy formulae and graphs for the most important beam parameters in colliding-beam storage rings using bunched p and/or $\bar{p}$ beams in head-on collision. The magnetic field B in the dipoles is used as an independent parameter because it strongly influences the design of the magnets.

The second part contains a collection of handy formulae and simple graphs describing effects due to synchtrotron radiation which may become important in machines with a beam energy of several tens of TeV.

2. Beam parameters in proton machines

The beam parameters are calculated by imposing values on the energy E, the luminosity L, the beam-beam tune shift ΔQ, and the bunch spacing s.

It is assumed that the beams are round at the collision points, i.e. that the beam size σ, the amplitude function β_t and the emittance ϵ has the same value in the two transverse direction. Then there is only one formula for the beam-beam tune shifts

$$\Delta Q = \frac{Nr_p\beta_t}{4\pi k\gamma\sigma^2} \tag{2.1}$$

Here N is the total number of particles, r_p is the classical proton radius, k is the number of bunches and γ is the usual relativistic factor. The luminosity for head-on collisions with the bunch length $\sigma_s < \beta$ is given by

$$L = \frac{N^2 f}{4\pi k\sigma^2} \tag{2.2}$$

Here f is the revolution frequency. By eliminating one power of N from (2.2), using (2.1), one obtains the standard formula

$$L = \frac{Nf\Delta Q\gamma}{r_p \beta t} \qquad \frac{I\Delta Q\gamma}{r_p e \beta t} \qquad (2.3)$$

In the second equation, I is the circulating current and e is the proton charge.

2.1 Stored beam current

By solving (2.3) for N one obtains:

$$N = \frac{Lr_p \beta t}{f\Delta Q\gamma} \qquad (2.4)$$

By replacing f by the obvious expression:

$$f = \frac{\beta c}{2\pi R} = \frac{\beta c}{2\pi} \frac{B}{B\rho} \frac{\rho}{R} = \frac{c^2 B}{2\pi\gamma} \frac{e}{E_p} \frac{\rho}{R} \qquad (2.5)$$

one finds

$$N = \frac{2\pi r_p}{c^2} \frac{E_p}{e} \frac{L\beta t}{\Delta Q B} \frac{R}{\rho} \qquad (2.6)$$

Here, $2\pi R$ is the circumference of the machine, ρ/R is the fraction of the machine occupied by dipoles, c is the velocity of light, and E_p/e is the rest voltage of the proton. It may be seen that N is independent of the energy. A handy formula is:

$$N/10^{12} = 33.563 \frac{(L/10^{32}cm^{-2}s^{-1})(\beta_t/m)}{(\Delta Q/0.003)(B/T)} \frac{R}{\rho} \qquad (2.7)$$

TABLE I :   Stored beam for various luminosities L and magnetic
fields B, in multiples of 10^{12} particles

$\beta_t = 1$ m $\Delta Q = 0.003$

$L/10^{32}cm^{-2}s^{-1}$	B/T				
	2	4	6	8	10
0.1	1.67	0.839	0.559	0.420	0.335
0.3	5.034	2.517	1.678	1.259	1.007
1.0	16.78	8.39	5.59	4.20	3.35
3.0	50.34	25.17	16.78	12.59	10.07
10.0	167.8	83.9	55.9	42.0	33.5

By solving (2.3) for the circulating current I one obtains

$$I = \frac{L r_p e \beta_t}{\Delta Q \gamma} \tag{2.8}$$

which can be written in the following handy form:

$$I/mA = 76.92 \frac{(L/10^{32} cm^{-2}s^{-1})(\beta_t/m)}{(\Delta Q/0.003)(E/TeV)} \tag{2.9}$$

It may be seen that the current is independent of the magnetic field B, but inversely proportional to the energy. A graph of the current I versus the energy E for several luminosities is shown in Fig. 1.

2.2 Beam emittance

Let us adopt the following definition of the normalised emittance:

$$\varepsilon = 4\pi\gamma\sigma^2/\beta_t \tag{2.10}$$

This definition is 3/2 of the definition used in Ref 1, but agrees with that used in Ref. 2. Combining (2.1) and (2.10), and solving for ε yields

$$\varepsilon = \frac{Nr_p}{k\Delta Q} \tag{2.11}$$

Using N as given in (2.6) and introducing the bunch spacing s

$$s = 2\pi R/k \tag{2.12}$$

The following expression for the emittance is obtained:

$$\varepsilon = \frac{r_p^2}{c} \frac{sL\beta_t}{\gamma\Delta Q^2} \tag{2.13}$$

In convenient units, this becomes:

$$\varepsilon/\pi\mu m = 0.261 \frac{(L/10^{32}cm^{-2}s^{-1})(\beta_t/m)(s/m)}{(\Delta Q/0.003)(E/TeV)} \tag{2.14}$$

The emittance is independent of the magnetic field, but proportional to $L\beta_t s/E\Delta Q$. It is shown as a function of energy for several values of $L\beta_t s$ in Fig. 2.

