Simulation of IBS (and cooling)

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- Intra-beam scattering rates for high-energy ion beams
- Why/when do we need kinetic simulations ?
- Interplay of IBS, cooling, collective effects
- 3D IBS simulations: Langevin equations
- 'Local' IBS model
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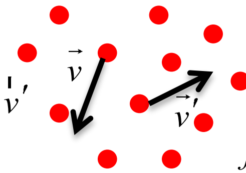
'Disclaimer:' I will report about IBS and cooling simulation schemes developed and validated for the HESR (15 GeV pbars), SIS-300 (35 GeV/u U^{92+}) and RHIC at BNL.

Multiple Coulomb Collisions in a Plasma

Fokker-Planck equation

$$\frac{df(\vec{v})}{dt} = -\sum_j \frac{\partial}{\partial v_i}(fK_j) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial v_i \partial v_j}(fD_{i,j})$$

$$\vec{u} = \vec{v} - \vec{v}'$$



$$f(\vec{x},\vec{v})$$

friction vector: $$\vec{K}(\vec{v}) = 8\pi\left(\frac{q^2}{4\pi\varepsilon_0 m}\right)^2 L_C \int f(\vec{v}')\frac{\vec{u}}{u^3}d^3v'$$

diffusion tensor: $$D_{i,j}(\vec{v}) = 4\pi\left(\frac{q^2}{4\pi\varepsilon_0 m}\right) L_C \int f(\vec{v}')\frac{u^2\delta_{i,j} - u_i u_j}{u^3}d^3v'$$

Coulomb logarithm: $$L_C = \ln\left(\frac{r_{beam}}{b_{90^0}}\right) \approx 10 - 20$$

(small angle collisions dominate)

Diffusion rate: $$\tau_{j,j}^{-1} \approx \left\langle \frac{D_{j,j}}{v_j^2} \right\rangle$$

Gaussian velocity distribution:

$$f(\vec{v},t) = \frac{n}{\pi\sqrt{2\pi}\Delta_\parallel \Delta_\perp^2}\exp\left(-\frac{v_\perp^2}{\Delta_\perp^2}\right)\exp\left(-\frac{v_\parallel^2}{2\Delta_\parallel^2}\right)$$

Anisotroptic velocity distribution:

$$\Delta_\perp \gg \Delta_\parallel$$

Longitudinal diffusion coefficient: $$D_\parallel \approx n\left(\frac{q^2}{4\pi\varepsilon_0 m}\right)^2 \frac{L_c}{\Delta_\perp}$$

Longitudinal diffusion rate: $$\tau_\parallel^{-1} \approx n\left(\frac{q^2}{4\pi\varepsilon_0 m}\right)^2 \frac{L_c}{\Delta_\perp \Delta_\parallel^2}$$

A. Sorensen, CERN Acc. School (1987)

IBS rates for high energy beams

Ratio of longitudinal/transverse velocities in the beam frame:

$$\frac{v_\parallel}{v_\perp} = \frac{\hat\beta_\perp \tilde\delta}{\gamma^2 \tilde\varepsilon_\perp} = 1 \quad \text{with} \quad \tilde\delta = \left\langle \frac{\Delta p}{p} \right\rangle$$

(rms momentum spread)

Longitudinal 'plasma' diffusion in the lab frame:

$$D_\parallel^{\text{ibs}} = \frac{r_i^2 c N L_C}{\pi R \beta_0^3 \gamma_0^3 \left\langle \hat\beta_\perp \right\rangle^{1/2} \tilde\varepsilon_\perp^{3/2}}$$

Longitudinal 'plasma' IBS heating rate:

$$\left(\tau_\parallel^{-1}\right)^{\text{ibs}} = \frac{1}{\tilde\delta}\frac{d\tilde\delta}{dt} = \frac{D_P^{IBS}}{\tilde\delta} = \frac{\Lambda_P^{IBS}}{\tilde\varepsilon_\perp^{3/2}\tilde\delta} \quad \text{with} \quad \Lambda_\parallel^{IBS} = \frac{N c r_i^2 L_C}{8\sqrt{\pi} R \beta_0^3 \gamma_0^3 \left\langle \hat\beta_\perp^{1/2} \right\rangle}$$

