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FUNCTIONAL INTEGRALS IN QUANTUM FIELD THEORY

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ABSTRACT

Functional integrals are useful in many branches of theoretical physics, and we can regard them as an "integral calculus" for modern physics. Solutions of differential or functional equations arising in diffusion theory, quantum mechanics, quantum field theory and quantum statistical mechanics may be written down as functional integrals. Nowadays functional integral methods are widely applied to problems connected with the quantization of gauge fields. There exist many interesting applications of functional integrals to the infra-red and ultra-violet asymptotic behaviours of Green functions in quantum field theory. Recently a new region for applying functional methods has arisen - namely the theory of extended objects (vortex-like excitations, solitons, instantons). In statistical physics functional methods are widely applied to the theory of phase transitions of the second kind, and they are very useful for problems in which we have to describe collective modes (long-wave phonons and quantum vortices in superfluids and superconductors, plasma oscillations in the theory of systems of charged particles and so on). So far, a rigorous mathematical theory of functional integrals in quantum field theory and quantum statistical mechanics still does not exist and neither does a rigorous theory in an operator formalism. Nevertheless, we can use functional methods as a powerful heuristic tool for building perturbation theory or to go from one perturbative scheme to another.

These lectures are devoted to some applications of functional integrals in quantum field theory.

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L E C T U R E 1

FUNCTIONAL INTEGRALS IN QUANTUM MECHANICS

Let us consider a one-dimensional mechanical system with a Hamiltonian $H(q,p)$. If we want to quantize it, we have to change q,p by the operators

$$q \to \hat{q} = q \quad , \quad p \to \hat{p} = -i\hbar \frac{\partial}{\partial q} \tag{1}$$

acting in Hilbert space of wave functions $\psi(q)$. As we know, the Schrödinger equation *)

$$i \frac{\partial \psi}{\partial t} = \hat{H} \psi \tag{2}$$

has a formal solution

$$\psi(t) = \hat{U}(t) \psi(t_0) \ , \tag{3}$$

where

$$\hat{U}(t) = \exp(-it\hat{H}) \tag{4}$$

is the co-called evolution operator.

It turns out that the following matrix element of the evolution operator

$$\langle q' | e^{-it\hat{H}} | q \rangle \tag{5}$$

can be written down as some average

$$\langle e^{iS} \rangle \tag{6}$$

over all trajectories $q(\tau)$, $p(\tau)$ in phase space Γ, such that

$$q(0) = q \ , \quad q(t) = q' . \tag{7}$$

S in Eq. (6) is an action functional

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*) We will use units with $\hbar = 1$.

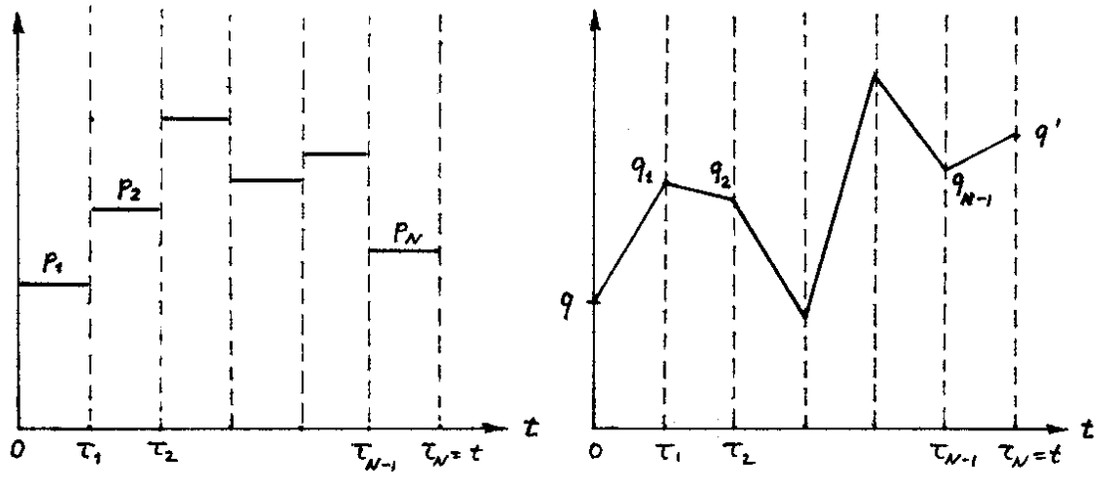
$$S = \int_0^t \left(p(\tau)\, \dot{q}(\tau) - H(q(\tau), p(\tau)) \right) d\tau \qquad (8)$$

$$\left(\dot{q}(\tau) = \frac{dq(\tau)}{d\tau} \right).$$

The average $< \exp i S >$ is nothing but a functional integral. It can be defined as a limit of some finite-dimensional integral $J_N(q,q',t)$ when $N \to \infty$ if we choose some finite-dimensional approximation for trajectories in phase space. We can use different ways of approximation. One of them is the following. Let us divide the time interval $[0,t]$ into N equal parts by points $\tau_1, \ldots, \tau_{N-1}$. We will consider trajectories for which the $p(\tau)$ function is constant in every small interval

$$(0, \tau_1), \ (\tau_1, \tau_2), \ \ldots \ (\tau_{N-1}, t) \qquad (9)$$

and $q(\tau)$ is a linear function in every small interval



We will know the trajectory if we know the $2N-1$ values

$$q_1, \ldots, q_{N-1}, \ p_1, \ldots, p_N \qquad (10)$$

The S functional turns out to be a function of q,q' and $(2N-1)$ variables (10).

Let us consider the finite-dimensional integral

$$(2\pi)^{-N} \int dp_1 dq_1 dp_2 dq_2 \ldots dq_{N-1} dp_N \exp iS = J_N(q, q', t) \tag{11}$$

over variables (10).

The main statement is that the limit of (11), when N goes to ∞, is equal to the matrix element (5) :

$$\lim_{N \to \infty} J_N(q, q', t) = \langle q' | e^{-it\hat{H}} | q \rangle \tag{12}$$

Let us check this statement for the two limiting cases :

1) H = H(q);
2) H = H(p).

If H = H(q), we will have :

$$S = \int_0^t (p\dot{q} - H(q))\, d\tau = p_1(q_1 - q) + p_2(q_2 - q_1) + \ldots + p_N(q' - q_{N-1}) - \int_0^t H(q(\tau))\, d\tau \tag{13}$$

Integrating over $p_1, \ldots, p_N$, we obtain

$$(2\pi)^{-N} \int dp_1 \ldots dp_N \exp i(p_1(q_1 - q) + \ldots + p_N(q' - q_{N-1})) = \delta(q - q_1)\, \delta(q_1 - q_2) \ldots \delta(q_{N-1} - q') \tag{14}$$

Due to the δ functions, it is easy to integrate over $q_1, \ldots, q_{N-1}$ and obtain the answer

$$J_N(q, q', t) = \delta(q - q') \exp(-it H(q)) \tag{15}$$

which is nothing but the matrix element (5) for H = H(q).

If H = H(p), we have

$$S = \int_0^t (p\dot{q} - H(p))\, d\tau = p_1(q_1 - q) + p_2(q_2 - q_1) + \ldots + p_N(q' - q_{N-1}) - \int_0^t H(p(\tau))\, d\tau \tag{16}$$

In this case we can integrate over $q_1, \ldots, q_{N-1}$

$$(2\pi)^{-N} \int dq_1 \ldots dq_{N-1} \exp i\left(p_1(q_1 - q) + p_2(q_2 - q_1) + \ldots + p_N(q' - q_{N-1})\right) =$$

$$= (2\pi)^{-1} \exp i(p_N q' - p_1 q)\ \delta(p_1 - p_2) \ldots \delta(p_{N-1} - p_N) \tag{17}$$

Then we can integrate over all p variables, except one, and obtain the result

$$(2\pi)^{-1} \int dp \exp\left(ip(q' - q) - it H(p)\right) \tag{18}$$

This is nothing but (5) for $H = H(p)$.

If H depends on both variables q, p, the problem of proving (12) becomes more difficult. It should be stressed that, in the general case, the limit (12) would depend on the way we do a finite-dimensional approximation. It is connected with the problem of the ordering of operators $\hat{q}$, $\hat{p}$ after the substitution of $q \to \hat{q}$, $p \to \hat{p}$ in $H(q,p)$. This problem has no definite solution, because there is no natural recipe for the ordering. Fortunately, in quantum mechanics we deal with Hamiltonians of the form

$$H_1(p) + H_2(q) \tag{19}$$

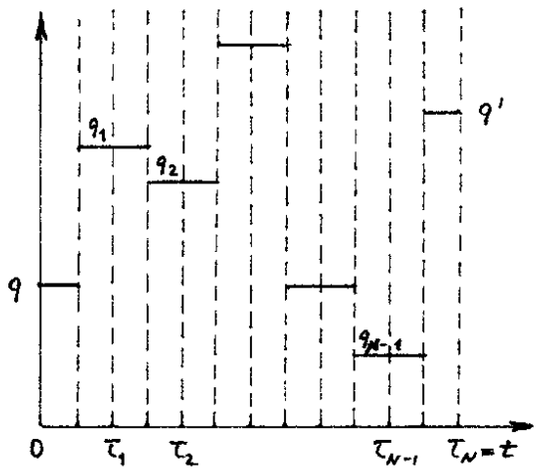
where $H_1(p)$ is a kinetic term, and $H_2(q)$ is a potential. In this case we have no problem of ordering, and the limit (12) does not depend on the approximation.

Let us now prove (12) for the case

$$H = \frac{p^2}{2m} + V(q) \tag{20}$$

typical in quantum mechanics. We will use a slightly different way of approximation. Namely, we will consider $q(\tau)$ functions which are constant in the intervals

$$\left(0, \frac{\tau_1}{2}\right), \left(\frac{\tau_1}{2}, \frac{\tau_1 + \tau_2}{2}\right), \ldots, \left(\frac{\tau_{N-1} + t}{2}, t\right) \tag{21}$$



Let us multiply $J_N(q,q',t)$ by some function $\phi_0(q)$ and integrate over q :

$$\phi(q',t) = \int J_N(q,q',t)\,\phi_0(q)\,dq \tag{22}$$

We may consider ϕ_0, ϕ as elements of the Hilbert space and rewrite (22) in the following form :

$$\phi_N(t) = e^{-i\frac{\tau}{2}\hat{V}}\, e^{-i\tau\hat{H}_0}\, e^{-i\tau\hat{V}} \cdots e^{-i\tau\hat{H}_0}\, e^{-i\frac{\tau}{2}\hat{V}}\,\phi_0 \tag{23}$$

where

$$\hat{V} = V(q)\,, \quad H_0 = \frac{\hat{p}^2}{2m} = -\frac{1}{2m}\frac{\partial^2}{\partial q^2} \tag{24}$$

Now it is not difficult to show that

$$\lim_{N\to\infty} \phi_N(t) = \phi(t) = e^{-iHt}\,\phi_0 \tag{25}$$

in the sense of a strong limit. Let us consider the difference

$$\phi_N(t) - \phi(t) = \left(e^{-i\frac{\tau}{2}\hat{V}}\, e^{-i\tau\hat{H}_0}\, e^{-i\tau\hat{V}} \cdots e^{-i\tau\hat{H}_0}\, e^{-i\frac{\tau}{2}\hat{V}} - e^{-i\hat{H}t}\right)\phi_0 \tag{26}$$

According to the well-known Lie-Trotter-Kato product t rmula, we have

$$\lim_{N\to\infty} \| e^{-i\frac{\tau}{2}\hat{V}} e^{-i\tau\hat{H}_0} e^{-i\tau\hat{V}} \dots e^{-i\tau\hat{H}_0} e^{-i\frac{\tau}{2}\hat{V}} - e^{-i\hat{H}t} \| = 0 \qquad (27)$$

if both $\hat{H}_0$ and $\hat{V}$ are self-adjoint operators. Eq. (27) means that

$$\lim_{N\to\infty} \| \phi_N(t) - \phi(t) \| = 0 \qquad (28)$$

and implies (25).

Let us denote the limit of $J_N(q,q';t)$ (the functional integral) as

$$\int_{q(0)=q}^{q(t)=q'} \exp(iS) \prod_{\tau} \frac{dq(\tau)dp(\tau)}{2\pi} \qquad (29)$$

Such a notation is useful in spite of the fact that the number of integrations over momenta (N) is not equal to the number of integrations over co-ordinates (N-1).

Generalization of the scheme described above to the case of an arbitrary finite number of variables is straightforward. We have to change

$$H(q,p) \to H(q^1,\dots,q^n,p_1,\dots,p_N)$$
$$(2\pi)^{-N} \to (2\pi)^{-Nn} \quad dq_k \to \prod_{i=1}^{n} dq_k^i \;, \quad dp_k \to \prod_{i=1}^{n} dp_{i,k} \qquad (30)$$

in Eq. (29). We shall denote the corresponding functional integral as

$$\int_{q(0)=q}^{q(t)=q'} \exp(iS) \prod_{\tau} \prod_{i=1}^{n} \frac{dq^i(\tau)dp_i(\tau)}{2\pi} \qquad (31)$$

if we fix $q^1,\dots,q^n$ on both ends of the time interval $[0,t]$.

Let me conclude this lecture by one example of the functional integral in the case where phase space is not R^{2n}.

Namely, let us consider $\Gamma = U(1)\times R$ (cylinder). We can calculate the corresponding functional integral :

$$\langle \varphi' | \exp(-i\hat{H}t) | \varphi \rangle = \int \exp i \int (p\,d\varphi - H\,d\tau) \prod_{\tau} \frac{d\varphi\,dp}{2\pi} \qquad (32)$$

if $H = H(p)$.