2.3 Phase-space density

As long as the damping times due to synchrotron radiation are longer than the beam lifetime between refills, the beam characteristics calculated earlier must be obtained by injecting beams with these characteristics. The question arises naturally whether these characteristics can be achieved.

An important parameter in this respect is the phase space density which can be defined in one, two and three degrees of freedom. For round beams it is reasonable to neglect the bunch area first, to consider only the two transverse degrees of freedom, and to introduce a phase space density D_2 defined as follows:

$$D_2 = N/(k\epsilon^2) \tag{2.15}$$

Hence, D_2 is the number of particles in a bunch divided by the square of the emittance. The density needed to drive the beam into the beam-beam limit is

$$D_2 = \frac{c}{r_p^3} \frac{\Delta Q^3 \gamma}{sL\beta_t} \tag{2.16}$$

In convenient units, this becomes:

$$D_2/10^{20} m^{-2} = 23.9 \frac{(\Delta Q/0.003)^3 (E/TeV)}{(s/m)(\beta_t/m)(L/10^{32} cm^{-2} s^{-1})} \tag{2.17}$$

A graph of D_2 versus energy E for several values of $Ls\beta_t$ is shown in Fig. 3.

It is instructive to compare the invariant phase-space density required in our machines with that actually achieved in existing proton machines which are shown in Table II.

TABLE II : Achieved phase space densities D_2

Machine	Part.	N/k	$\varepsilon/\pi\mu m$	D_2	ε_ℓ/eVs
AA[3]	$\bar{p}$	5×10^{10}	1.0	5×10^{21}	0.5
PS[4]	p	10^{12}	56*	3.2×10^{19}	1.0
	$\bar{p}$	1.5×10^{10}	15	6.8×10^{18}	0.5
ISR[5]	p	1.5×10^{11}	11*	1.2×10^{20}	0.3
	$\bar{p}$				
SPS[6]	p	1.4×10^{11}	20	3.5×10^{19}	0.5
	$\bar{p}$	1.5×10^{10}	15	6.8×10^{18}	0.5
FNAL[7]	p	2.4×10^{10}	20	6×10^{18}	0.3
	$\bar{p}$				
Tevatron[7]	p	2.4×10^{10}	20	6×10^{18}	0.25
	$\bar{p}$				
Dedicated Collider[1]	p / $\bar{p}$	10^{11}	24	1.76×10^{19}	3

*geometric mean of ε_x and ε_z

3. Effects due to synchtrotron radiation

For energies of the order of several tens of TeV, synchrotron radiation is no longer negligible even in proton machines. Handy formula for several effects related to synchrotron radiation are given below.

3.1 Synchrotron radiation loss per turn

The synchrotron radiation loss per turn U_S is given by

$$U_s/e = (4\pi/3)r_p\beta^2\gamma^3 Bc \tag{3.1}$$

In convenient units, this becomes numerically

$$U_s/eV = 2.33357(E/TeV)^3 B(T) \tag{3.2}$$

A graph of the synchrotron radiation loss is shown in Fig. 4.

3.2 Radiation damping time.

The synchrotron radiation damping time τ for horizontal or vertical betatron oscillations and for synchrotron oscillation is given by

$$\tau = 3 \left(\frac{E_p}{e}\right)^2 \frac{R}{\rho} \frac{1}{r_p\beta\gamma B^2c^3 J} \tag{3.3}$$

Here J is the damping partition number for the oscillation under consideration. For betatron oscillation J is typically one, for synchrotron oscillation it is typically two. In convenient units this becomes

$$J(\tau/h) \left(\frac{\rho}{R}\right) = \frac{1.6644\times10^4}{(E/TeV)(B/T)^2} \tag{3.4}$$

A graph of the damping time τ is shown in Fig. 5

3.3 Critical photon energy

The critical photon energy E_c is defined such that half the synchrotron radiation loss occurs in the form of photons with higher (or lower) energy. It is given by

$$E_c = \frac{3}{2}\, \lambdabar_p\, \gamma^2\, eBc \tag{3.5}$$

Here $\lambdabar_p$ is the Compton wavelength of the proton:-

$$\lambdabar_\rho = \hbar c / E_\rho \tag{3.6}$$

In convenient units, E_c becomes:

$$E_c / eV = 0.10742 (E/TeV)^2 (B/T) \tag{3.7}$$

A graph of E_c is shown in Fig. 6.

The ratio E/E_c is roughly the number of critical-energy photons in a damping time. It may be seen from (3.7) that it is typically about 10^{12}.

3.4 Equilibrium energy spread

The relative equilibrium energy spread σ_e is given by

$$J_e\, \sigma_e^2 = \frac{55}{32\sqrt{3}}\ \frac{e}{E_p}\ \lambdabar\, Bc \tag{3.8}$$

Here J_e is the damping partition number for synchrotron oscillations. In convenient units this becomes:

$$\sigma_e = 2.6658 \times 10^{-7} \left((E/TeV)(B/T)/J_e \right)^{\frac{1}{2}} \tag{3.9}$$

A graph of the equilibrium energy spread σ_e is shown in Fig. 7.