(corresponds to the Bjorken-Mtingwa result for high energies)

J.D. Bjorken, S.K. Mtingwa, Part. Accel. 13 (1983)

RHIC: A.V. Fedotov et al., PAC 2005

HESR: O. Boine-Frankenheim et al., NIM 2006

Touschek loss rate: $\left(\tau_{\parallel,loss}^{-1}\right)^{ibs} = \frac{D_P^{ibs}}{L_C \delta_{\max}^2}$

Transverse IBS heating rates:

$$\left(\tau_h^{-1}\right)^{ibs} \approx \left(\tau_\parallel^{-1}\right)^{ibs} \frac{\tilde\delta^2}{\varepsilon_\perp} \left\langle \frac{D_x^2 + \tilde D_x^2}{\hat\beta_x} \right\rangle$$

with $\tilde D_x = D_x \hat\alpha_x + D_x' \hat\beta_x$

Cooling equilibrium for a constant cooling time :

$$\tau_c = \tau_\parallel^{IBS}: \qquad \tilde\delta^2 = \frac{\tau_c \Lambda_P^{ibs}}{\varepsilon_\perp^{3/2}} \propto N^{2/5}$$

$$\tau_c = \tau_\perp^{IBS}: \qquad \varepsilon_\perp = \left(\tau_c \Lambda_\parallel^{ibs} \left\langle \frac{D_x^2 + \tilde D_x^2}{\beta_x} \right\rangle \right)^{2/5}$$

Why/When do we need kinetic IBS simulation ?

1. **Ultra-cold beams**, strong correlations (L_C<1).

2. **Interplay of IBS with:**

- nonlinear resonances (beam-beam, e-clouds, space charge)
- impedances and wakes
- internal targets
- (nonlinear) cooling force and particle losses