The above limiting expression in the exponent is equal to

$$i \sum_{i=1}^{N} p_i (\varphi_{i+1} - \varphi_i) - i \sum_{i=1}^{N} H(p_i) \Delta\tau \qquad (33)$$

We have to integrate $\exp(iS)$ over N variables p_i and over $(N-1)\varphi_i$. It is clear that we must integrate over p_i from $-\infty$ to ∞. But what about the limits of integrations over φ_i ? We can understand $\varphi(t)$ as

$$\varphi(t) = \varphi(0) + \int_0^t d\varphi \qquad (34)$$

The right-hand side of (34) can be equal to an arbitrary real number. This is why we have to integrate over φ_i from $-\infty$ to ∞. But then we must sum over the functions $\varphi(t)$ which obey conditions

$$\varphi(0) = \varphi \;,\; \varphi(t) = \varphi' + 2\pi n \;,\; n \text{ is integer} \qquad (35)$$

This is why we obtain the following expression :

$$(2\pi)^{-N} \sum_{n=-\infty}^{\infty} \int \prod_{i=1}^{N} dp_i \prod_{i=1}^{N-1} d\varphi_i \; \exp\Big\{ i \sum_i p_i (\varphi_{i+1} - \varphi_i) + \qquad (36)$$

$$+ i p_N 2\pi n - i \sum_i H(p_i) \Delta\tau \Big\}$$

The integral over $\varphi_1, \ldots, \varphi_{N-1}$ is equal to

$$(2\pi)^{-N} \int \prod_{i=1}^{N-1} d\varphi_i \exp\Big\{ i \sum_{i=1}^{N} p_i (\varphi_{i+1} - \varphi_i) \Big\} = \qquad (37)$$

$$= (2\pi)^{-1} \Big\{ \prod_{i=1}^{N-1} \delta(p_i - p_{i+1}) \Big\} \exp i (p_N \varphi' - p_1 \varphi)$$

Now, if we integrate over all momentum variables except one, we obtain

$$(2\pi)^{-1}\int dp \sum_{n=-\infty}^{\infty} \exp\left\{ip(\varphi'-\varphi) + 2\pi n p i - i t H(p)\right\} \tag{38}$$

Using the formula

$$\sum_{n=-\infty}^{\infty} \exp(2\pi n p i) = \sum_{m=-\infty}^{\infty} \delta(p-m) \tag{39}$$

we came to the answer :

$$(2\pi)^{-1} \sum_{n=-\infty}^{\infty} \exp im(\varphi'-\varphi) \cdot \exp(-i t H(m)) \tag{40}$$

This is the sum over all possible values of the momentum variable $p = m$ in compact co-ordinate space.

Problem :

Consider the case of thorus $\Gamma(0 \leq x \leq a, \ 0 \leq y \leq b)$ and try to obtain the condition $ab = 2\pi k$ (k is integer). (Quantization of the volume of a compact phase space).

L E C T U R E 2

QUANTIZATION AND FUNCTIONAL INTEGRALS FOR SYSTEMS WITH CONSTRAINTS

We may consider field theory as an infinite-dimensional analogue of a mechanical system. We can quantize a field system if we generalize the method of quantization developed for finite-dimension systems. In such an approach, gauge field theory is nothing but the infinite-dimensional generalization of the finite-dimensional system with constraints. Therefore if we want to deal with gauge fields, let us consider finite-dimensional systems with constraints. The classical action functional for such a system is equal to

$$S = \int d\tau \left(\sum_{i=1}^{n} p_i \dot{q}^i - H(q,p) - \sum_{a=1}^{m} \lambda_a \varphi^a(q,p) \right) \qquad (1)$$

The set of variables q,p form the phase space Γ, $\dim \Gamma = 2n$, λ_a $(a = 1,\dots,m)$ are the Lagrange multipliers and φ^a are constraints. We will assume that the following conditions are fulfilled :

$$\{H, \varphi^a\} = \sum_e C^a_e \varphi^e$$
$$\{\varphi^a, \varphi^e\} = \sum_d C^{ae}_d \varphi^d \qquad (2)$$

Here

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) \qquad (3)$$

is the Poisson bracket.

By demanding S to have an extremum, we obtain the following system of equations :

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \sum_{a=1}^{m} \lambda_a \frac{\partial \varphi^a}{\partial p_i} \qquad i = 1,\dots,n$$
$$\dot{p}_i = -\frac{\partial H}{\partial q^i} - \sum_{a=1}^{m} \lambda_a \frac{\partial \varphi^a}{\partial q^i} \qquad i = 1,\dots,n \qquad (4)$$
$$\varphi^a(q,p) = 0 \qquad a = 1,\dots,m$$

Constraint equations show that some variables are extra ones, and should be excluded. In practice, such an excluding problem is very complicated in many cases. This is why we want to have a method which does not require an explicit solution of constraint equations.

Because of conditions (2) any trajectory in phase space cannot leave submanifold M, defined by the constraint equations $\varphi^a = 0$, if its initial point is on M. Really, if $\varphi^a = 0$ for $t = 0$ we have

$$\dot{\varphi}^a = \{H, \varphi^a\} + \sum_\beta \lambda_\beta \{\varphi^\beta, \varphi^a\} = \\ = \sum_\beta c^a_\beta \varphi^\beta + \sum_{\beta,d} \lambda_\beta C^{\beta a}_d \varphi^d \tag{5}$$

$\dim M = 2n-m$. Let us now restrict once more the space of variables. Namely, we shall call physical variables only those variables which obey the system of equations :

$$\{f, \varphi^a\} = \sum_\beta d^a_\beta \varphi^\beta \tag{6}$$

If this system is fulfilled, the time derivative

$$\dot{f} = \{H, f\} + \sum_a \lambda_a \{\varphi^a, f\} \tag{7}$$

does not depend on a choice of Lagrange multipliers λ_a. System (6) defines trajectories on submanifold M such that $f = \text{const}$ along every trajectory. Therefore we know f if we know it on the surface of dimension $(2n-m)-m = 2(n-m)$ which intersects every trajectory in one point. Such a surface may be defined by equation :

$$\chi_a(q, p) = 0 \qquad a = 1, \ldots, m \tag{8}$$

The necessary condition is

$$\det \| \{\chi_a, \varphi^\beta\} \| \neq 0 \tag{9}$$

Furthermore, it will be suitable to demand that the χ_a commute with each other

$$\{\chi_a, \chi_\beta\} = 0 \tag{10}$$

If this is the case, we can perform a canonical transformation which turns the χ_a into new canonical momenta

$$\chi_a(q,p) = p_a \qquad a = 1,\dots,m \tag{11}$$

Let q^a be the co-ordinates conjugate to p_a, and q^*,p^* be the rest of the canonical variables. In the new variables, condition (9) can be rewritten in the following form :

$$\det \left\| \frac{\partial \varphi^a}{\partial q^b} \right\| \neq 0 \tag{12}$$

which implies that we can solve the system $\varphi^a = 0$, $a = 1,\dots,n$ and find $q^a = q^a(q^*,p^*)$.

So the constraint conditions $\varphi^a = 0$ and the supplementary conditions χ_a define the physical phase space Γ^*. We have for Γ^*

$$p_a = 0 \,, \qquad q^a = q^a(q^*, p^*) \tag{13}$$

We can now construct a functional integral scheme for our system with constraints. Let us choose the supplementary conditions $\chi_a = 0$ so that (9) and (10) are fulfilled.

The main statement is : the matrix element of the evolution operator can be expressed as the following functional integral :

$$\int \exp\left\{ i \int_0^t d\tau \left(\sum_{i=1}^n p_i \dot{q}^i - H(q,p) \right) \right\} \prod_\tau d\mu(q(\tau), p(\tau)) \tag{14}$$

where the measure is equal to

$$d\mu(q(\tau),p(\tau)) = (2\pi)^{m-n} \det \| \{\chi_a, \varphi^b\} \| \prod_a \delta(\chi_a)\,\delta(\varphi^a) \prod_{i=1}^n dq^i(\tau)\, dp_i(\tau) \tag{15}$$

In order to prove this statement let us go to the new variables q^a, q^*, p_a, p^* mentioned above, performing a canonical transformation in Γ. In terms of new variables we have

$$d\mu = (2\pi)^{m-n} \det \left\| \frac{\partial \varphi^a}{\partial q^b} \right\| \prod_a \delta(p_a)\, \delta(\varphi^a) \prod_{i=1}^{n} dq^i\, dp_i = \tag{16}$$

$$= \prod_a \delta(p_a)\, \delta(q^a - q^a(q^*, p^*))\, dq^a\, dp_a \prod_{j=1}^{n-m} \frac{dq^{*j}\, dp_j^*}{2\pi}$$

We can now integrate over q^a, p_a due to the δ functions, and we obtain the integral

$$\int \exp\left\{ i \int_0^t \left(\sum_j p_j^*\, \dot{q}^{*j} - H(q^*, p^*) \right) d\tau \right\} \prod_\tau \prod_{j=1}^{n-m} \frac{dq^{*j}\, dp_j^*}{2\pi} \tag{17}$$

It is nothing but a functional integral for a non-degenerate mechanical system, described in the first Lecture.

We can rewrite the integral (16) in the following form

$$\int \exp\left\{ i \int_0^t \left(\sum_i p_i\, \dot{q}^i - H - \sum_a \lambda_a \varphi^a \right) d\tau \right\} \cdot$$

$$\cdot \prod_\tau \det \left\| \{ \chi_a, \varphi^b \} \right\| (2\pi)^{m-n} \prod_a \delta(\chi_a) \prod_{i=1}^{n} dq^i\, dp_i \prod_b \frac{\Delta\tau\, d\lambda_b}{2\pi} \tag{18}$$

The symbol $\prod_b (\Delta\tau\, d\lambda_b)/2\pi$ shows that before the limiting process we have the following integral over λ_a variables :

$$\int \exp\left(-i \sum_{i,a} \lambda_a(\tau_i)\, \varphi^a(q(\tau_i), p(\tau_i))\, \Delta\tau \right) \prod_{i,b} \frac{\Delta\tau\, d\lambda_b}{2\pi} = \tag{19}$$

$$= \prod_{i,a} \delta(\varphi^a(q(\tau_i), p(\tau_i)))$$

This means that we can integrate over λ_b in (18) and go back to (16).

Let us now show that the integral (16) does not depend on the concrete choice of supplementary conditions χ_a. Let $\delta\chi_a$ be an infinitesimal addendum to the supplementary condition χ_a. We can write :

$$\delta\chi_a = \{ \Phi, \chi_a \} + \sum_b c_{ab}\, \varphi^b$$

where

$$\phi = \sum_a h_a \varphi^a ,$$

$$\sum_b \{\chi_a, \varphi^b\} h_b = -\delta\chi_a$$

On the submanifold M $(\varphi^a = 0)$ we can consider $\delta\chi_a$ as the result of an infinitesimal canonical transformation with a generator ϕ.

Now we have

$$\delta\varphi^a = \sum_b A^a_b \varphi^b$$

where

$$A^a_b = \{h_b, \varphi^a\} - \sum_c h_c C^{ac}_b \quad ;$$

$$\chi_a \to \chi_a + \delta\chi_a \quad , \quad \varphi^a \to \varphi^a + \sum_b A^a_b \varphi^b , \quad H \to H ,$$

$$\prod_a \delta(\varphi^a) \to \prod_a \delta(\varphi^a + \delta\varphi^a) = (1 + A^a_a)^{-1} \prod_a \delta(\varphi^a) ,$$

$$\det \| \{\chi_a, \varphi^b\} \| \to \det \| \{\chi_a + \delta\chi_a , \varphi^b + \delta\varphi^b\} \| =$$

$$= \det \| \{\chi_a + \delta\chi_a, \varphi^b\} \| \det \left\| \frac{\partial(\varphi^a + \delta\varphi^a)}{\partial\varphi^b} \right\| =$$

$$= \det \| \{\chi_a + \delta\chi_a, \varphi^b\} \| \ (1 + A^a_a) .$$

Therefore all the change in the measure is a change $\chi_\alpha \to \chi_\alpha + \delta\chi_\alpha$. It means that our integral is invariant if we replace $\chi_\alpha \to \chi_\alpha + \delta\chi_\alpha$.

L E C T U R E 3

FUNCTIONAL INTEGRALS AND DIAGRAM TECHNIQUES IN QUANTUM FIELD THEORY

We may consider quantum field theory as a theory of a mechanical system with an infinite number of degrees of freedom. There are different ways to introduce functional integrals into field theory. The first one is to start with the Hamiltonian form of the action functional and then to deal with the functional integral over all the trajectories in the phase space of our infinite-dimensional system. The second way is to start with the so-called functional integral "over all fields" in order to obtain the explicit relativistic theory. Every way has its own advantages and demerits. If we use the Hamiltonian approach, the relativistic invariance is hidden and is difficult to prove in many cases. As to unitarity - it is evident in the Hamiltonian approach - but not in the formalism of integration over all fields.

First of all, let me dwell upon the formalism of integration over all fields. We will use this formalism in order to obtain the perturbative scheme for Green functions. It is appropriate to consider the example of self-interacting neutral scalar fields with the following action functional

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \frac{g}{3!} \varphi^3 \right) \qquad (1)$$

At the end of this Lecture I shall show how to go from the integral over all fields to the Hamiltonian form of the functional integral.

Let us take a big cubic volume V in Minkowski space-time and divide it in N^4 equal small cubes v_i $(i = 1, \ldots, N^4)$. Let us choose the approximation of $\varphi(x)$ by functions which are constant in every cube v_i

$$\varphi(x) = const \qquad x \in v_i \qquad (2)$$

For derivatives we shall use the following approximation :

$$\partial_\mu \varphi \approx \frac{1}{\Delta l} \left[\varphi(x_\nu + \delta_{\mu\nu} \Delta l) - \varphi(x_\nu) \right] \qquad (3)$$

If the approximations (2) and (3) are made, S will be a function of N^4 variables $\varphi_i(x)$ and we may consider the integral

$$\int \exp(iS) \prod_{\substack{i=1 \\ x\in V_i}}^{N^4} d\varphi_i(x) \tag{4}$$

We may now define the one-particle Green function in the following way :

$$G(x,y) = -i\langle \varphi(x)\varphi(y)\rangle =$$

$$= -i \lim_{\substack{V_i\to 0 \\ V\to\infty}} \frac{\int \exp(iS)\,\varphi(x)\varphi(y) \prod_{i=1}^{N^4} d\varphi_i(x)}{\int \exp(iS) \prod_{i=1}^{N^4} d\varphi_i(x)} \tag{5}$$

For the right-hand side of (5) we shall use the notation

$$-i \frac{\int e^{iS}\,\varphi(x)\varphi(y) \prod_x d\varphi(x)}{\int e^{iS} \prod_x d\varphi(x)} \tag{6}$$

It is a difficult problem, which is as yet unsolved, to prove that the limit (5) does indeed exist. Nevertheless we can calculate Green functions in the free field theory (when the coupling constant g vanishes) and then develop a perturbative scheme.