References

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3. E.J.N. Wilson, private communication.

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5. G. Guignard, CERN 77-10 (1977).

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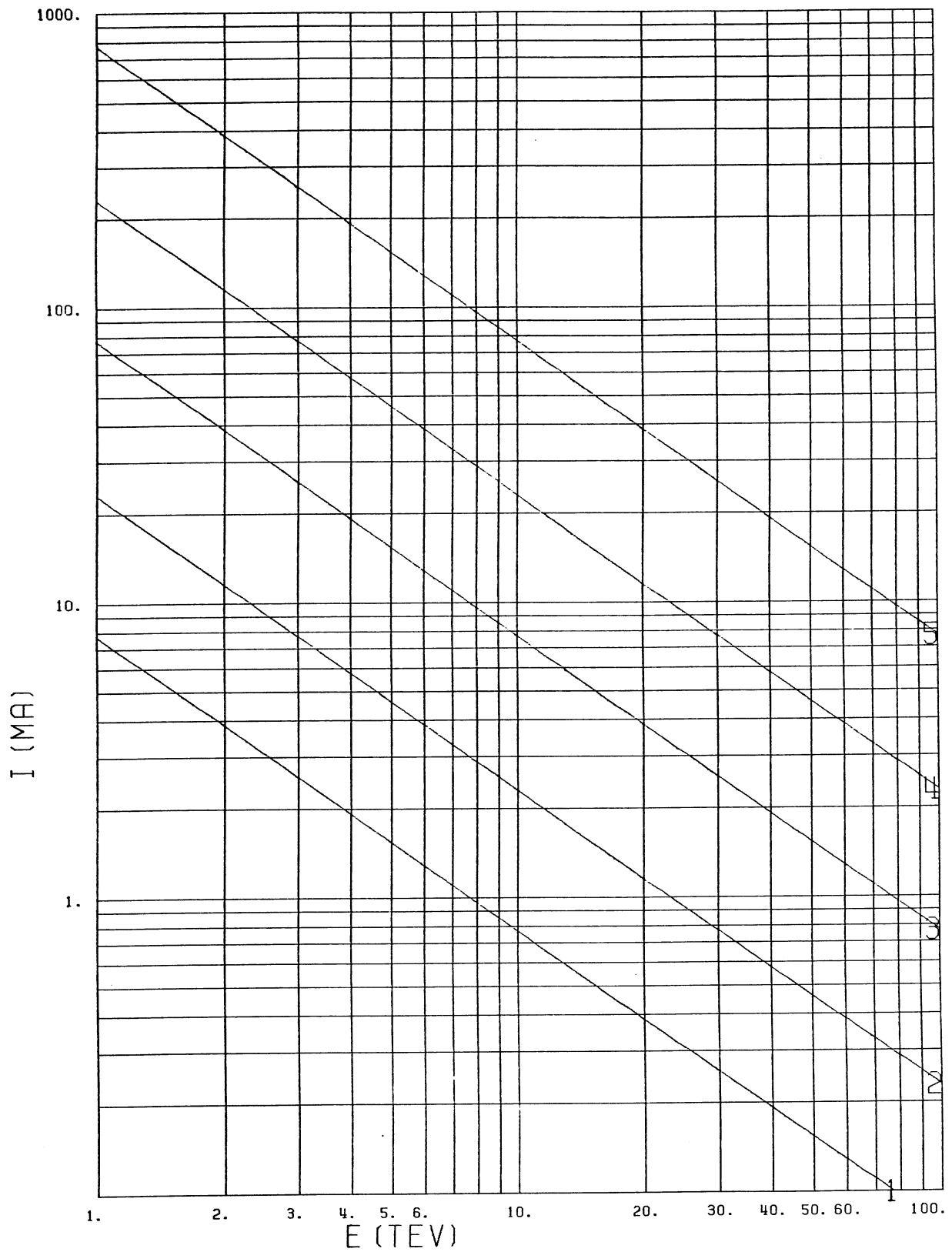