-> non-Gaussian distribution functions

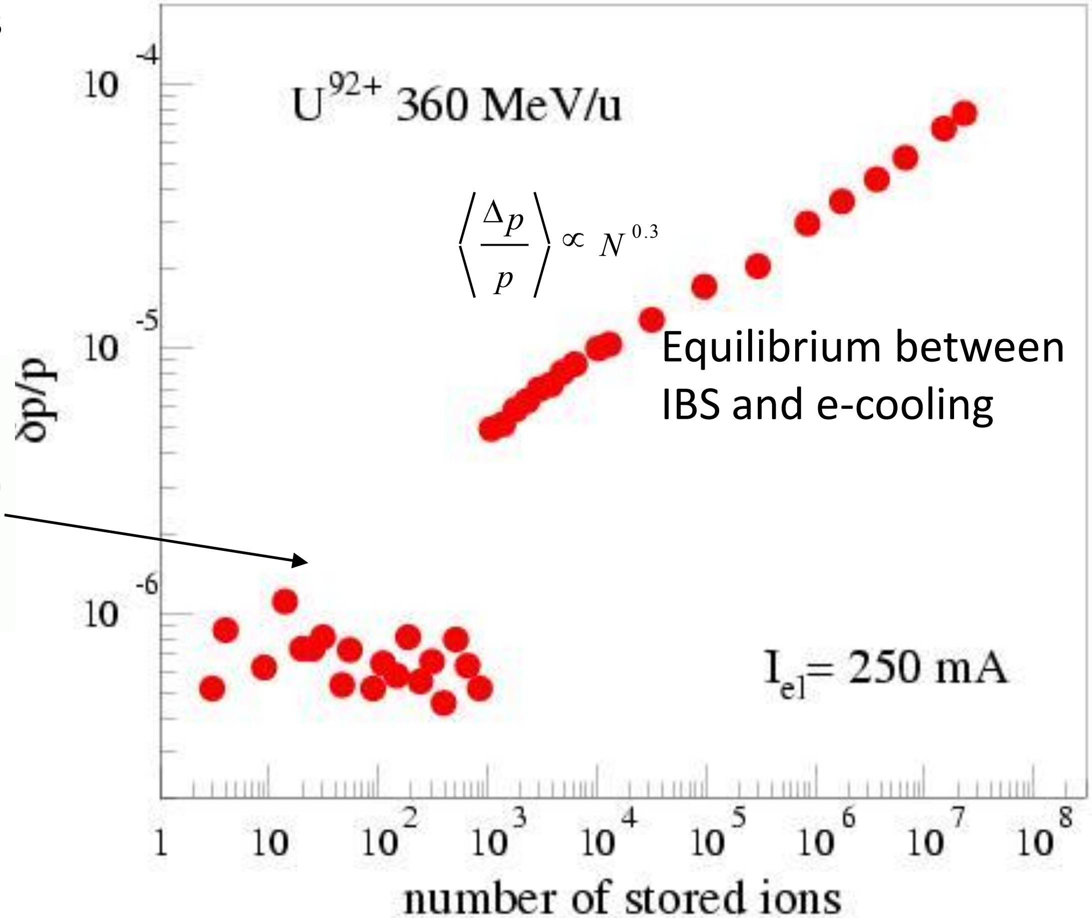
Coulomb strings in coasting ion beams

$L_C \leq 1$:

-Molecular dynamics simulations
-Tree codes

Ultra-cold beams

Equilibrium momentum spread vs. number of ions



Experiment: M. Steck et al., Phys. Rev. Lett. 1996
Theory/Simulations: R. Hasse Phys. Rev. Lett. 1999, 2001, 2003

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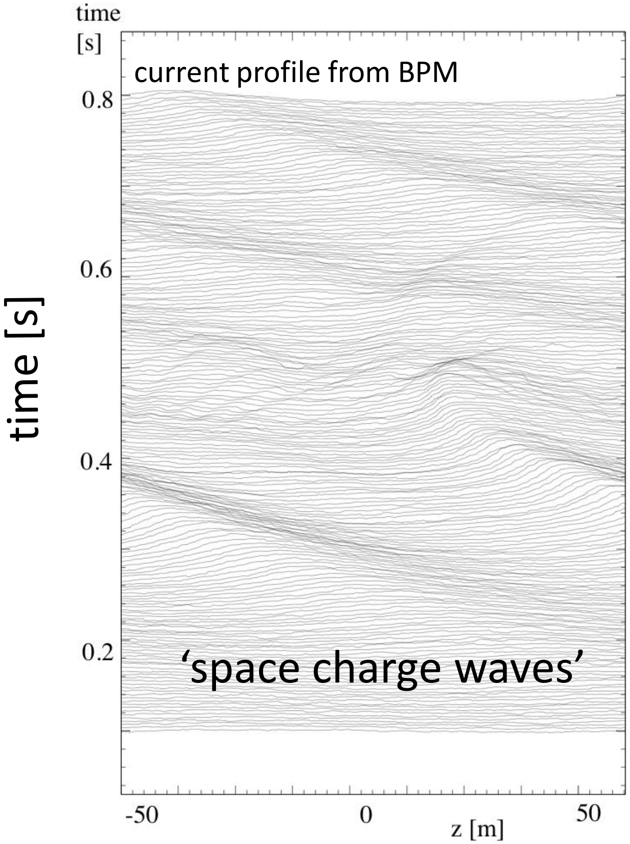
Self-bunching in a coasting beam (observed in the ESR) due to the rf cavity impedance:

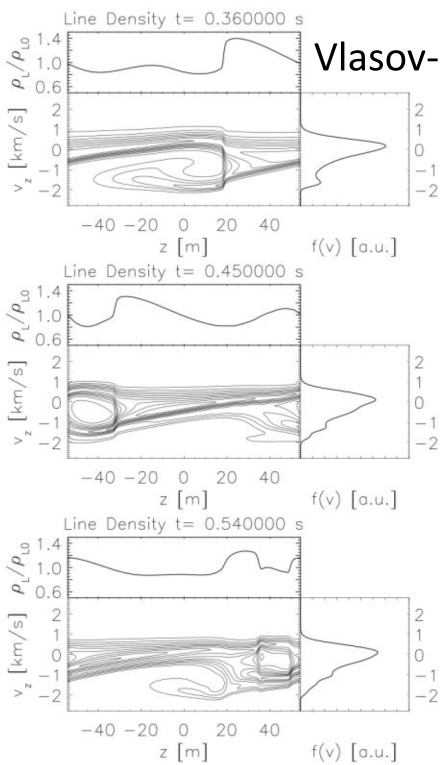


ESR circumference

1D Vlasov-Fokker-Planck equation for $f(z,\delta,t)$

$$\frac{\partial f}{\partial t} - \eta_0 v_0 \delta \frac{\partial f}{\partial z} + \frac{qV(z,t)}{p_0}\frac{\partial f}{\partial \delta} = -\frac{\partial}{\partial \delta}\left(F_{\parallel}^{e}(\delta) f\right) + D_{\parallel}^{ibs}\frac{\partial^2 f}{\partial \delta^2}$$

Impedances (harmonic n): $V_n(t) = \left(Z_n^{sc} + Z_n^{cav}\right) I_n(t)$



Vlasov-F-P simulation results

Direct numerical solution of the V-F-P equation on a grid in (z,δ) phase space.

O. Boine-Frankenheim, I. Hofmann, G. Rumolo, Phys. Rev. Lett. 1999

Simplified 3D Fokker-Planck approach

Assumptions:
- Gaussian distribution
- Constant diffusion coefficients $D_{i,j}$
- No vertical dispersion

Averaging of the FP coefficients over the field and test particles:

$$\langle K_i \rangle = \frac{A_0 L_C}{2} \left\langle \frac{u_i^2}{u^3} \right\rangle \qquad \langle D_{i,j} \rangle = A_0 L_C \left\langle \frac{u^2 \delta_{i,j} - u_i u_j}{u^3} \right\rangle$$

Diffusion tensor:

$$D_{i,j} = \begin{pmatrix} D_{x,x} & 0 & 0 \\ 0 & D_{y,y} & 0 \\ D_{z,x} & 0 & D_{z,z} \end{pmatrix}$$

Calculation of the coefficients using the B-M formalism:

$$D_{i,j} = A_N \left(\delta_{i,j} \sum_{i=1}^{3} INT_{i,i} - INT_{i,j} \right)$$

with

$$INT_{z,z} = 4\pi \int_0^\infty \frac{\sqrt{\lambda}(a_2 + \lambda)\, d\lambda}{[(a_1+\lambda)(a_2+\lambda) - \alpha^2]^{3/2} (a_3+\lambda)^{1/2}}$$

$$INT_{z,x} = -8\pi\alpha \int_0^\infty \frac{\sqrt{\lambda}\, d\lambda}{[(a_1+\lambda)(a_2+\lambda) - \alpha^2]^{3/2} (a_3+\lambda)^{1/2}}$$

$$INT_{y,y} = 4\pi \int_0^\infty \frac{\sqrt{\lambda}\, d\lambda}{[(a_1+\lambda)(a_2+\lambda) - \alpha^2]^{1/2} (a_3+\lambda)^{3/2}}$$

$$INT_{x,x} = 4\pi \int_0^\infty \frac{\sqrt{\lambda}(a_1 + \lambda)\, d\lambda}{[(a_1+\lambda)(a_2+\lambda) - \alpha^2]^{3/2} (a_3+\lambda)^{1/2}}$$

and

$$\frac{a_1}{2} = \frac{1}{\varepsilon_z} + \frac{\gamma^2 (D_x^2 + \tilde{D}_x^2)}{\beta_x \varepsilon_x}, \quad \frac{a_2}{2} = \frac{\beta_x}{\varepsilon_x}, \quad \frac{a_3}{2} = \frac{2\beta_y}{\varepsilon_y}$$

$$\alpha = \frac{2\gamma \tilde{D}_x}{\varepsilon_x}, \quad \tilde{D}_x = D'_x \beta_x + D_x \alpha_x$$