Let us consider the generating functional

$$Z[\eta] = \frac{\int \exp\left(iS + i\int \eta(x)\varphi(x)d^4x\right) \prod_x d\varphi(x)}{\int \exp(iS) \prod_x d\varphi(x)} \tag{7}$$

We can obtain all the Green functions by differentiating $Z[\eta]$. For instance, we have

$$G(x,y) = i \frac{\delta}{\delta\eta(x)} \frac{\delta}{\delta\eta(y)} Z[\eta]\Big|_{\eta=0} \tag{8}$$

First of all, let us obtain the generating functional $Z_0[\eta]$ for the free field $(g=0)$. We can do it by performing the shift transformation

$$\varphi(x) \to \varphi(x) + \varphi_0(x) \tag{9}$$

in the integral in the numerator of (7), in order to cancel the linear functional $\int \eta\varphi d^4x$ in the exponent. The cancellation condition

$$-(\Box + m^2)\,\varphi_0(x) = -\eta(x) \tag{10}$$

can be solved in the form

$$\varphi_0(x) = -\int D(x,y)\,\eta(y)\,d^4y \tag{11}$$

where $D(x,y)$ is a Green function of the equation :

$$(-\Box_x - m^2)\,D(x,y) = -\delta(x-y) \tag{12}$$

We obtain the following formula

$$Z_0[\eta] = \left\langle \exp i\int \eta(x)\varphi(x)d^4x \right\rangle_0 =$$

$$= \exp\left\{-\frac{i}{2}\int d^4x\,d^4y\;\eta(x)\,D(x,y)\,\eta(y)\right\} \tag{13}$$

The Green function $D(x,y)$ is still not defined by Eq. (12) because we may add an arbitrary solution f of the homogeneous equation $(-\Box_x - m^2)f = 0$ to any solution of (12). Nevertheless, there exists the most natural form of $D(x,y)$, which can be justified in many ways. One of them is the following. Let us note that $\exp i S$ is an oscillating functional, and $|\exp(iS)| = 1$. So we cannot hope the integral converges well. Let us try to improve convergence by changing :

$$S_0 \to S_{0\varepsilon} = \frac{1}{2}\int \varphi\,(-\Box - m^2 + i\varepsilon)\,\varphi\,d^4x \tag{14}$$

where ε is an arbitrarily small positive parameter. The absolute value of $\exp i S_{0\varepsilon}$ is less than one and vanishes if $\int \varphi^2 dx \to \infty$. If we use the "improved" action $S_{0\varepsilon}$ instead of S_0, and then let ε go to $+0$, we obtain the following Green function

$$D(x-y) = \lim_{\varepsilon \to +0} \frac{1}{(2\pi)^4} \int \frac{d^4k \; e^{i(k, x-y)}}{k^2 - m^2 + i\varepsilon} = D_F(x-y) \qquad (15)$$

This is the so-called Feynman (or causal) Green function. Now it is clear that we have to take $D(x,y)$ in (13) as $D_F(x-y)$.

If we differentiate (13) twice and then put $\eta = 0$, we will have

$$\langle \varphi(x) \varphi(y) \rangle_0 = i \, D_F(x-y) \qquad (16)$$

It is easy to derive now formulae for averages of an arbitrary number of field variables in a free field theory. It is evident that an average of an odd number of fields is equal to zero. For the case of an even number, we arrive at :

Wick's theorem :

> The average of an even number of field variables in a free field theory is equal to the sum of the products of all possible pair averages (the sum over all possible ways to get n pairs from $2n$ objects).

For instance, if $n = 4$, we will have

$$\langle \varphi(x_1) \varphi(x_2) \varphi(x_3) \varphi(x_4) \rangle_0 = \langle \varphi(x_1) \varphi(x_2) \rangle_0 \langle \varphi(x_3) \varphi(x_4) \rangle_0 +$$

$$+ \langle \varphi(x_1) \varphi(x_3) \rangle_0 \langle \varphi(x_2) \varphi(x_4) \rangle_0 + \langle \varphi(x_1) \varphi(x_4) \rangle_0 \langle \varphi(x_2) \varphi(x_3) \rangle_0 \qquad (17)$$

So we may consider Eq. (13) to be equivalent to Wick's theorem.

We are now ready to derive the perturbation theory. Let us expand $\exp i S$:

$$\exp iS = \exp iS_0 \exp iS_1 = \exp iS_0 \sum_{n=0}^{\infty} \frac{(iS_1)^n}{n!} \qquad (18)$$

where

$$S_1 = -\frac{g}{3!}\int \varphi^3(x)\,dx \tag{19}$$

Substituting Eq. (18) into Eq. (6), we come to the following formula for the Green function

$$G(x,y) = \\ = -i\,\frac{\sum_{n=0}^{\infty}\frac{(-ig)^n}{n!(3!)^n}\int e^{iS_0}\varphi(x)\varphi(y)\int dx_1\ldots dx_n\,\varphi^3(x_1)\ldots\varphi^3(x_n)\prod_x d\varphi(x)}{\sum_{n=0}^{\infty}\frac{(-ig)^n}{n!(3!)^n}\int e^{iS_0}\int dx_1\ldots dx_n\,\varphi^3(x_1)\ldots\varphi^3(x_n)\prod_x d\varphi(x)} \tag{20}$$

Dividing both the numerator and the denominator of (20) by the same factor $\int\exp(iS_0)\prod_x d\varphi(x)$, we obtain the averages

$$\langle\varphi^3(x_1)\ldots\varphi^3(x_n)\rangle_0 = \frac{\int e^{iS_0}\,\varphi^3(x_1)\ldots\varphi^3(x_n)\prod_x d\varphi(x)}{\int e^{iS_0}\prod_x d\varphi(x)} \tag{21}$$

in the denominator of (20), and the averages

$$\langle\varphi(x)\varphi(y)\varphi^3(x_1)\ldots\varphi^3(x_n)\rangle_0 \tag{22}$$

in the numerator.

With the help of Wick's theorem we can calculate all the expressions (21) and (22). Feynman noticed that it is convenient to relate some picture (diagram) to every term of the sum in Wick's theorem.

We can arrive at diagrams in the following way. First of all let us relate the diagram

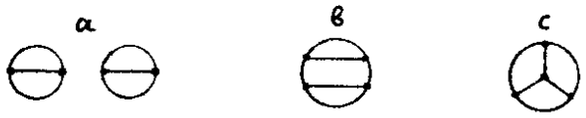
(23)

which consists of n points (every point has three tails), to the denominator average (21). The corresponding diagram for the numerator average has the following form :

(24)

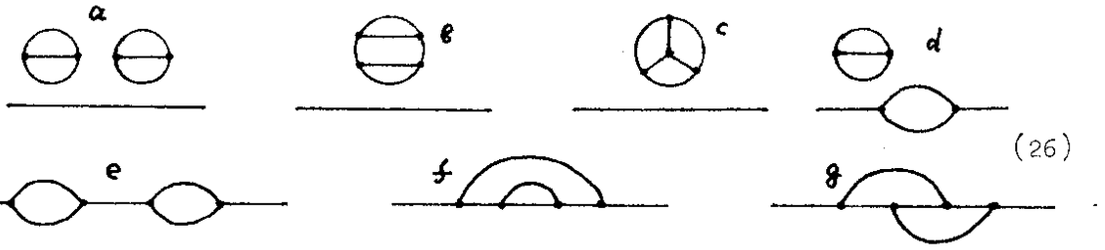
In order to distinguish the pictures, (23) and (24), from real Feynman diagrams, let us call them <u>prediagrams</u>. Each prediagram has its own symmetry group. The order of the group is equal to $R_n = n!(3!)^n$ due to the fact that we may rearrange every pair of points $x_1,\dots,x_n$ and every pair of tails attached to a given point without changing a prediagram. Let us note that we have just $(-ig)^n R_n^{-1}$ factors in series in both the numerator and the denominator in the right-hand side of (20). According to Wick's theorem, each average (21) or (22) is equal to the sum of the products of pair averages. We can obtain a diagram corresponding to a given way of getting n pairs from $2n$ field variables if we link each pair of points (x_i,x_j) by line if there is an average $\langle\varphi(x_i)\varphi(x_j)\rangle_0$ in the product of pair averages. The number of lines is equal to the number of pairs.

For instance, if $n=4$ we obtain the following set of diagrams from prediagram (23) :



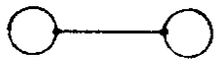
(25)

and the following one for (24) :

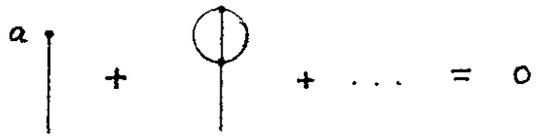


(26)

In (25) and (26) we do not take into account tadpole diagrams like this one :


(27)

We can exclude such diagrams by introducing the counter term $a\int\varphi(x)dx$ in the action functional and by choosing a in order to cancel all tadpole contributions



(28)

In the operator formalism we exclude the tadpole diagrams by choosing the normal form of field operators $:\hat{\varphi}^3(x):$.

Now we can calculate the expression for a given diagram if we integrate the product

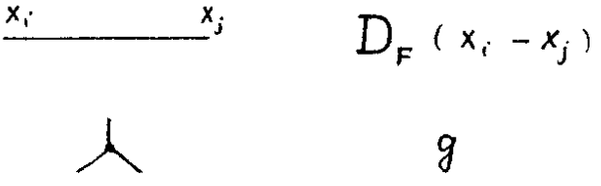
$$\prod \langle \varphi(x_i)\,\varphi(x_j)\rangle_0 \tag{29}$$

over $x_1,\dots,x_n$ and then multiply the result by $(-ig)^n R_n^{-1}\cdot N$. Here N is the number of ways to get a given diagram from a prediagram. It is easy to see that

$$N = \frac{R_n}{r} \tag{30}$$

where r is the order of a group symmetry of the diagram.

We thus come to corresponding rules which allow us to calculate an expression corresponding to a given diagram. Let us relate Green's function $D_F(x_i-x_j)$ to each line connecting points x_i and x_j, and the coupling constant g to every vertex :



(31)

In order to obtain the expression corresponding to a given diagram, we have to take the product of the expressions corresponding to its elements (lines and vertices), and then to integrate this product over all co-ordinates of the vertices. We then have to multiply the result by

$$i^{l-n}\, r^{-1} \tag{32}$$

for diagrams of the denominator, and by

$$i^{l-n-1}\, r^{-1} \tag{33}$$

for diagrams of the numerator. r in (32) and (33) is an order of the symmetry group of a diagram.

It is a well-known fact that the denominator of (20) may be written in an exponential form :

$$\exp \sum_i D_i^{(vac)} \tag{34}$$

Here D_i^{vac} is the contribution of the i^{th} connected vacuum diagram (i.e., a diagram without outgoing lines). The exponent function is due to the symmetry factor

$$\prod_i \left(n_i!\, \tau_i^{n_i}\right)^{-1} \tag{35}$$

for any diagram which consists of n_1 connected components of the first sort, n_2 components of the second sort, and so on.

The numerator of (20) is equal to the factor (34), multiplied by the sum of all connected diagrams without vacuum components. This is why the factor (34) in the numerator cancels that in the denominator, and we may take into account only connected diagrams without vacuum components.

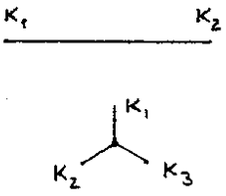
As for concrete calculations, it is more suitable to go to the Fourier transform

$$\varphi(x) = \frac{1}{(2\pi)^4} \int e^{iKx}\, \tilde{\varphi}(K)\, d^4K \tag{36}$$

and to consider Green functions

$$\langle \tilde{\varphi}(K_1) \dots \tilde{\varphi}(K_n) \rangle_0 \tag{37}$$

The elements of the diagram technique in momentum space are the following ones :



$$\delta(K_1+K_2)\,(K_1^2 - m^2 + i0)^{-1}$$

$$g\,\delta(K_1+K_2+K_3) \tag{38}$$

We shall obtain an expression corresponding to a given diagram if we take a product of the expressions corresponding to all diagram elements and then integrate this product over all inner momenta and multiply the result by

$$\frac{1}{z}\left(\frac{i}{(2\pi)^4}\right)^{\ell-n-1} \tag{39}$$

$\ell-n-1 = c$ is nothing but the number of independent loops in a given diagram.

Every Green function in k space has the form

$$G(k_1,\ldots,k_n) = \delta\left(\sum_{i=1}^{n} k_i\right) M(k_1,\ldots,k_n) \tag{40}$$

Here $\delta(\sum_i k_i)$ is due to the energy-momentum conservation law.

If we know the Green functions, we can obtain the S matrix elements by using the well-known formula :

$$S(k_1,\ldots,k_n) = \lim_{k_i^2\to m^2} M(k_1,\ldots,k_n)\left\{\prod_{i=1}^{n}(k_i^2-m^2)\,\theta(\pm k_i^0)\,|2k_i^0|^{-1/2}(2\pi)^{-3/2}\right\} \tag{41}$$

Let me conclude this Lecture by demonstrating how we can go from the integral over all fields to the Hamiltonian form of the functional integral. Let us show the integral over φ fields

$$\int \exp(iS[\varphi]) \prod_x d\varphi(x) \tag{42}$$

to be equivalent to the functional integral

$$\int \exp(iS[\varphi,\pi]) \prod_x d\varphi(x)\,d\pi(x) \tag{43}$$

over φ and π fields. Here

$$S[\varphi,\pi] = \int\left(\pi\partial_0\varphi - \frac{\pi^2}{2} - \frac{(\vec{\partial}\varphi)^2}{2} - \frac{m^2}{2}\varphi^2 - \frac{g}{3!}\varphi^3\right)d^4x \tag{44}$$

The functional $S[\varphi,\pi]$ is equal to $S[\varphi]$ if we change $\pi \to \partial_0\varphi$. The integral (43) is a Hamiltonian functional integral. The corresponding Hamiltonian is equal to

$$H = \int d^3x \left(\frac{\pi^2}{2} + \frac{(\vec{\partial}\varphi)^2}{2} + \frac{m^2\varphi^2}{2} + \frac{g}{3!}\varphi^3 \right) \tag{45}$$

Here $\varphi(x)$ plays the role of a co-ordinate density, $\pi(x)$ is the momentum density.