1. Circulating current I in mA versus beam energy E in TeV. The parameter is $L\beta_t$, taking values 10^{33}, 3.10^{33}, 10^{34}, 3.10^{34}, 10^{35} cm^{-1}s^{-1} for curves 1, 2, 3, 4, 5 respectively. For all curves $\Delta Q = 0.003$.
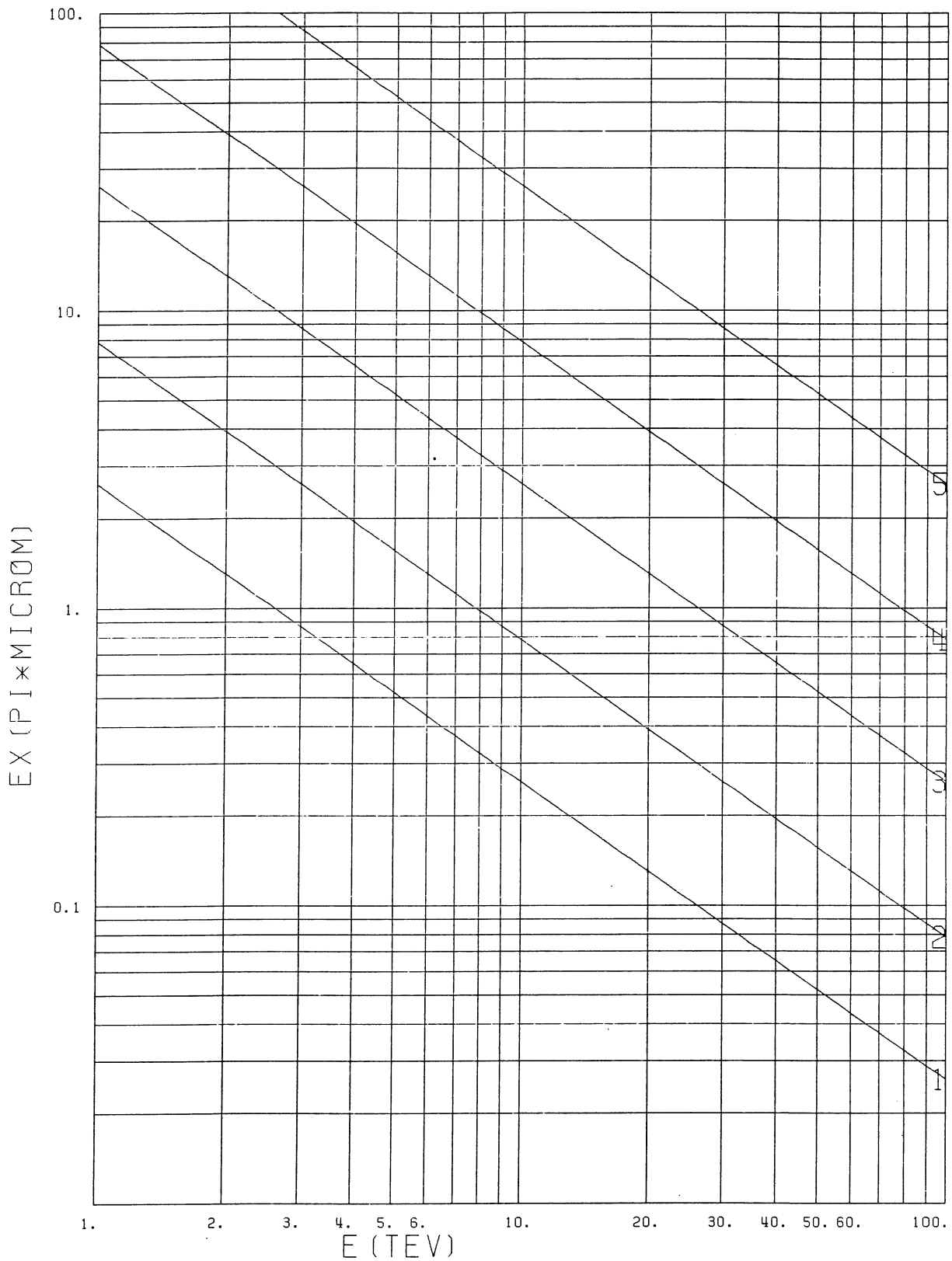
2. Beam emittance ε in $\pi\mu$m versus beam energy E in TeV. The parameter is $L\beta_t$s, taking values 