P. Zenkevich, O. Boine-Frankenheim, A. Bolshakov, NIM A (2006)

3D numerical solution of the Fokker-Planck equation:
Langevin equations

$$\vec{P} = \begin{pmatrix} x' \\ y' \\ \frac{1}{\gamma}\frac{\Delta p}{p} \end{pmatrix}$$

Langevin equation:

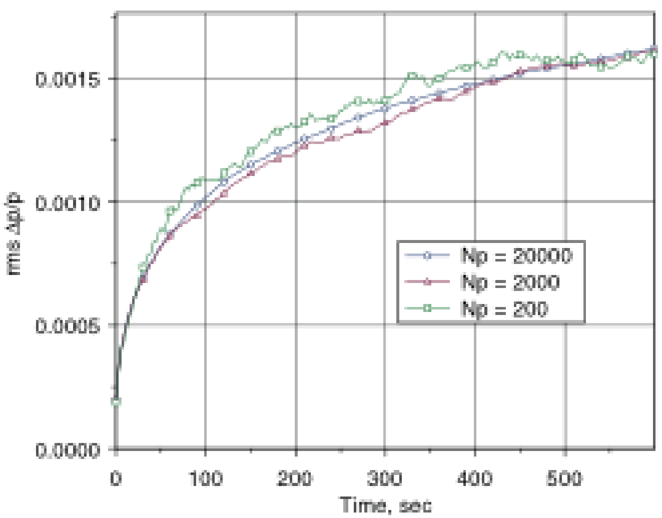
$$P_i(t+\Delta t) = P_i(t) - K_i P_i(t)\Delta t + \sqrt{\Delta t}\sum_{j=1}^{3} C_{i,j}\xi_j$$

ξ_i : Random numbers with Gaussian distribution

Relation between Langevin and diffusion coefficients:

$$\sum_{k=1}^{3} C_{i,k}C_{j,k} = D_{i,j}$$

(e.g. H. Risken, *The Fokker Planck equation,* 1984)



$$D_{i,j} = \begin{pmatrix} D_{x,x} & 0 & 0 \\ 0 & D_{y,y} & 0 \\ D_{z,x} & 0 & D_{z,z} \end{pmatrix} \Rightarrow$$

$$C_{11} = \sqrt{D_{x,x} - C_{13}^2}$$

$$C_{22} = \sqrt{D_{y,y}}$$

$$C_{33} = \sqrt{D_{z,z}}$$

$$C_{13} = \frac{1}{C_{33}}$$

$$C_{31} = 0$$

Choice of C_{ij} gives correct growth of the emittances (B-M theory)

Outline of the algorithm:

1. Calculation of $C_{i,j}$ at every lattice elements
2. Three random numbers ξ for each macro-particle
3. Apply Langevin kick.
4. Transport particles through the lattice element

P. Zenkevich, O. Boine-Frankenheim, A. Bolshakov, NIM A (2006)

General diffusion tensor: Meshkov et al., BNL Report, 2007, **BETACOOL code**

Local IBS model

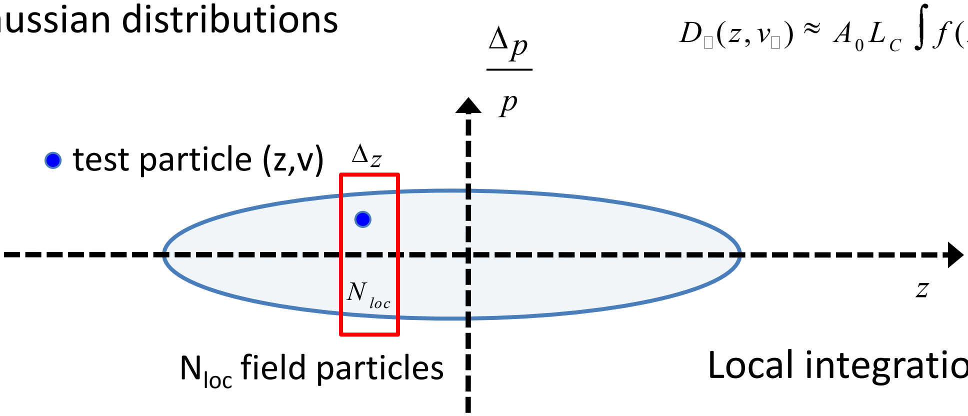
a 1D illustration

Motivation:

-accurately treat tail/halo particles in non-Gaussian distributions

Longitudinal diffusion function:

$$D_{\parallel}(z,v_{\parallel}) \approx A_0 L_C \int f(z,v'_{P}) \frac{\langle u_{\perp}^2 \rangle - (v_P - v'_P)^2}{\langle u_{\perp}^3 \rangle} dv'_P \qquad \text{with} \qquad \vec{u} = \vec{v} - \vec{v}'$$



Local integration over field particles (index m'):

$$D_{\parallel}^{\Delta z}(v_P) \approx A_0 L_C \frac{1}{N_{loc}} \sum_{m'=0}^{N_{loc}} \frac{\langle u_{\perp}^2 \rangle - (v_P - v'_{P,m'})^2}{\langle u_{\perp}^3 \rangle}$$

Langevin equation for macro-particles (index m):

$$v_{\parallel,m}(z,v,t+\Delta t) = v_{P,m}(z,v,t) - \xi_m \sqrt{\Delta t D_P(z,v_P)}$$

Outline/problems of the algorithm:

-Diffusion coefficient has to be calculated every time step for every macro-particle

- N_{loc} large enough to reduce **numerical noise**

Plasma: Manheimer, Lampe, Joyce, J. of Comput. Phys. (1997)

RHIC: Meshkov et al., BNL Report, 2007; Fedotov, Proc. of ICFA-HB2010

IBS and internal targets

Momentum kick: $$\Delta\delta_j = -F_P^e(\delta_i) + Q_j(t)\sqrt{D_P^{ibs}\Delta t} - \frac{\Delta \acute{U}_j(t)}{\beta_0^2 E_0}$$

e-cooling force — IBS diffusion (Langevin force) — target

Simulation of the interaction of an electron cooled bunch in a barrier bucket with an internal target.

Target energy kick:

Integrated probability:

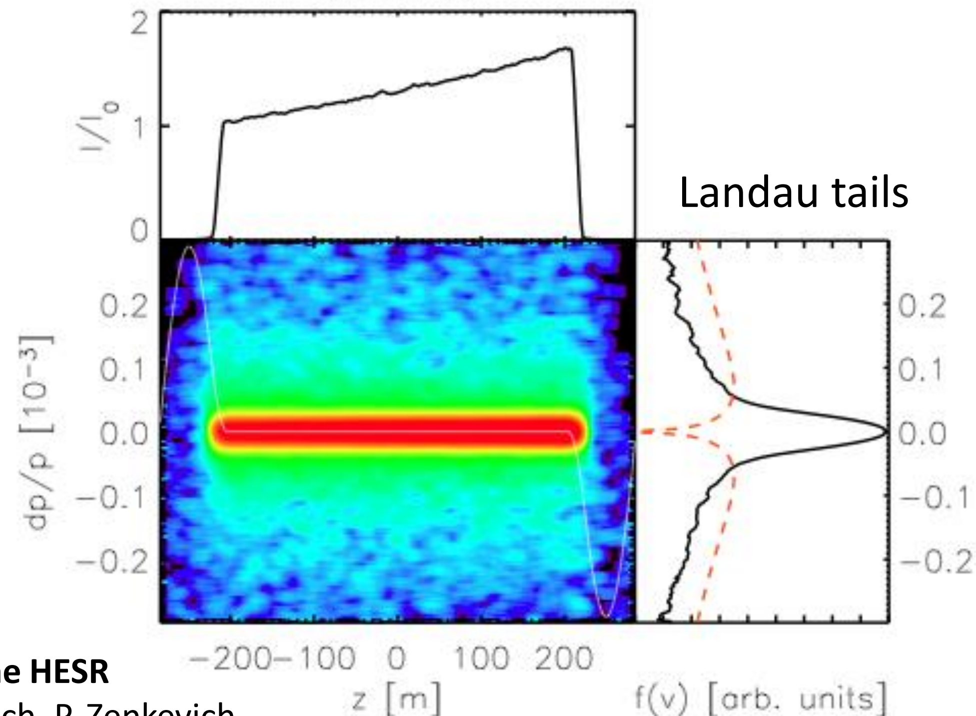
$$P(\Delta\epsilon_j) = \frac{\Delta t}{T_0}\int_I^{\Delta\epsilon} w(\epsilon)d\epsilon$$