We can go back from (43) to (42) by performing the shift transformation

$$\pi(x) \to \pi(x) + \partial_0 \varphi(x) \tag{46}$$

Instead of (43) we obtain the integral (42) multiplied by

$$\int \exp\left(-\frac{i}{2} \int \pi^2(x) d^4x \right) \prod_x d\pi(x) \tag{47}$$

We may consider (47) as some normalized factor. In integrals for the Green functions we have factors (47) in both the numerator and the denominator and they cancel one another.

So we see that it is possible to go to the Hamiltonian form of the functional integral by introducing the integral over a new variable (momentum density) into the integral over all fields.

L E C T U R E 4

THE CASE OF FERMIONS; FUNCTIONAL INTEGRALS IN QUANTUM STATISTICAL MECHANICS

In the previous Lecture we have considered the quantization scheme for a Bose field in the functional integral approach. Quantization of Fermi fields may be realized with the help of the integration scheme over anticommuting variables.

We can define the integral over a Fermi field as a limit of the integral over the Grassmann algebra with a finite (even) number of generators

$$x_i \,,\; x_i^* \qquad i = 1, \ldots, n \tag{1}$$

which anticommute with each other :

$$x_i x_j + x_j x_i = 0 \;,\quad x_i^* x_j^* + x_j^* x_i^* = 0 \;,\quad x_i x_j^* + x_j^* x_i = 0 \tag{2}$$

According to Eq. (2) we have $x_i^2 = 0$, $x_j^{*2} = 0$, and each element of the algebra can be written down in the following form :

$$f(x, x^*) = \sum_{a_i, b_i = 0,1} C_{a_1 \ldots a_n, b_1 \ldots b_n}\; x_1^{a_1} \ldots x_n^{a_n}\, x_n^{*b_n} \ldots x_1^{*b_1} \,. \tag{3}$$

Here $C_{a_1 \cdots a_n, b_1 \cdots b_n}$ are complex coefficients.

Let us define the involution operation in the algebra by the following formula :

$$f \to f^* = \sum_{a_i, b_i = 0,1} \bar{C}_{a_1 \ldots a_n, b_1 \ldots b_n}\; x_1^{b_1} \ldots x_n^{b_n}\, x_n^{*a_n} \ldots x_1^{*a_1} \tag{4}$$

We can now introduce the functional integral over the algebra

$$\int f(x, x^*)\, dx^* dx = \int f(x_1, \ldots, x_n, x_1^*, \ldots, x_n^*)\; dx_1^* dx_1 \ldots dx_n^* dx_n \tag{5}$$

We can do it in a unique way, by demanding that the following conditions are valid

$$\int dx_i = 0 \ , \ \int dx_i^* = 0 \ , \ \int x_i dx_i = 1 \ , \ \int x_i^* dx_i^* = 1 \qquad (6)$$

$$\int (c_1 f_1 + c_2 f_2)\, dx^* dx = c_1 \int f_1 dx^* dx + c_2 \int f_2 dx^* dx \qquad (7)$$

Symbols dx_i, dx_i^* must anticommute with each other and with the generators of the algebra. The C_1, C_2 coefficients in (7) are complex numbers.

So if we integrate the polynomial function (3), we obtain the following result

$$\int f\, dx^* dx = C_{1\ldots1,1\ldots1} \qquad (8)$$

The following two formulae are important for future applications to quantum field theory

$$\int \exp(-x^* A x)\, dx^* dx = \det A \qquad (9)$$

$$\frac{\int \exp(-x^* A x + \eta^* x + x^* \eta)\, dx^* dx}{\int \exp(-x^* A x)\, dx^* dx} = \exp \eta^* A^{-1} \eta \qquad (10)$$

where x^*Ax is a quadratic form

$$x^* A x = \sum_{i,k} a_{ik} x_i^* x_k \qquad (11)$$

$\eta^* x$, $x^* \eta$ are linear forms

$$\eta^* x = \sum_i \eta_i^* x_i \ , \quad x^* \eta = \sum_i x_i^* \eta_i \qquad (12)$$

The symbols η_i, η_i^* anticommute with each other and with the generators x_i, x_i^*, This is why we may consider the set η_i, η_i^*, x_i, x_i^* as a set of generators of a wider algebra.

We shall prove (10) if we expand the exp function and then notice that only the n^{th} term gives a contribution to the integral. As for Eq. (11), we can prove it with the help of the shift transformation $x \to x+\eta$, $x^* \to x^* + \tilde{\eta}^*$ which cancel the linear terms in the exponent.

Let me now dwell upon the functional integrals in quantum statistical mechanics.

First of all, let us consider the system of Bose particles at finite temperature T in the finite cubic valume $V = L^3$ *). It turns out that we can associate a functional integral over a space of complex functions $\psi(\vec{x},\tau)$ ($\vec{x} \in V$, $\tau \in [0,\beta]$, $\beta = T^{-1}$) with our Bose system.

We shall demand $\psi(\vec{x},\tau)$ to be a periodic function, so $\psi(\vec{x},0) = \psi(\vec{x},\beta)$. Besides this it is suitable to impose periodic boundary conditions on the $\vec{x}$ variable. We may thus expand $\psi(\vec{x},\tau)$, $\bar{\psi}(\vec{x},\tau)$ in Fourier series

$$\psi(\vec{x},\tau) = \frac{1}{\sqrt{\beta V}} \sum_{\vec{k},\omega} e^{i(\vec{k}\vec{x}+\omega\tau)} a(\vec{k},\omega)$$

$$\bar{\psi}(\vec{x},\tau) = \frac{1}{\sqrt{\beta V}} \sum_{\vec{k},\omega} e^{-i(\vec{k}\vec{x}+\omega\tau)} a^+(\vec{k},\omega) \qquad (13)$$

Here $a(\vec{k},\omega)$, $a^+(\vec{k},\omega)$ are Fourier coefficients

$$\omega = 2\pi n/\beta, \quad k_i = 2\pi n_i/L, \quad n, n_1, n_2, n_3 \text{ are integers} \qquad (14)$$

Let us introduce the functional

$$S = \int_0^\beta d\tau \int d^3x \, \bar{\psi}(\vec{x},\tau) \partial_\tau \psi(\vec{x},\tau) - \int_0^\beta H'(\tau)\, d\tau \qquad (15)$$

which plays the role of an action functional of the system. Here $H'(\tau)$ is the so-called generalized Hamiltonian

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*) I will use the system of units where $\hbar = k_B = 1$ ($\hbar$ is the Planck constant and k_B is the Boltzmann constant).

$$H'(\tau) = \int d^3x \left(\frac{1}{2m} \vec{\partial} \bar{\psi}(\vec{x},\tau) \vec{\partial} \psi(\vec{x},\tau) - \lambda \bar{\psi}(\vec{x},\tau) \psi(\vec{x},\tau) \right) +$$

$$+ \frac{1}{2} \int d^3x d^3y \, u(\vec{x}-\vec{y}) \, \bar{\psi}(\vec{x},\tau) \, \bar{\psi}(\vec{y},\tau) \, \psi(\vec{y},\tau) \, \psi(\vec{x},\tau) \qquad (16)$$

$u(\vec{x}-\vec{y})$ is a pair potential function, λ is a chemical potential coefficient.

We can now define the Green function of the system by :

$$G(\vec{x},\tau ; \vec{y},\tau') = - \langle \psi(\vec{x},\tau) \bar{\psi}(\vec{y},\tau') \rangle \qquad (17)$$

where

$$\langle \psi(\vec{x},\tau) \bar{\psi}(\vec{y},\tau') \rangle = \frac{\int e^{S} \psi(\vec{x},\tau) \bar{\psi}(\vec{y},\tau') \, d\bar{\psi} d\psi}{\int e^{S} \, d\bar{\psi} \, d\psi} \qquad (18)$$

is a quotient of two functional integrals over a space of complex functions. We denote the measure of integration as $d\bar{\psi} d\psi$.

It is convenient to use the Fourier decomposition (13) and to rewrite the S functional in terms of the Fourier coefficients a, $a^{\dagger}$

$$S = S_0 + S_1 \qquad (19)$$

where

$$S_0 = \sum_{p} \left(i\omega - \frac{\kappa^2}{2m} + \lambda \right) a^{+}(p) \, a(p) \, , \qquad (20)$$

$$S_1 = - \frac{1}{4\beta V} \sum_{p_1+p_2=p_3+p_4} \left(V(\vec{\kappa}_1-\vec{\kappa}_3) + V(\vec{\kappa}_1-\vec{\kappa}_4) \right) a^{+}(p_1) a^{+}(p_2) a(p_4) a(p_3) \qquad (21)$$

Here p denotes the set $(\vec{k},\omega)$, $V(\vec{k})$ is the Fourier transform of $u(\vec{x})$ which is defined by :

$$u(\vec{x}) = \frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\vec{x}} V(\vec{k}) \tag{22}$$

In terms of a, a^+ we have the following explicit form for the measure

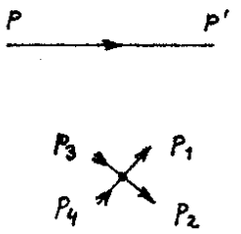
$$d\bar{\psi}\,d\psi = \prod_{p} da^+(p)\,da(p) \tag{23}$$

We can now build up the perturbative scheme for the Green functions like we did for the relativistic Bose field in Lecture 3. First of all we can calculate the generating functional for the free field theory [in the case of a vanishing pair potential $u(\vec{x}-\vec{y})$]. The corresponding formula is equivalent to the Wick theorem. We then use the expansion :

$$e^{S} = e^{S_0} e^{S_1} = e^{S_0} \sum_{n=0}^{\infty} \frac{S_1^n}{n!} \tag{24}$$

where S_0 is a quadratic form, S_1 is a fourth degree form in the S functional.

So we arrive at diagram techniques with the following elements :



$$\delta_{pp'}\left(i\omega - \frac{k^2}{2m} + \lambda\right)^{-1}$$

$$V(\vec{k}_1-\vec{k}_3) + V(\vec{k}_1-\vec{k}_4) \tag{25}$$

In order to obtain an expression corresponding to a given diagram, we have to take a product of the expressions corresponding to all diagram elements, and then sum up over all independent four-momenta $p=(\vec{k},\omega)$ and multiply the result by

$$\frac{1}{2}\left(\frac{-1}{\beta V}\right)^{l-n} \tag{26}$$

for vacuum diagrams, and by

$$\frac{1}{r}\left(\frac{-1}{\beta V}\right)^{\ell-n-1} \tag{27}$$

for diagrams corresponding to a one-particle Green function. r in Eqs. (26) and (27) is an order of a symmetry group of a given diagram.

The exponentiation of contributions of vacuum diagrams takes place in statistical mechanics as well as in relativistic field theory. This is the inherent feature of a perturbative scheme independent of concrete details of the theory. Therefore we may take into account only connected diagrams (without vacuum components) for the Green functions.

However, the sum of all vacuum diagrams has a physical meaning in statistical mechanics. Namely, we have the formula :

$$\frac{\int e^{S}\, d\bar{\psi}\, d\psi}{\int e^{S_0}\, d\bar{\psi}\, d\psi} = \frac{Z}{Z_0} = \exp \beta(\Omega_0 - \Omega) \tag{28}$$

Here Z_0, Z are partition functions for ideal and non-ideal systems, $\Omega_0 = -p_0 V$, $\Omega = -pV$, where p_0 is the pressure in an ideal system, and p is the pressure in a non-ideal one. The left-hand side of (18) is equal to

$$\exp\left(\sum_i D_i^{(vac)}\right) \tag{29}$$

where $\sum_i D_i^{(vac)}$ is the sum of all connected vacuum diagrams. So we arrive at the formula :

$$P = P_0 + \frac{1}{\beta V}\sum_i D_i^{(vac)} \tag{30}$$

If we want to deal with a Fermi system at finite temperature T in the volume $V = L^3$, we have to use the anticommutative field functions $\psi(\vec{x},\tau)$, $\bar{\psi}(\vec{x},\tau)$. It turns out that ψ, $\bar{\psi}$ must be antiperiodic :

$$\psi(\vec{x},\beta) = -\psi(\vec{x},0) \quad , \quad \bar{\psi}(\vec{x},\beta) = -\bar{\psi}(\vec{x},0) \tag{31}$$

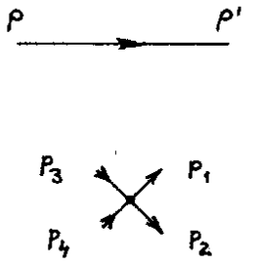
So we have the following Fourier series for the ψ, $\bar{\psi}$ in the Fermi case :

$$\psi(\vec{x},\tau) = \frac{1}{\sqrt{\beta V}} \sum_p e^{i(\omega\tau + \vec{k}\vec{x})} a(p)$$

$$\bar{\psi}(\vec{x},\tau) = \frac{1}{\sqrt{\beta V}} \sum_p e^{-i(\omega\tau + \vec{k}\vec{x})} a^+(p) \qquad (32)$$

where $\omega = (2n+1)\pi/\beta$ are the "Fermi frequencies". Let us notice that the Fourier coefficients $a(p)$, $a^+(p)$ in (32) may be considered as generators of an infinite-dimensional Grassmann algebra.

The S functional and the Hamiltonian are of the same form – (15), (16) – as those for the Bose system. So we arrive at diagram techniques with the elements



$$\delta_{pp'} \left(i\omega - \frac{\kappa^2}{2m} + \lambda \right)^{-1} \qquad (\omega = (2n+1)\pi/\beta)$$

$$V(\vec{k}_1 - \vec{k}_3) - V(\vec{k}_1 - \vec{k}_4) \qquad (33)$$

The "Fermi" diagram techniques differ from the "Bose" ones on the following points :

1) Fermi frequencies $\omega = (2n+1)\pi/\beta$ are odd instead of even Bose ones $\omega = 2n\pi/\beta$;

2) we have an antisymmetrized potential $V(\vec{k}_1-\vec{k}_3) - V(\vec{k}_1-\vec{k}_4)$ in (33) instead of a symmetrized one in (25);

3) the factors with which we have to multiply the results afer summation over independent momenta are

$$(-1)^F \frac{1}{2} \left(\frac{-1}{\beta V} \right)^{\ell - n} \qquad (34)$$

for vacuum diagrams, and

$$(-1)^F \frac{1}{2} \left(\frac{-1}{\beta V}\right)^{l-n-1} \tag{35}$$

for diagrams corresponding to the one-particle Green function. There is an extra factor $(-1)^F$ in (34) and (35), where F is a number of independent closed fermion loops.