10^{37}, 3.10^{37}, 10^{38}, 3.10^{38}, $10^{39}s^{-1}$ for curves 1, 2, 3, 4, 5 respectively. For all curves $\Delta Q = 0.003$.
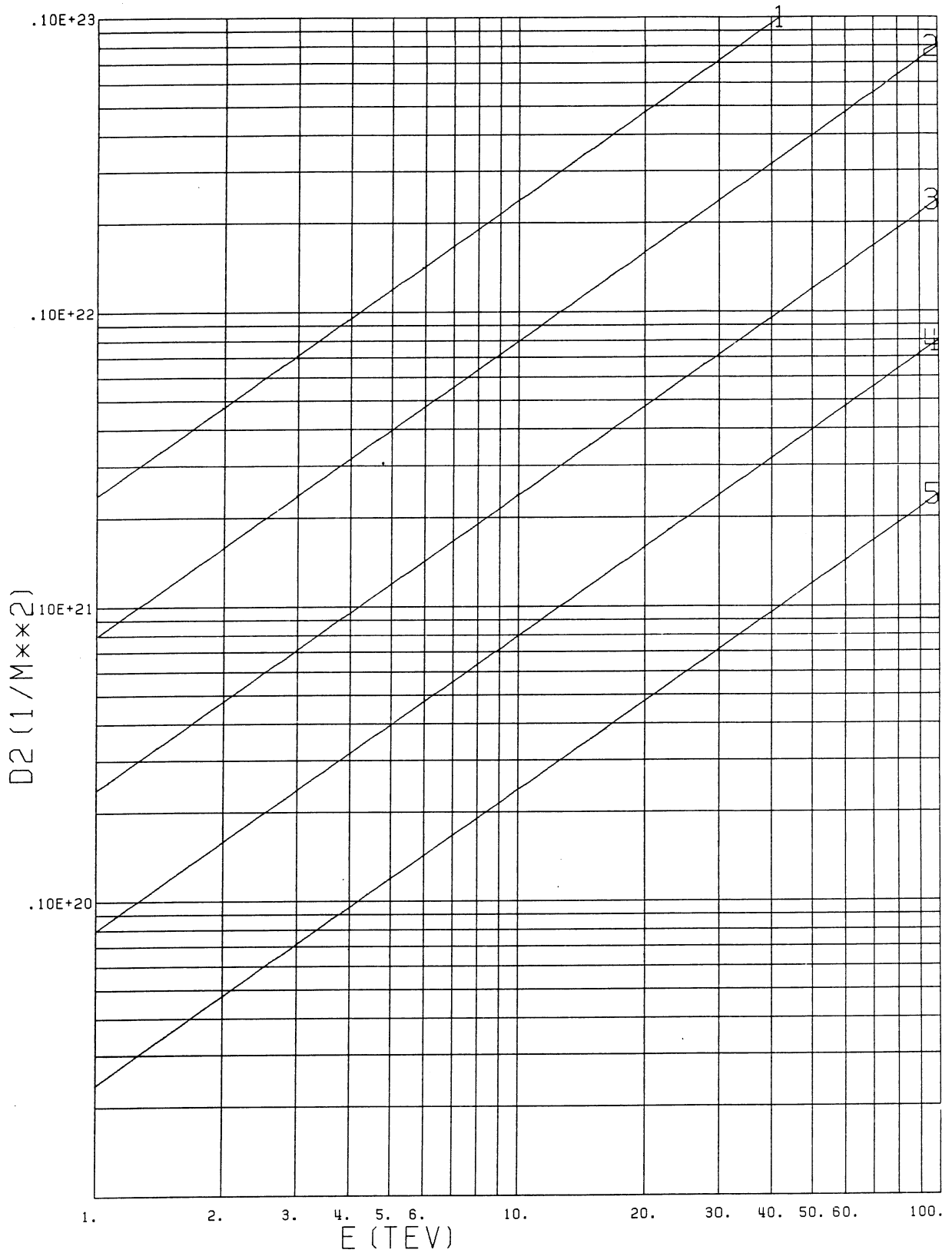
3. Phase space density D_2 in m^{-2} versus beam energy E in TeV. The parameter is $L\beta_t s$, taking values 10^{37}, 3.10^{37}, 10^{38}, 3.10^{38}, $10^{39} s^{-1}$ for curves 1, 2, 3, 4, 5 respectively. For all curves $\Delta Q = 0.003$.
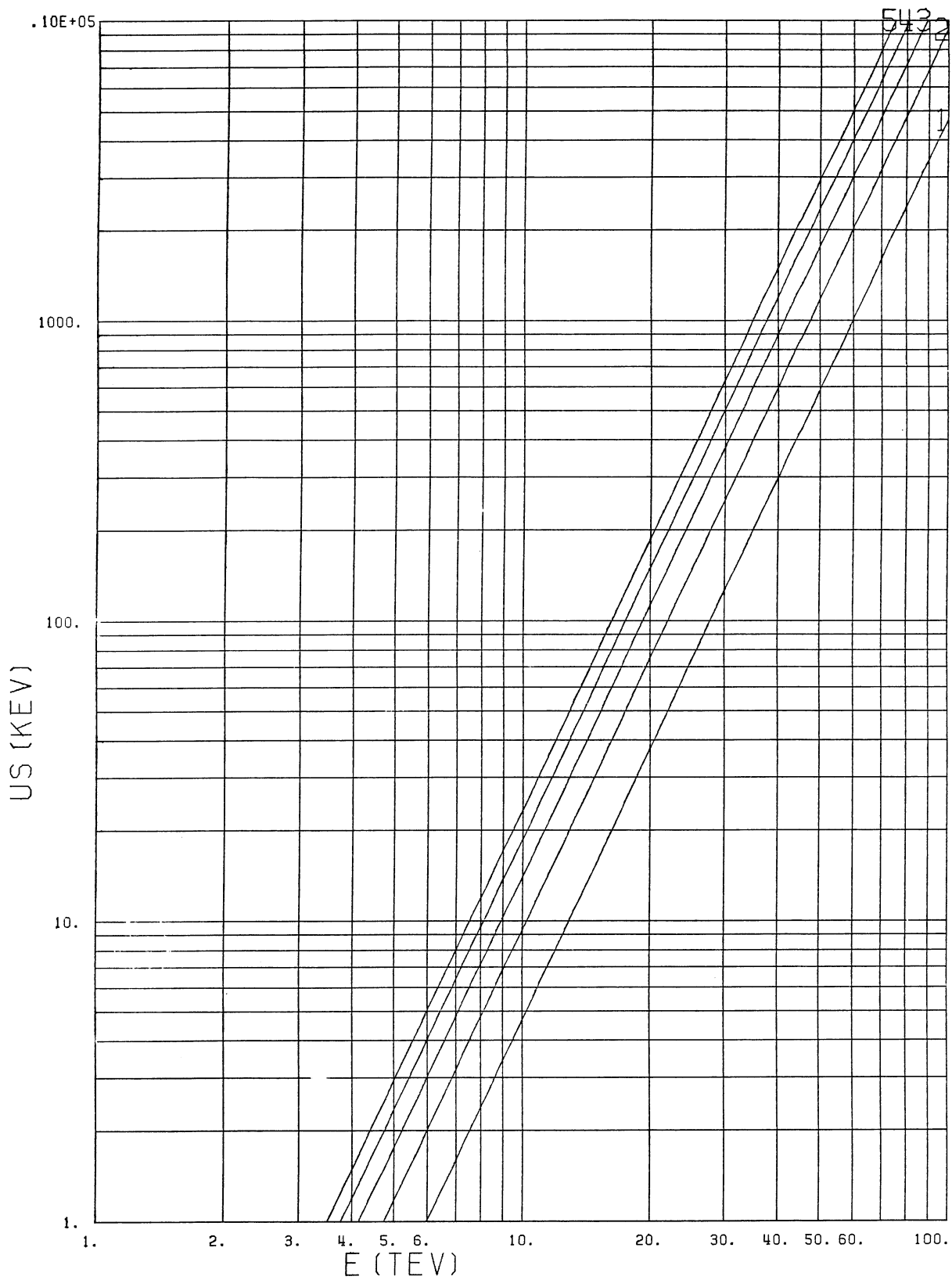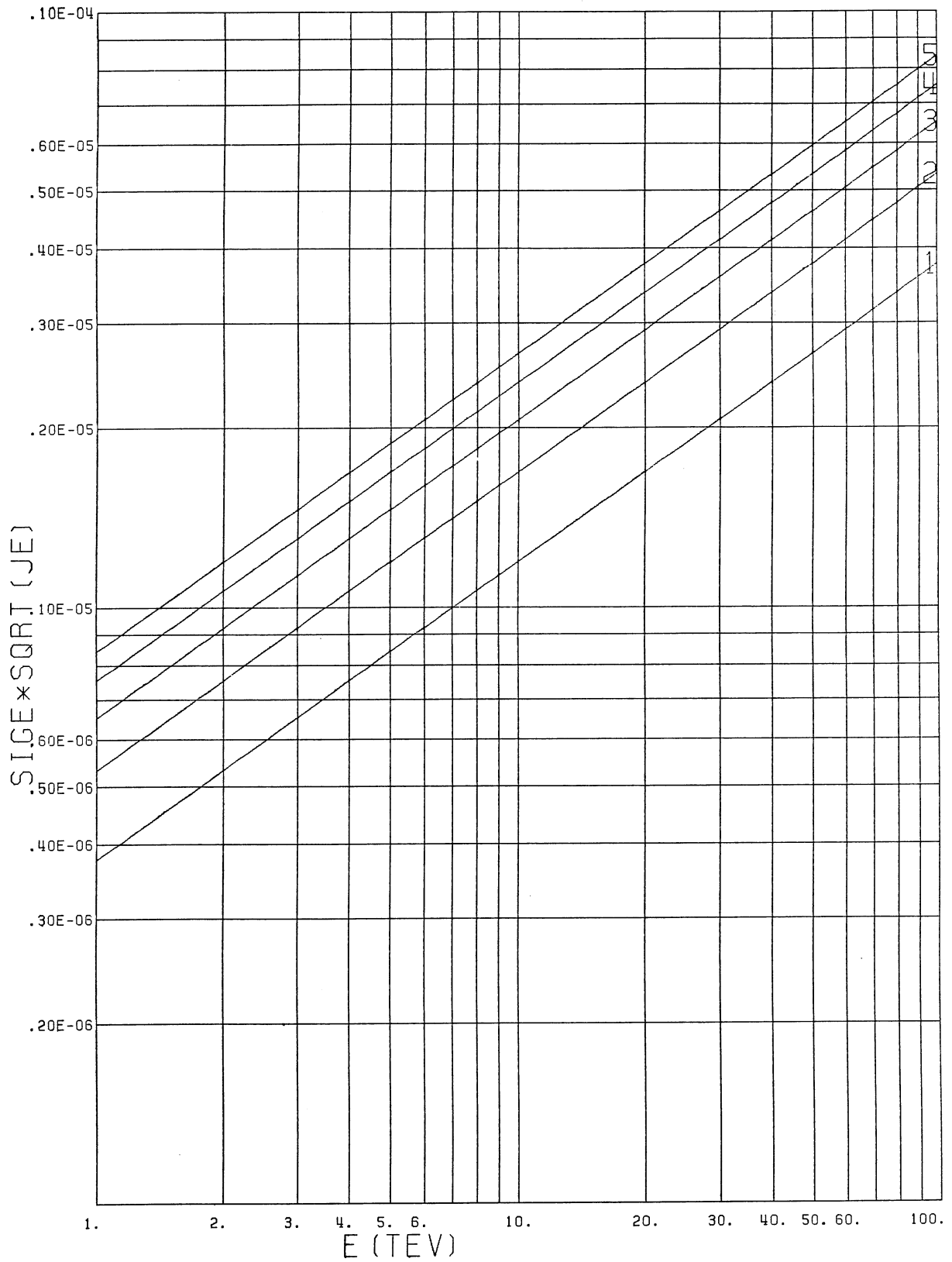
4. Synchrotron radiation loss U_S in keV versus beam energy E in TeV. The parameter is the dipole field B, taking values 2, 4, 6, 8, 10T for curves 1, 2, 3, 4, 5 respectively.