Random number:

$$P(\Delta\epsilon_j) = \xi_j(t)$$

'Thin target' requirement:

$$\Delta t \ll T_0 \frac{I}{\xi}$$



Beam equilibrium and beam loss with internal targets in the HESR
O. Boine-Frankenheim, R. Hasse, F. Hinterberger, A. Lehrach, P. Zenkevich
Nucl. Inst. and Methods A 560 (2006) 245.

Validating IBS simulation modules

Global IBS: IBS emittance growth rates for Gaussian beams

IBS for non-Gaussian distributions: Integration of the stationary F-P equation

$$\frac{\partial}{\partial v}\left(K_{\square}^{cool}(v) f_0(v)\right) + \frac{\partial^2}{\partial v^2}\left(D_{\square}^{ibs}(v) f_0(v)\right) = 0 \quad \Rightarrow \quad f_0(v)$$

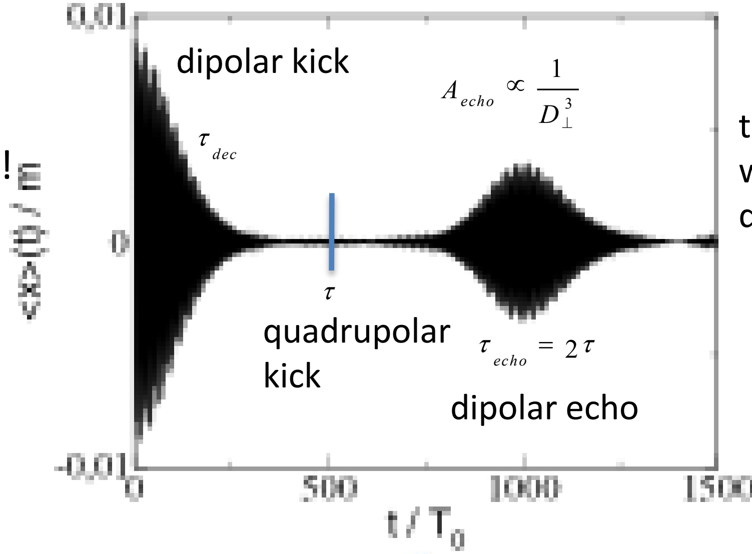
Problem in beam dynamics simulations with IBS modules:

Artificial numerical diffusion vs. IBS diffusion in (detailed) IBS simulations

Test case: (transverse) beam echo amplitudes

Echo amplitude depends very sensitive on diffusion !

$$A_{\text{echo,rel}} = \frac{\langle x \rangle_{\text{echo,max}}}{\langle x \rangle_0} = \frac{q}{\alpha^3} \frac{\tau}{\tau_{\text{dec}}} \qquad \alpha = \frac{2}{3} \frac{\tau}{\tau_{\text{diff}}} \left(\frac{\tau}{\tau_{\text{dec}}} \right)^2 + 1 \qquad D_\perp = \frac{\varepsilon_\perp}{\tau_{\text{diff}}}$$



tracking with constant diffusion

Simulation of beam echoes and code validation:

Sorge, Boine-Frankenheim, W. Fischer, Proc. ICAP 2006

Al-khateeb, Boine-Frankenheim, Hasse, PRST-AB 2003

Conclusions

IBS simulation schemes based on the Fokker-Planck equation have been presented.

a. Direct solution of the Vlasov-Fokker-Planck equation in 1D

b. Langevin equations in 1D-3D:
- Constant diffusion coefficients from B-M theory
- Local coefficents for non-Gaussian distributions

Code validation for IBS effect is an important issue (numerical noise vs. IBS) !

For most applications the Langevin equations with constant diffusion coefficients might the method of choice.