The diagram techniques which we have just obtained in the functional integral approach are nothing but the well-known Matzubura-Abrikosov-Gorkov-Dzyaloshinsky perturbation theory for the temperature Green functions. The functional integral method gives us the shortest way to obtain it. Moreover, functional methods are very useful in many cases when the perturbative scheme is not applicable in its standard form, for instance for the superfluid Bose or Fermi systems.

LECTURE 5

QUANTIZATION OF GAUGE FIELDS

Gauge fields are geometrical objects. They may be considered as connection functions of some fiber bundles. The base of a bundle is a Minkowski space-time and the fiber is a finite-dimensional "isotopic" space.

If we quantize a gauge field we must take into account its geometrical nature. If two fields can be obtained one from the other by gauge transformation, they are non-distinguishable geometrically and physically. Therefore it is natural to unite in one class all the fields which we can obtain from a given one by gauge transformations. Obviously we need a theory which deals with classes of fields rather than with fields themselves. Such a theory can be developed in a functional integral approach if we integrate not over all fields but over some surface which has only one common point with every class.

Let A be a gauge field with A^a_μ components, where $\mu = 0,1,2,3$ is a space-time index, and a is an "isotopic" index. Let us denote by A^Ω the field which one obtains from A by gauge transformation Ω. The set A^Ω, where A is fixed and Ω goes over G, is the so-called orbit of the gauge group.

The action functional of the gauge field is gauge invariant

$$S[A^\Omega] = S[A] \tag{1}$$

We will suppose a local measure of the function integration

$$d_\mu[A] = \prod_x \prod_{\mu,a} dA^a_\mu(x) \tag{2}$$

to be gauge invariant as well as $S[A]$. This is really the case for both the electromagnetic and Yang-Mills fields which we will consider below. The gauge invariance of the action functional and the measure implies a naïve functional integral

$$\int e^{iS[A]}\, d_\mu[A] \tag{3}$$

to be proportional to the "group volume"

$$\int \prod_x d\Omega(x) \tag{4}$$

Thus it is natural to extract factor (4) from (3). We can do it by going to the integral over some surface

$$f(A) = 0 \tag{5}$$

which has one and only one common point with every gauge group orbit. It means that the equation $f(A^\Omega) = 0$ has one and only one solution Ω for any fixed A. Let us define a gauge invariant functional $\Delta_f[A]$ by the equation

$$\Delta_f[A] \int \prod_x \delta\left(f\left(A^{\Omega(x)}\right)\right) d\Omega(x) = 1 \tag{6}$$

Here $\prod_x \delta(f(A(x)))$ is the so-called δ functional which is defined by the rule of how to integrate it with other functionals.

Now let us put the left-hand side of (6) into (3) and, after a change of variables $A^\Omega \to A$, we obtain just the factor (4) multiplied by the integral

$$\int e^{iS[A]} \Delta_f[A] \left[\prod_x \delta(f(A))\right] d\mu[A] \tag{7}$$

This integral is a starting point for gauge field quantization. It does not depend on a choice of the surface $f(A) = 0$. We can prove it by substituting another "unity"

$$1 = \Delta_g[A] \int \prod_x \delta\left(g\left(A^{\Omega(x)}\right)\right) d\Omega(x) \tag{8}$$

into (7). Making the change of variables : $A^\Omega \to A$ and $A \to A^{\Omega^{-1}}$, we obtain the integral which differs from (2) by making the following change : $f(A) \to g(A)$ and $\Delta_f[A] \to \Delta_g[A]$.

We can generalize (7), replacing the δ functional by an arbitrary non-gauge invariant functional $F[A]$ and $\Delta_f[A]$ by $\phi[A]$ which is defined by the equation :

$$\phi[A] \int F[A^{\Omega}] \prod_x d\Omega(x) = 1 \tag{9}$$

So we come to the following functional integral

$$\int e^{iS[A]} F[A] \phi[A] d_\mu[A] \tag{10}$$

We will justify the outlined quantization scheme if we show the integral (7) [or (10)] to be equivalent to the Hamiltonian form of the functional integral. We shall do this for both the electromagnetic and Yang-Mills fields considered below.

The electromagnetic field is the simplest gauge field. Its action functional

$$S = -\frac{1}{4} \int (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 d^4x \tag{11}$$

is invariant under the Abelian group of transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x) \tag{12}$$

as well as under the local integration measure

$$d_\mu[A] = \prod_x \prod_\mu d A_\mu(x) \tag{13}$$

Let us now choose a non-gauge invariant functional $F[A]$. We will consider three possible ways of doing it

$$F_1[A] = \prod_x \delta(\partial_\mu A_\mu(x)) \qquad \text{Lorentz-Landau gauge}$$

$$F_2[A] = \prod_x \delta(\partial_i \cdot A_i(x)) \qquad \text{Coulomb gauge} \tag{14}$$

$$F_3[A] = \exp\left(-\frac{i}{2d_\ell} \int (\partial_\mu A_\mu(x))^2 d^4x\right) \qquad \text{Longitudinal gauge}$$

The corresponding ϕ functionals are defined by the formulae

$$\phi_1^{-1} = \int \prod_x \delta(\partial_\mu (A_\mu + \partial_\mu \lambda))\, d\lambda(x)$$

$$\phi_2^{-1} = \int \prod_x \delta(\partial_i (A_i + \partial_i \lambda))\, d\lambda(x) \qquad (15)$$

$$\phi_3^{-1} = \int \exp\left\{-\frac{i}{2d\ell}\int (\partial_\mu (A_\mu + \partial_\mu \lambda))^2 d^4x\right\} \prod_x d\lambda(x)$$

These functionals do not depend on A_μ at all. We can check it, performing a shift transformation $\lambda \to \lambda - \Box^{-1}\, \partial_\mu A_\mu$ in the first and third integrals in (15) and $\lambda \to \lambda - \Delta^{-1}\, \partial_i A_i$ in the second one. This is why we may put

$$\phi_1 = \phi_2 = \phi_3 = 1 \qquad (16)$$

Thus we come to the following functional integrals

$$\text{a)} \int \exp(iS[A]) \left(\prod_x \delta(\partial_\mu A_\mu)\right) d\mu[A] \qquad \text{Lorentz-Landau gauge}$$

$$\text{b)} \int \exp(iS[A]) \left(\prod_x \delta(\partial_i A_i)\right) d\mu[A] \qquad \text{Coulomb gauge} \qquad (17)$$

$$\text{c)} \int \exp\left\{i\left(S[A] - \frac{1}{2d\ell}\int (\partial_\mu A_\mu)^2 d^4x\right)\right\} d\mu[A] \qquad \text{Longitudinal gauge}$$

Green's functions of the electromagnetic field are different in different gauges. We can obtain them from a generating functional. For instance, for the Lorentz-Landau gauge we have

$$Z[\eta] = \frac{\int \exp\left(iS[A] + i\int \eta_\mu A_\mu d^4x\right)\left(\prod_x \delta(\partial_\mu A_\mu)\right) d\mu[A]}{\int \exp(iS[A])\left(\prod_x \delta(\partial_\mu A_\mu)\right) d\mu[A]} \qquad (18)$$

We can calculate (18) by performing a shift transformation $A_\mu \to A_\mu + A^{(o)}_\mu$ in order to cancel the linear functional $\int \eta_\mu A_\mu d^4x$ in the exponent. The $A^{(o)}_\mu$ field must obey the gauge equation $\partial_\mu A^{(o)}_\mu = 0$ as well as A_μ. Therefore

$$Z[\eta] = \exp\left(-\frac{i}{2}\int d^4x\, d^4y\, \eta_\mu(x)\, D_{\mu\nu}(x-y)\, \eta_\nu(y)\right) \tag{19}$$

which is equivalent to the Wick theorem. Here

$$D_{\mu\nu}(x-y) = \frac{1}{(2\pi)^4}\int d^4k\, e^{i(k,x-y)}\, \frac{-\delta_{\mu\nu}k^2 + k_\mu k_\nu}{(k^2+i0)^2} \tag{20}$$

is a Green function of the electromagnetic field in the Lorentz–Landau gauge.

In quantum electrodynamics we deal with both electromagnetic and electron-positron fields, and we have the following action functional

$$S[\bar\psi, \psi, A] = \int\left(\bar\psi(i\gamma_\mu(\partial_\mu - ieA_\mu) - m)\psi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2\right)d^4x \tag{21}$$

ψ, $\bar\psi$ in (21) are the four-component spinors which can be regarded as elements of the infinite-dimensional Grassmann algebra. We can quantize this system by using a functional integral over both electromagnetic and spinor fields. For instance, in the Lorentz–Landau gauge, we have

$$\int \exp(iS[\bar\psi, \psi, A]) \prod_x \delta(\partial_\mu A_\mu(x)) \prod_\alpha d\bar\psi_\alpha(x)\, d\psi_\alpha(x) \prod_\mu dA_\mu(x) \tag{22}$$

Functional integrals and perturbative schemes for Green functions are different for different gauges. Nevertheless all physical results do not depend on gauge at all.

In the conclusion, let me dwell upon the Hamiltonian form of the functional integral for an electromagnetic field. Here it is suitable to use a functional integral in the Coulomb gauge [the second one in (17)]. This integral is equivalent to

$$\int \exp(i S[A_\mu, F_{\mu\nu}]) \prod_x \delta(\partial_i A_i) \prod_\mu dA_\mu(x) \prod_{\mu<\nu} dF_{\mu\nu}(x) \tag{23}$$

where

$$S[A_\mu, F_{\mu\nu}] = \int \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} F_{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) \right) d^4x \tag{24}$$

In (23) we integrate over A_μ and $F_{\mu\nu}(x)$ as over independent variables. In order to prove that (23) is equivalent to (17.b), we can do a shift transformation $F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu$ and obtain the product of two integrals

$$\int \exp\left(\frac{i}{4} \int F_{\mu\nu} F_{\mu\nu} \, d^4x \right) \prod_{x,\mu<\nu} dF_{\mu\nu}(x) \cdot$$

$$\cdot \int \exp\left(-\frac{i}{4} \int (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 d^4x \right) \prod_x \delta(\partial_i A_i) \prod_\mu dA_\mu(x) \tag{25}$$

instead of (23).

But let us now transform (23) in another way. First of all let us rewrite (24) in a three-dimensional form

$$S[A,F] = \int \left(E_i \partial_0 A_i - \frac{1}{2} \vec{E}^2 + \frac{1}{2} \vec{H}^2 - (\vec{H}, \operatorname{rot} \vec{A}) + A_0 \partial_i E_i \right) d^4x \tag{26}$$

Here

$$E_i = F_{0i} \quad i = 1,2,3 \; ; \quad H_1 = F_{23}, \; H_2 = F_{31}, \; H_3 = F_{12} \tag{27}$$

We can integrate over H_i, $(i = 1,2,3)$. It is equivalent to changing

$$\vec{H} \to \operatorname{rot} \vec{A} \tag{28}$$

in (26). Then we can integrate over A_0. This integral is proportional to

$$\prod_x \delta(\partial_i E_i) \tag{29}$$

So from (23) we get the following integral

$$\int \exp(iS[A_i, E_i]) \prod_x \delta(\partial_i A_i)\, \delta(\partial_i E_i) \prod_i dA_i(x)\, dE_i(x) \tag{30}$$

where

$$S[A_i, E_i] = \int \left(E_i \partial_0 A_i - \frac{1}{2} \vec{E}^2 - \frac{1}{2} (\mathrm{rot}\, \vec{A})^2 \right) d^4x \tag{31}$$

This is an action functional in a Hamiltonian form. The corresponding Hamiltonian is

$$H = \frac{1}{2} \int d^3x \left(\vec{E}^2 + (\mathrm{rot}\, \vec{A})^2 \right) \tag{32}$$

The functional integral (30) is an infinite dimensional analogue of functional integrals for systems with constraints (see Lecture 2). Here the vector potential $\vec{A}$ plays the role of a co-ordinate, the electric field $\vec{E}$ is an analogue of the conjugate momentum. Equation $\partial_i A_i = 0$ is a supplementary condition. In (30) we integrate over the transverse components of vectors $\vec{A}$, $\vec{E}$, which are real dynamical variables and correspond to two independent polarizations of the electromagnetic field.

Problems :

1. Obtain Green functions of an electromagnetic field in both Coulomb and longitudinal gauges.
2. Obtain a perturbative scheme for spinor electrodynamics in the Lorentz-Landau gauge (22).
3. Consider a Hamiltonian form of the functional integral for spinor electrodynamics.

L E C T U R E 6

THE YANG-MILLS FIELD

The Yang-Mills field is the simplest case in which we deal with a non-Abelian gauge group.

It is convenient to describe a Yang-Mills vector field $b_\mu^a(x)$ by matrices in a Lie algebra

$$B_\mu(x) = \sum_{a=1}^{n} b_\mu^a(x)\,\tau_a \tag{1}$$

We suppose τ_a to be normalized by the condition

$$\mathrm{tr}\,\tau_a\tau_b = -2\,\delta_{ab} \tag{2}$$

Choosing an adjoint representation of the Lie algebra, we will have

$$(B_\mu)_{ab} = (\tau_c)_{ab}\,b_\mu^c = t_{abc}\,b_\mu^c \tag{3}$$

where t_{abc} are structure coefficients of the Lie group. The action functional of a Yang-Mills field

$$S[B] = \frac{1}{g}\int \mathrm{tr}\,F_{\mu\nu}F_{\mu\nu}\,d^4x \tag{4}$$

where

$$F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + \varepsilon\,[B_\mu, B_\nu] \tag{5}$$

is invariant under the gauge transformations

$$B_\mu \to \Omega\,B_\mu\,\Omega^{-1} + \frac{1}{\varepsilon}\,\partial_\mu\Omega\cdot\Omega^{-1} \tag{6}$$

Let us apply the general scheme of quantization outlined in the previous Lecture to the Yang-Mills field. We will consider the functional integral in different gauges and different purturbative schemes. In the conclusion we we dwell upon the Hamiltonian form of the functional integral and discuss Gribov's ambiguity of the Coulomb gauge.