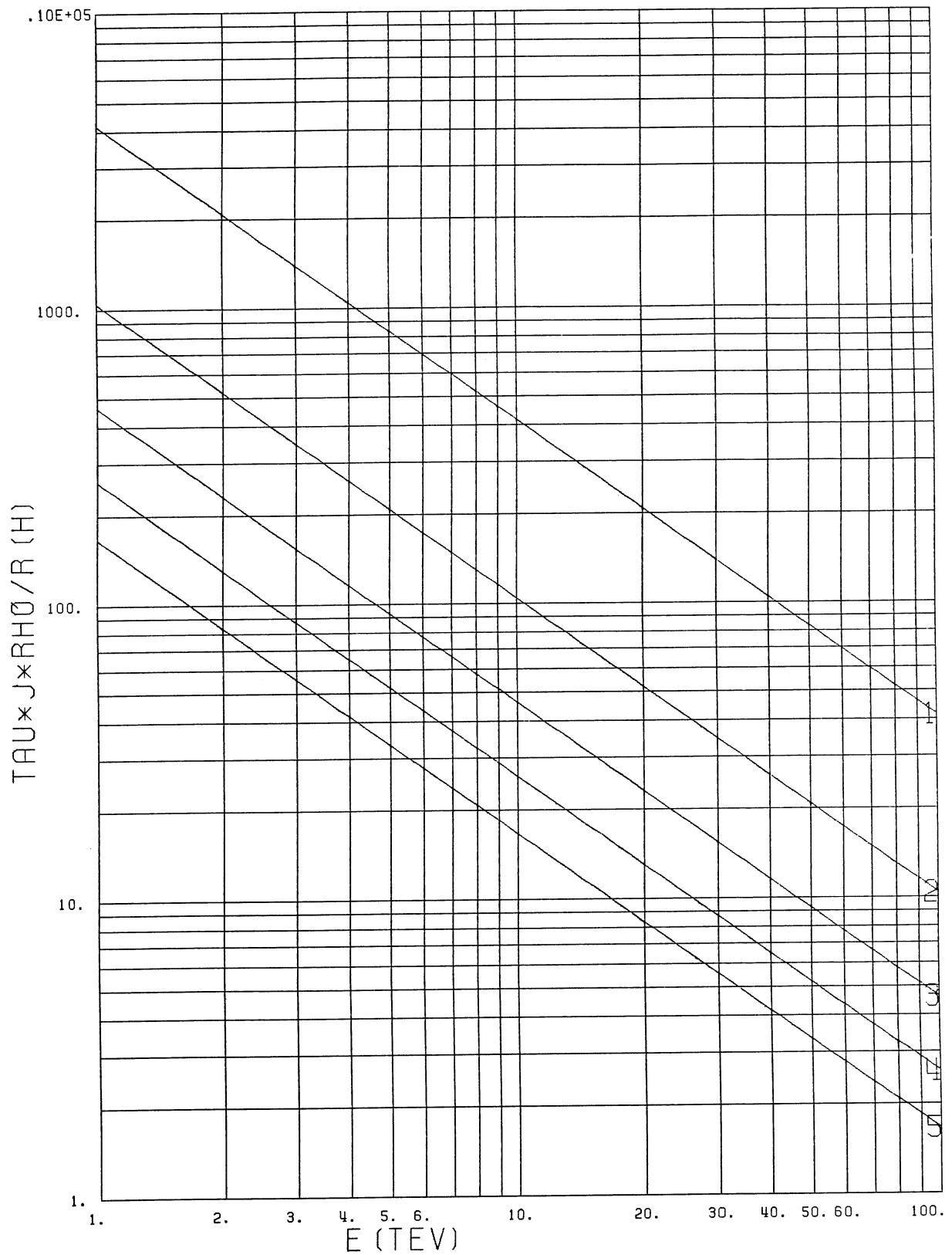
5. Product $\tau J\rho/R$ for synchrotron radiation damping in hours versus beam energy E in TeV. The parameter is the dipole field B, taking values 2, 4, 6, 8, 10T for curves 1, 2, 3, 4, 5 respectively.
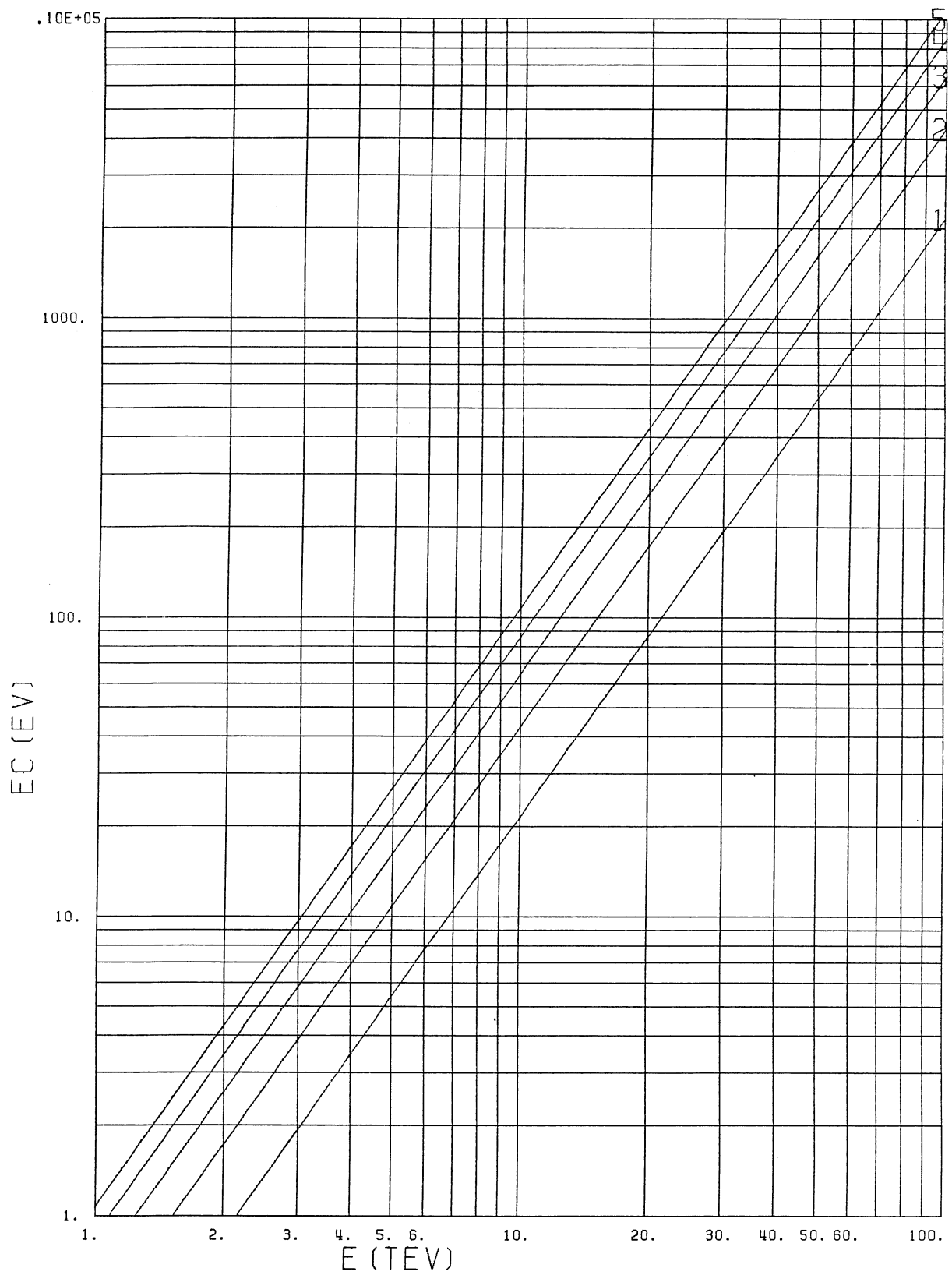
6. Critical synchrotron radiation photon energy E_C in eV versus
   beam energy E in TeV. The parameter is the dipole field B,
   taking values 2, 4, 6, 8, 10T for curves 1, 2, 3, 4, 5 respec-
   tively.

7. Product $\sigma_e J_e^{1/2}$ for equilibrium energy spread versus beam energy E in TeV. The parameter is the dipole field B, taking values 2, 4, 6, 8, 10T for curves 1, 2, 3, 4, 5 respectively.