We will consider two different gauge conditions :

$$\partial_\mu B_\mu = 0 \qquad \text{Lorentz-Landau gauge} \qquad (7.a)$$

$$\partial_i B_i = 0 \qquad \text{Coulomb gauge} \qquad (7.b)$$

The corresponding factors $\Delta_L[B]$, $\Delta_R[B]$ depend on B non-trivially in a non-Abelian case. Nevertheless, we can obtain them in a closed form as the determinants of some operators.

First of all, let us consider the equation for $\Delta_L[B]$

$$\Delta_L[B] \int \prod_x \delta(\partial_\mu B_\mu^\Omega)\, d\Omega(x) = 1 \qquad (8)$$

We really need the Δ_L functional only for fields obeying the Lorentz-Landau equation (7.a). In this case the whole contribution to the integral over the gauge group comes from an arbitrarily small neighbourhood of the unit element of the group itself. In this neighbourhood we have

$$\Omega = I + \varepsilon u(x)$$

$$\partial_\mu B_\mu^\Omega = \partial_\mu (B_\mu + \varepsilon [u, B_\mu] + \partial_\mu u) = \qquad (9)$$

$$= \Box u - \varepsilon [B_\mu, \partial_\mu u] = \hat{A} u$$

So we have

$$\Delta_L[B] = \det (\Box - \varepsilon B_\mu \partial_\mu) \qquad (10)$$

Analogously, for the Coulomb gauge, we obtain

$$\Delta_R[B] = \det (\Delta - \varepsilon B_i \partial_i) \qquad (11)$$

We can write down the determinant (10) as a functional integral over some anticommutative fields $\eta^a(x)$, $\bar{\eta}^a(x)$

$$\det (\Box - \varepsilon B_\mu \partial_\mu) =$$

$$= \int \exp \left\{ i \int L[B_\mu, \eta^a, \bar{\eta}^a]\, d^4x \right\} \prod_{x,a} d\bar{\eta}^a(x)\, d\eta^a(x) \qquad (12)$$

where

$$L[B_\mu, \eta^a, \bar{\eta}^a] = -\frac{1}{2} \operatorname{tr} \bar{\eta} (\Box - \varepsilon B_\mu \partial_\mu) \eta = \tag{13}$$

$$= \bar{\eta}^a \Box \eta^a - \varepsilon t_{abc} B_\mu^c \bar{\eta}^a \partial_\mu \eta^b$$

We may say that the Yang-Mills vector field interacts with some fictitious scalar Fermi fields $\eta^a(x)$, $\bar{\eta}^a(x)$. We may consider $L[B_\mu, \eta^a, \bar{\eta}^a]$ as an addendum to the Lagrangian of a Yang-Mills field, and we may speak about an effective action functional

$$S[B, \bar{\eta}, \eta] = \int d^4x \left(\frac{1}{8} \operatorname{tr} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} \operatorname{tr} \bar{\eta} (\Box - \varepsilon B_\mu \partial_\mu) \eta \right) \tag{14}$$

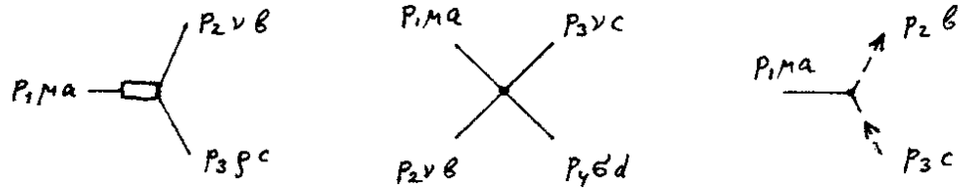
So we come to the functional integral

$$\int \exp(i S[B, \bar{\eta}, \eta]) \prod_x \delta(\partial_\mu B_\mu) \prod_\mu dB_\mu \, d\bar{\eta} \, d\eta \tag{15}$$

and can develop a perturbative scheme for its calculation. This scheme can be formulated as a diagram technique with the following elements

$G^{ab}_{\mu\nu}$ $\qquad$ G^{ab}

(16)



$V^{abc}_{\mu\nu\rho}$ $\qquad$ $V^{abcd}_{\mu\nu\rho\sigma}$ $\qquad$ V^{abc}_{μ}

where

$$
\begin{aligned}
G^{ab}_{\mu\nu}(p) &= -\delta_{ab}\,(p^2\delta_{\mu\nu} - p_\mu p_\nu)\,(p^2+i0)^{-2} \\
G^{ab}(p) &= -\delta_{ab}\,(p^2+i0)^{-1} \\
V^{abc}_{\mu,\nu\rho} &= i\varepsilon\, t_{abc}\,(p_{1\nu}\delta_{\mu\rho} - p_{1\rho}\delta_{\mu\nu}) \\
V^{abcd}_{\mu\nu\rho\sigma} &= \varepsilon^2\, t_{abe}\, t_{cde}\,(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) \\
V^{abc}_{\mu} &= \frac{i\varepsilon}{2}\, t_{abc}\,(p_3 - p_2)_\mu
\end{aligned}
\tag{17}
$$

We shall obtain an expression corresponding to a given diagram if we take the product of the expressions corresponding to all its elements (lines and vertices) and then integrate this product over all independent four-momenta, sum over all inner indices and multiply the result by

$$
r^{-1}\left(\frac{i}{(2\pi)^4}\right)^{\ell-v-1}(-1)^s \tag{18}
$$

here v is the number of vertices, ℓ is the number of lines, s is the number of closed fermion loops and r is an order of the symmetry group of the diagram.

This perturbation theory is not the only possible one. We can get a different form of diagram techniques in the so-called first-order formalism. Let us consider the integral

$$
\int \exp(iS[B,F])\;\Delta_L[B]\prod_x \delta(\partial_\mu B_\mu)\prod_\mu dB_\mu \prod_{\mu<\nu} dF_{\mu\nu} \tag{19}
$$

where

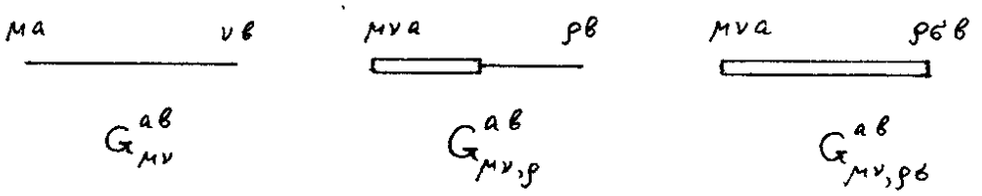
$$S[B,F] = \int d^4x \left(-\frac{1}{8} \operatorname{tr} F_{\mu\nu} F_{\mu\nu} + \frac{1}{4} \operatorname{tr} F_{\mu\nu} \left(\partial_\nu B_\mu - \partial_\mu B_\nu + \varepsilon [B_\mu, B_\nu] \right) \right) \tag{20}$$

The integral (19) is equivalent to (15), and come back to (15) after a shift transformation

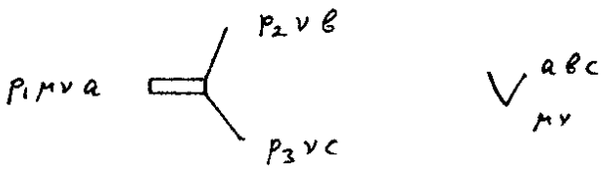
$$F_{\mu\nu} \to F_{\mu\nu} + \partial_\nu B_\mu - \partial_\mu B_\nu + \varepsilon [B_\mu, B_\nu] \tag{21}$$

in (19).

If we try to build up a perturbative scheme for the integral (19) (after writing down $\Delta_L[B]$ as an integral over a scalar fermion field) we get a diagram technique with the following elements :



(22)



here one line corresponds to a B_μ field and a double line to a $F_{\mu\nu}$ field.

Expressions corresponding to the diagram elements (22) are :

$$G^{ab}_{\mu\nu} = \delta_{ab} \left(-p^2 \delta_{\mu\nu} + p_\mu p_\nu \right) \left(p^2 + i0 \right)^{-2} \tag{23}$$

$$G^{ab}_{\mu\nu,\rho} = i\delta_{ab}(p_\nu\delta_{\mu\rho} - p_\mu\delta_{\nu\rho})(p^2+i0)^{-1}$$

$$G^{ab}_{\mu\nu,\rho\sigma} = \delta_{ab}\{\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} -$$

$$- (p^2+i0)^{-1}(\delta_{\mu\rho}p_\nu p_\sigma + \delta_{\nu\sigma}p_\mu p_\rho - \delta_{\mu\sigma}p_\nu p_\rho - \delta_{\nu\rho}p_\mu p_\sigma)\} \qquad (23)$$

Ctd.

$$V^{abc}_{\mu\nu} = \varepsilon t_{abc}$$

Lines of fictitious particles and vertices, describing their interaction with the vector B_μ field, are the same as in the second-order formalism (16) and (17). The first-order formalism is convenient if we want to obtain a canonical quantization of the Yang-Mills field. Let me demonstrate how we can go to the explicit Hamiltonian form of the theory. The starting point is the generating functional for Green's functions in the Coulomb gauge.:

$$Z[\eta] = \frac{\int \exp\left(iS[B,F] + i\int(\eta_{\mu a}f^a_\mu + \frac{1}{2}\eta_{\mu\nu a}f^a_{\mu\nu})d^4x\right)\Delta_R[B]\prod_x\delta(\partial_i B_i)dBdF}{\int \exp(iS[B,F])\,\Delta_R[B]\prod_x\delta(\partial_i B_i)\,dBdF} \qquad (24)$$

We will consider three-dimensional transverse fields B_i, F_{oi} as dynamical variables. We suppose the following conditions to be valid

$$\eta_{oa} = \eta_{ika} = \partial_i\eta_{oia} = \partial_i\eta_{ia} = 0 \qquad (25)$$

These conditions mean that there are only source terms of dynamical variables in $Z[\eta]$. Using three-dimensional notations, we can write the Lagrangian in the following form :

$$tr\left(-\frac{1}{8}F_{ik}F_{ik} + \frac{1}{4}F_{oi}F_{oi} + \frac{1}{4}F_{ik}(\partial_k B_i - \partial_i B_k + \varepsilon[B_i,B_k])\right.$$
$$\left. - \frac{1}{2}F_{oi}\partial_o B_i - \frac{1}{2}B_o(\partial_i F_{oi} - \varepsilon[B_i,F_{oi}])\right) \qquad (26)$$

We can integrate over B_o, F_{oi} due to the absence of sources in these variables. Integration over F_{ik} is equivalent to the change

$$F_{ik} \to H_{ik} = \partial_k B_i - \partial_i B_k + \varepsilon [B_i, B_k] \tag{27}$$

in (26). Integration over B_o gives us a δ functional

$$\prod_x \delta(\partial_i F_{oi} - \varepsilon [B_i, F_{oi}]) \tag{28}$$

Let us now substitute into the integral over B_i, F_{oi} the integral

$$\int \prod_x \delta(\Delta C + \partial_i F_{oi})\, dC(x) \tag{29}$$

which really does not depend on F_{oi} at all. Then, after a shift transformation $F_{oi} \to F_{oi} - \partial_i C$, we have

$$\prod_x \delta(\Delta C + \partial_i F_{oi}) \to \prod_x \delta(\partial_i F_{oi}) \tag{30}$$

$$\prod_x \delta(\partial_i F_{oi} - \varepsilon [B_i, F_{oi}]) \to \prod_x \delta(\Delta C - [B_i, \partial_i C] + \varepsilon [B_i, F_{oi}]) \tag{31}$$

Let $C_o(x)$ be a solution of the equation

$$\Delta C - \varepsilon [B_i, \partial_i C] = -\varepsilon [B_i, F_{oi}] \tag{32}$$

We can solve this equation if we know the Green's function $D(x,y;B)$ of the operator in the left-hand side of (32). Namely, we have

$$C_o(x) = -\varepsilon \int D(x,y;B) [B_i(y), F_{oi}(y)]\, d^3y \tag{33}$$

After the $C \to C + C_o$ shift, we obtain

$$\prod_x \delta(\Delta C - \varepsilon [B_i, \partial_i C]) \tag{34}$$

and can put $C = 0$ elsewhere, except in this functional. The integral over the C variables

$$\int \prod_x \delta'(\Delta C - \varepsilon [B_i, \partial_i C]) \, dC(x) \tag{35}$$

cancels the factor $\Delta_R[B]$.

So we come to the integral

$$\frac{\int \exp\left\{ i S[B_i, F_{oi}] + i \int (\eta_{ai} b_i^a + \eta_{aoi} f_{oi}^a) d^4x \right\} \prod_x \delta(\partial_i B_i) \delta(\partial_i F_{oi}) \prod_i dB_i dF_{oi}}{\int \exp\left\{ i S[B_i, F_{oi}] \right\} \prod_x \delta(\partial_i B_i) \delta(\partial_i F_{oi}) \prod_i dB_i dF_i} \tag{36}$$

Here

$$S[B_i, F_{oi}] = \int dx_o \left(\int f_{oi}^a \partial_o b_i^a \, d^3x - H \right) \tag{37}$$

$$H = \int d^3x \left(\frac{1}{4} h_{ik}^a h_{ik}^a + \frac{1}{2} f_{oi}^a f_{oi}^a + \frac{1}{2} \partial_i C_o^a \partial_i C_o^a \right) \tag{38}$$

The action functional $S[B_i, F_{oi}]$ has a Hamiltonian form, where b_i^a and f_{oi}^a play the role of co-ordinates and momenta, respectively. The formalism of a functional integration over canonical conjugate variables is equivalent to a canonical quantization. For our case we shall obtain a canonical operator formalism by replacing the fields $b_i^a(x)$, $f_{oi}^a(x)$ by operators obeying the following commutation relations

$$\begin{aligned} \left[\hat{b}_i^a(\vec{x}), \hat{f}_{oj}^b(\vec{y}) \right] &= i \delta_{ab} \delta_{ij}^{tr}(\vec{x} - \vec{y}) = \\ &= \frac{i \delta_{ab}}{(2\pi)^4} \int d^3k \, e^{i(\vec{k}, \vec{x} - \vec{y})} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) \end{aligned} \tag{39}$$

After such a change, the Hamiltonian (38) would be a self-adjoint and positive energy operator.

In the conclusion, let me dwell upon the ambiguity of the Coulomb gauge recently discovered by Gribov. The point is that the solution $C_o(x)$ of Eq. (32) may be non-unique if the field b_u^a is not small in some sense. If this is the case, a non-trivial solution of the homogeneous equation

$$\Delta C - \varepsilon [B_i, \partial_i C] = 0 \tag{40}$$

may exist. The natural condition for such a solution is

$$\int \partial_i C^a \partial_i C^a \, d^3x < \infty \tag{41}$$

because the left-hand side of (41) is just the energy of the longitudinal part of the f_{oi} field.

First of all let us look for a solution of (40) where B_i can be obtained from zero by some gauge transformation

$$B_i = (O_i)^{\Omega} = \frac{1}{\varepsilon} \partial_i \Omega \, \Omega^{-1} \tag{42}$$

Demanding that B_i obeys the Coulomb gauge condition, we obtain the following equation

$$\partial_i (\partial_i \Omega \cdot \Omega^{-1}) = 0 \tag{43}$$

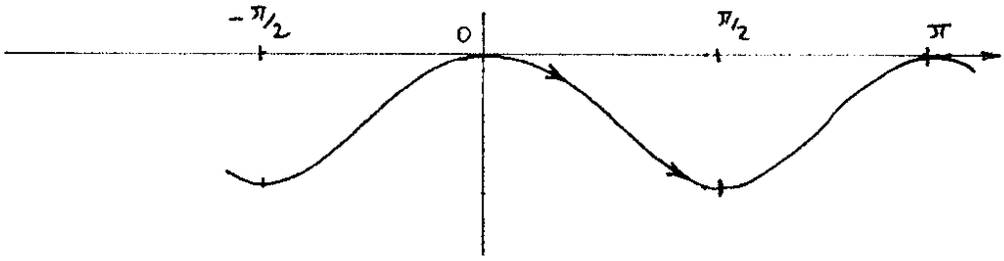
Gribov has obtained a spherical symmetric solution of (43) of the form

$$\Omega = \exp\left(\tau_a x_a \frac{\theta(r)}{r}\right) \tag{44}$$

where $\theta(r)$ depends only on r. Substituting (44) into (43) gives us the following equation for θ :

$$\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} - \sin 2\theta = 0 \tag{45}$$

where $t = \ln(r/r_o)$. This equation describes a pendulum with friction. The potential energy of this pendulum has a form like :



Let us look for a solution $\theta(r)$ which behaves like ar for small r, and $a > 0$. We see that θ must go to $\pi/2$ [the bottom of a potential pit for $r \to \infty$ $(t \to \infty)$ due to friction]. For large r we must have

$$\theta \approx \frac{\pi}{2} + \frac{\beta}{r^{1/2}} \cos\left(\frac{\sqrt{7}}{2} \ln \frac{r}{r_0}\right) \tag{46}$$

It is interesting that Gribov's solution belongs to a non-zero homotopy class and cannot be obtained from the zero field in a continuous way.

Let us now substitute Gribov's solution into the homogeneous equation (40). We can search for a solution in the form

$$C_\alpha(x) = \frac{x_\alpha}{r} f(r) \quad , \quad r = |\vec{x}| \tag{47}$$

which behaves like $\sim r$ for $r \to 0$. The equation for f

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{2\cos^2\theta(r)}{r^2} f = 0 \tag{48}$$

may have a non-trivial solution. Nevertheless, the condition of finiteness of the energy (41) is not valid for any solution which behaves as $\sim r$ at small r. Let us show this. For $r \to \infty$ we must have

$$f = a + o(1) \quad (a \neq 0) \quad \text{or} \quad f = \frac{\beta}{r} + o\left(\frac{1}{r}\right) \quad (\beta \neq 0) \tag{49}$$

In order to exclude the second possibility, we multiply (48) by $r^2 f$ and integrate over from $r = 0$ to $r = R$. We thus obtain

$$R^2 f(R) f'(R) = \int_0^R \left(r^2\left(\frac{df}{dr}\right)^2 + 2 f^2 \cos^2\theta\right) dr \tag{50}$$

If $f = (b/r) + o(1/r)$, the left-hand side of (50) vanishes if $R \to \infty$. But the right-hand side is a positive value. This is why f must go to some non-zero constant a if $r \to \infty$, and the energy functional (41) is infinite. So we see that the homogeneous equation corresponding to Gribov's field $B_i = (1/\epsilon)\partial_i \Omega \Omega^{-1}$ still has a trivial solution with finite energy.

Nevertheless, we can choose a field $b_i^a(x)$ – belonging to the zero homotopic class – such that there exists a non-trivial finite energy solution of (40). Gribov suggested to take a field of the form

$$b_i^a(x) = \frac{1}{\epsilon} \varepsilon_{iae} \frac{x_e}{r} g(r) \tag{51}$$

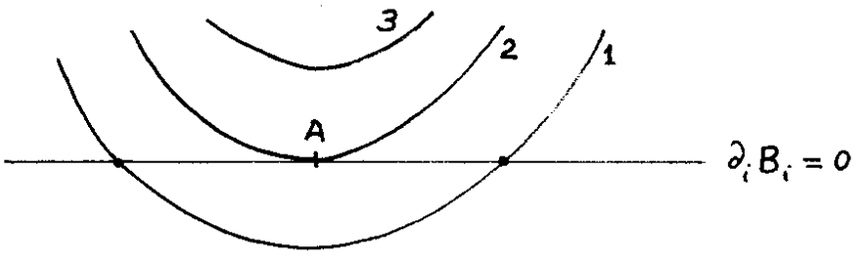
which obeys the Coulomb gauge condition automatically. Looking for a solution of (40), in the form (47), we come to the equation :

$$\frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} - \frac{2 g(r)}{r^2} f = 0 \tag{52}$$

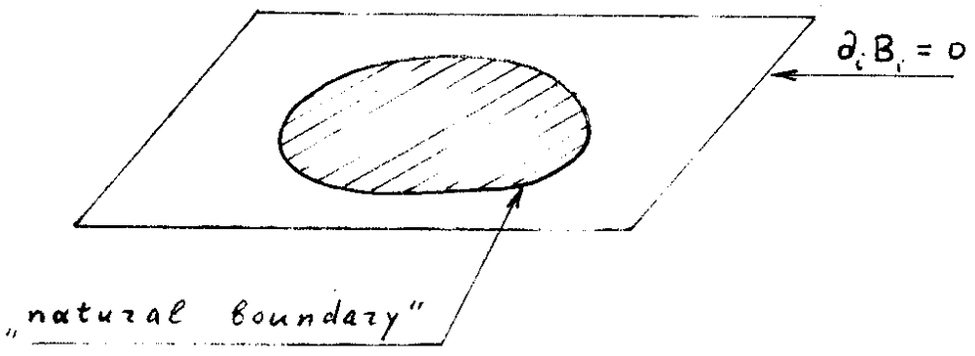
It is clear that we can choose a function $g(r)$ such that this equation has a non-trivial solution with finite energy.

The examples considered are interesting in many respects. First of all, they are "strong" fields. Formally, we have an inverse coupling constant ϵ^{-1} in front of both Eqs. (42) and (51). So we are beyond the perturbation theory when we deal with such fields. In other words, the formal perturbative scheme is still valid for "non-strong" fields. The first example, Eq. (42), gives us a field which belongs to a non-zero homotopy class. We can say that this field corresponds to another vacuum state with non-zero topological charge. The second example, Eq. (51), shows that for a sufficiently large field, the Coulomb gauge condition is not suitable for field parametrization.

Let us try to represent the situation by the following picture



Here the straight line is a "Coulomb gauge surface" in functional space, the curves are the orbits of the gauge group. The first orbit has two common points with a gauge surface, the second one is tangent to the surface, and the third one has no common points with the gauge surface at all. The point A in the picture corresponds to the case when Eq. (40) has a non-trivial solution. We can consider A as a critical point on the gauge surface. The set of critical points gives a natural boundary on the gauge surface, confining the region where a Coulomb gauge parametrization is still valid.



For this piece of phase space we have a unitary theory, and the transverse fields B_i and F_{oi} are natural dynamical variables. As for fields whose orbits do not intersect the gauge surface, we have to look for another parametrization for them. If the orbit has more than one common point with a gauge surface, we may still use this gauge if we would take into account only one point, for instance the point inside the "natural boundary". In this case we may use the functional

$$\Theta(B) \prod_x \delta(f(B)) \tag{53}$$

where $\Theta(B) = 1$ inside the natural boundary, and $\Theta(B) = 0$ outside. The corresponding equation for a weight factor $\Delta_f[B]$ has the following form :

$$\Delta_f[B] \int \theta(B^\Omega) \prod_x \delta(f(B^\Omega(x))) \, d\Omega(x) = 1 \tag{54}$$

Therefore a real problem arises only for those fields whose orbits do not intersect a gauge surface at all. It is natural to suppose that such field configurations are connected with extended objects in the Yang-Mills field theory.

L E C T U R E 7

INFRA-RED ASYMPTOTICS OF GREEN'S FUNCTIONS IN QUANTUM FIELD THEORY

Non-trivial infra-red asymptotics of Green functions is characteristic for theories with massless particles. In this case we have to sum up an infinite number of diagrams of a standard perturbative scheme in order to obtain the correct result. The functional integral formalism allows us to perform such a summation in a shorter way than standard perturbative methods.

One of the possible approaches to the infra-red problem is the method of integration first of all over "fast" and then over "slow" variables. The main idea is to go to some new perturbative scheme when we integrate over "slow" variables.

The question arises as how to define "fast" and "slow" variables. For instance, we can use the Fourier transform of the field variable $\psi(x)$

$$\psi(x) = \int e^{ikx}\,\tilde{\psi}(k)\,d^4k \qquad (1)$$

where $kx = k_0 x_0 - \vec{k}\vec{x}$, $k^2 = k_0^2 - \vec{k}^2$. We can then define the "fast" part of $\psi(x)$ as an integral over k with $|k^2| = |k_0^2 - \vec{k}^2| \geq k_0^2$, and the "slow" part as an integral over $|k^2| \leq k_0^2$, where k_0 is some parameter.

It is perhaps more convenient to go to the Euclidean field theory and define the "slow" part $\psi_0(x)$ and the "fast" part $\psi_1(x)$ of $\psi(x)$ by :

$$\psi(x) = \psi_0(x) + \psi_1(x)$$

$$\psi_0(x) = \int_{|K|<K_0} e^{iKx}\,\tilde{\psi}(K)\,dK\,, \qquad \psi_1(x) = \int_{|K|>K_0} e^{iKx}\,\tilde{\psi}(K)\,dK \qquad (2)$$

where

$$Kx = \sum_{i=1}^{4} K_i x_i\,, \quad K^2 = \sum_{i=1}^{4} K_i^2\,, \quad |K| = \sqrt{K^2}\,.$$

If we obtain the asymptotic formula for Green's functions in a Euclidean theory, we can go to the pseudo-Euclidean case by an analytical continuation. The parameter k_0 which separates "slow" variables from "fast" ones is not defined exactly. There exists only one most reasonable choice of the order of k_0. It is clear that this parameter, which occurs in intermediate calculations, must drop out from the final result.

Let me consider first the one-electron Green function in quantum electrodynamics (QED). We will then obtain asymptotic formulae for the Green functions of an arbitrary number of fields in QED. Such formulae are valid not only in the infra-red region, but can be used in some problems of high energy scattering. In the conclusion I will dwell upon one example of a theory with infra-red asymptotics different from those in QED.

Therefore we start with the one-electron Green function of Euclidean quantum electrodynamics in the Lorentz-Landau gauge.

$$G(x,y) = -\langle \psi(x)\,\bar{\psi}(y)\rangle = -\frac{\int \psi(x)\,\bar{\psi}(y)\,e^{S}\prod_x \delta(\partial_\mu A_\mu)\,d\bar{\psi}\,d\psi\,dA}{\int e^{S}\prod_x \delta(\partial_\mu A_\mu)\,d\bar{\psi}\,d\psi\,dA} \qquad (3)$$

Here we will use the shorthand notation $d\bar{\psi}d\psi dA$ instead of

$$\prod_\alpha d\bar{\psi}_\alpha(x)\,d\psi_\alpha(x)\prod_\mu dA_\mu(x)$$

The S functional in (3) is

$$S = -\frac{1}{4}\int (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 d^4x - \int \bar{\psi}(\gamma_\mu(\partial_\mu - ieA_\mu) - m)\psi\, d^4x \qquad (4)$$

where γ_μ are the Euclidean Dirac metrices obeying the anticommutation condition

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu} \qquad (5)$$

We want now to obtain an asymptotic formula for (3) for large $r = |x-y|$ ($r \gg m^{-1}$).

We will integrate Eq. (3) in three steps :

1. first of all we integrate over the Fermi fields $\psi, \bar{\psi}$;
2. then we integrate over the "fast" part $A_\mu^{(1)}$ of the electromagnetic field A_μ;
3. finally we integrate over the "slow" part $A_\mu^{(0)}$ of the electromagnetic field by using some asymptotic formula for the one-electron Green function in the "slow" part $A_\mu^{(0)}$.

We define the $A_\mu^{(0)}$, $A_\mu^{(1)}$ fields in the following way

$$A_\mu(x) = A_\mu^{(0)}(x) + A_\mu^{(1)}(x)$$

$$A_\mu^{(0)}(x) = \int_{|K|\leq K_0} e^{iKx}\, \tilde{A}_\mu(K)\, d^4K \quad , \quad A_\mu^{(1)}(x) = \int_{|K|\geq K_0} e^{iKx}\, \tilde{A}_\mu(K)\, d^4K \tag{6}$$

choosing the order of k_o according to the following inequalities

$$K_0 \ll m \quad , \quad \frac{e^2}{4\pi^2} \ln \frac{m}{K_0} \ll 1 \tag{7}$$

The first step of our programme (integration over Fermi fields) may be carried out in closed form :

$$\int e^S \prod_x d\bar{\psi}\, d\psi = e^{S_0[A]} \det(\gamma_\mu(\partial_\mu - ie A_\mu) - m) \tag{8}$$

$$\int \psi(x)\, \bar{\psi}(y)\, e^S \prod_x d\bar{\psi}\, d\psi = e^{S_0[A]} G(x,y|A) \det(\gamma_\mu(\partial_\mu - ieA_\mu) - m) \tag{9}$$

Here $S_o[A]$ is an action of the free electromagnetic field, $\det[\gamma^\mu(\partial_\mu - ieA_\mu) - m]$ is a determinant of the Dirac operator for a spinor particle in the electromagnetic field A_μ. We can regularize this determinant if we divide it by the determinant of the free Dirac operator $\det(\gamma^\mu \partial_\mu - m)$. $G(x,y|A)$ is a Green function of the electron in the electromagnetic field, defined by :

$$(\gamma^\mu(\partial_\mu - ie A_\mu) - m)\, G(x,y|A) = \delta(x-y) \tag{10}$$

We cannot integrate the functionals (8) and (9) in closed form and we need an approximation method to do it. For the first non-trivial approximation, let us use the following formulae

$$\det\left(I + \frac{1}{\hat{\partial} - m} ie A_\mu\right) \approx 1 \tag{11}$$

$$G(x,y|A^{(0)}+A^{(1)}) \approx G(x,y|A^{(0)}) \exp\left(ie\int_x^y A_\mu^{(0)}(z)\,dz_\mu\right) \tag{12}$$

The first formula means that we neglect vacuum polarization effects in the first approximation. The second one allows us to take into account the "slow" part of the electromagnetic field in an explicit form. The integral in the right-hand side of (12) is a linear integral along the straight line, connecting points x and y. This formula is a well-known one. It is exact for $A_\mu = \partial_\mu \lambda$ and must be asymptotically exact in the limit $k_0 \to 0$.

Approximations (11) and (12) imply the Green function to be a product of two factors

$$G(x,y) = -\frac{\int \exp(S_0[A^{(1)}])\,G(x,y|A^{(1)}) \prod_x \delta(\partial_\mu A_\mu^{(1)})\,dA^{(1)}}{\int \exp(S_0[A^{(1)}]) \prod_x \delta(\partial_\mu A_\mu^{(1)})\,dA^{(1)}} \tag{13}$$

$$\frac{\int \exp\left(S_0[A^{(0)}] + ie\int_x^y A_\mu^{(0)}(z)\,dz_\mu\right) \prod_x \delta(\partial_\mu A_\mu^{(0)})\,dA^{(0)}}{\int \exp(S_0[A^{(0)}]) \prod_x \delta(\partial_\mu A_\mu^{(0)})\,dA^{(0)}}$$

The whole non-trivial asymptotics is due to the second factor in the right-hand side of (13). Let me call it the "infra-red factor". This factor is a Gaussian functional integral because we can write down the linear integral as a four-dimensional one :

$$ie\int_x^y A_\mu^{(0)}(z)\,dz_\mu = ie\int \ell_\mu(z)\,A_\mu^{(0)}(z)\,d^4z \tag{14}$$

If we cancel this linear functional in the exponent by the shift transformation $A_\mu \to A_\mu + A_\mu^{(0)}$, we obtain the following formula for the "infra-red factor" :

$$\exp\left\{-\frac{e^2}{2}\int \ell_\mu(z_1)\,D_{\mu\nu}(z_1-z_2)\,\ell_\nu(z_2)\,d^4z_1\,d^4z_2\right\} =$$

$$= \exp\left\{-\frac{e^2}{2}\int_x^y dz_{1\mu}\int_x^y dz_{2\nu}\,D_{\mu\nu}(z_1-z_2)\right\} \tag{15}$$

Here

$$D_{\mu\nu}(z) = \frac{1}{(2\pi)^4} \int_{|K| \leq K_0} e^{iKz} \frac{K^2 \delta_{\mu\nu} - K_\mu K_\nu}{K^4} d^4K \tag{16}$$

By performing the linear integrations

$$\int_X^Y e^{iKz} dz_\mu = \frac{e^{iKY} - e^{iKX}}{i(K, X-Y)} (X-Y)_\mu \tag{17}$$

we obtain that the exponent in (15) is equal to

$$-\frac{e^2}{(2\pi)^4} \int_{|K| \leq K_0} \frac{d^4K}{K^4} \frac{K^2(X-Y)^2 - (K, X-Y)^2}{(K, X-Y)^2} \left(1 - \cos(K, X-Y)\right) \tag{18}$$

Using polar co-ordinates $\left[|x-y| = r, \quad (x-y, k) = kr\cos\theta\right]$ we will have

$$-\frac{e^2}{4\pi^3} \int_0^\pi \frac{\sin^4\theta \, d\theta}{\cos^2\theta} \int_0^{K_0 r|\cos\theta|} \frac{1 - \cos x}{x} dx \tag{19}$$

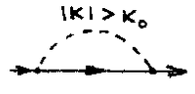
Now it is not difficult to obtain the asymptotics of (19) if $k_0 r \gg 1$. For the "infra-red factor" we obtain the formula

$$\exp\left\{-\frac{e^2}{4\pi^2} K_0 r + \frac{3e^2}{8\pi^2} \ln K_0 r + \frac{3e^2}{16\pi^2}(1 + 2C - 2\ln 2)\right\} \tag{20}$$

where C is the Euler constant. This function depends on the parameter k_0 which must drop out after we multiply (20) by the first factor in the right-hand side of (13). We can calculate it by taking into account only the one-loop diagram for the self-energy part according to the formulae

$$G_{K_0}(X-Y) = \frac{1}{(2\pi)^4} \int \frac{e^{i(P, X-Y)} d^4p}{i\hat{p} + m - \Sigma(p, K_0)} \tag{21}$$

$$\Sigma(p,\kappa_0) = \quad (22)$$



In the integral over k in the one-loop diagram we have to integrate only over $|k| \geq k_0$. So we obtain for $p^2 \approx -m^2$

$$G_{\kappa_0}(p) \approx a_{\kappa_0} \frac{m_{\kappa_0} - i\hat{p}}{m_{\kappa_0}^2 + p^2} \quad (23)$$

where

$$a_{\kappa_0} = 1 - \frac{7e^2}{16\pi^2} - \frac{3e^2}{8\pi^2} \ln \frac{\kappa_0}{m}$$

$$m_{\kappa_0} = m - \frac{e^2}{4\pi^2}\kappa_0 \quad (24)$$

$$m = m_0 \left(1 + \frac{3e^2}{16\pi^2}\left(\ln \frac{\Lambda^2}{m^2} + 1\right)\right)$$

Here Λ is the ultra-violet cut-off in the integral for the self-energy part, m_{k_0} is the electron mass if we do not take into account its interaction with "slow" field $A_\mu^{(0)}$. So we have

$$G_{\kappa_0}(x-y) = \frac{1}{(2\pi)^4}\int e^{i(p,x-y)} G_{\kappa_0}(p)\, d^4p \approx \quad (25)$$

$$\approx \frac{a_{\kappa_0}}{2}\left(\frac{m}{2\pi r}\right)^{3/2}(I + \hat{n})\, e^{-m_{\kappa_0} r}$$

where

$$\hat{n} = \frac{x_\mu \gamma_\mu}{r} \quad (26)$$

Multiplying (25) by the "infra-red factor" (20), we come to :

$$G(x-y) = \frac{a}{2}\left(\frac{m}{2\pi}\right)^{3/2}(mz)^{-\frac{3}{2}+\frac{3e^2}{8\pi^2}}(1+\hat{n})e^{-mz} \tag{27}$$

which does not contain k_0. Here

$$a = 1 + \frac{e^2}{8\pi^2}(3C - 3\ln 2 - 2) \tag{28}$$

The corresponding formula in p space is

$$G(p) \approx \frac{m - i\hat{p}}{m^2\left(1+\frac{p^2}{m^2}\right)^{1+\frac{3e^2}{8\pi^2}}}\left(1 - \frac{e^2}{4\pi^2}\right) \tag{29}$$

Let me now explain why the asymptotics obtained

$$G(p) \sim \left(1+\frac{p^2}{m^2}\right)^{-1-\frac{3e^2}{8\pi^2}}$$

must be exact if we understand e as a renormalized coupling constant. Let us choose k_0 arbitrarily small. In this case we can use (12) with $e = e_0$ (bare coupling constant). We can integrate over the "fast" field $A_u^{(1)}$ according to the formulae

$$\int G(x,y|A^{(1)})\, e^{S_0[A]} \det\left(\hat{\partial} - m - ie_0(\hat{A}^{(0)} + \hat{A}^{(1)})\right) \prod_x \delta(\partial_\mu A_\mu^{(1)})\, dA^{(1)} \approx$$

$$\approx G_{k_0}(x-y)\, \exp S[A^{(0)}] \tag{30}$$

where

$$G_{k_0}(x-y) = \frac{\int G(x,y|A^{(1)}) \exp S_0[A^{(1)}] \det(\hat{\partial} - m - ie_0\hat{A}^{(1)}) \prod_x \delta(\partial_\mu A_\mu^{(1)})\, dA^{(1)}}{\int \exp S_0[A^{(1)}] \det(\hat{\partial} - m - ie_0\hat{A}^{(1)}) \prod_x \delta(\partial_\mu A_\mu^{(1)})\, dA^{(1)}} \tag{31}$$

is a quantum Green function for the electron in the "fast" field,

$$\exp(S[A^{(0)}]) = \int \exp(S_0[A]) \det(\hat{\partial} - m - ie_0(\hat{A}^{(0)} + A^{(1)})) \prod_x d(\partial_\mu A_\mu^{(1)}) dA^{(1)} \quad (32)$$

is a result of the integration of $\exp S$ over the Fermi fields ψ, $\bar{\psi}$ and over the "fast" field $A_\mu^{(1)}$. $S[A^{(0)}]$ is an effective action functional of the "slow" field. It can be written in the form

$$\tilde{C} - \frac{1}{4Z} \int (\partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)})^2 d^4x + \dots , \quad (33)$$

where $\tilde{C}$ is some constant,

$$Z = Z_3 = e^2 / e_0^2 \quad (34)$$

is nothing but a charge renormalization factor (or the renormalization constant of the photon Green function). Now we have to integrate the expression

$$\exp\left(S[A^{(0)}] + ie_0 \int_x^y A_\mu^{(0)}(z) dz_\mu \right) \quad (35)$$

over the "slow" field $A_\mu^{(0)}$. The result is just the infra-red factor (15), where $e^2 = e_0^2 Z$ is the square of the renormalized coupling constant. As for $G_{k_0}(x-y)$ it has the form (25). So we obtain the asymptotic formula (27) for $G(x-y)$, and (29) for $G(p)$. The e value in these equations is a renormalized coupling constant.

We can apply the outlined method to any theory of minimal electromagnetic interaction, obtained by the change

$$\partial_\mu \to \nabla_\mu = \partial_\mu - ieA_\mu \quad (36)$$

For instance, let us consider scalar electrodynamics with an action functional

$$-\int \left(|(\partial_\mu - ieA_\mu)\psi|^2 + m^2 |\psi|^2 \right) d^4x - \frac{1}{4} \int (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 d^4x \quad (37)$$

The one-particle Green function of the theory has the form

$$G(p) = \frac{Z_2}{m}\left(1+\frac{p^2}{m^2}\right)^{-1-\frac{3e^2}{8\pi^2}} \tag{38}$$

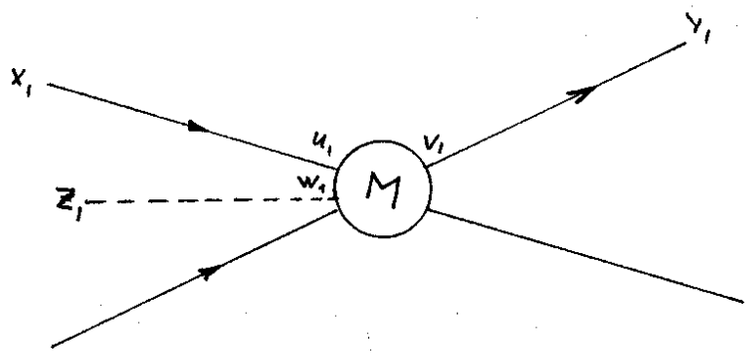
The point is that the asymptotic formula (12) for the Green function of a particle in the "slow" field $A_\mu^{(0)}$ is a universal one for any "minimal" theory. It implies the same "infra-red" factor for any such theory.

Let us now consider the Green function of an arbitrary number of fields in QED

$$\langle \psi(x_1)\dots\psi(x_n)\bar\psi(y_1)\dots\bar\psi(y_n)A_{\mu_1}(z_1)\dots A_{\mu_k}(z_k)\rangle_L \tag{39}$$

(in the Lorentz-Landau gauge). We are interested in the asymptotics of (39) if the distance between any pair of arguments $x_1,\dots,x_n$, $y_1,\dots,y_n$, $z_1,\dots,z_k$ goes to infinity.

After integration over the Fermi fields ψ, $\bar\psi$ and over the fast electromagnetic field $A_\mu^{(1)}$ we will obtain the sum of all connected diagrams of the form



(40)

Here M is an irreducible part of a given diagram.

The expression corresponding to (40) is

$$\prod_{i=1}^{n} G_{K_0}(x_i,u_i|A^{(0)})\,G_{K_0}(v_i,y_i|A^{(0)})\prod_{j=1}^{K} D_{\mu_j\nu_j}\;M_{\nu_1\dots\nu_K} \tag{41}$$

Let us now use the approximation

$$G_{K_0}(x,u\,|A^{(0)}) \approx G_{K_0}(x-u)\exp\left(ie_0\int_u^x A_\mu^{(0)}(z)\,dz_\mu\right) \tag{42}$$

for the electron Green function. As for $D_{\mu\nu}$ and M we may consider them as independent of $A_\mu^{(0)}$. So we have to integrate the following expression

$$\exp K =$$

$$= \exp\left(-\frac{1}{4Z_3}\int(\partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)})^2\,d^4x + i\sum_{i=1}^{2n} e_{i0}\int_{u_i}^{x_i} A_{\mu_i}^{(0)}(z_i)\,dz_{i\mu}\right) \tag{43}$$

over the "slow" field $A_\mu^{(0)}$. Here

$$\begin{aligned} &x_i = x_i\,, \quad u_i = u_i\,, \quad e_{i0} = e_0 && 1 \le i \le n \\ &x_i = y_i\,, \quad u_i = v_i\,, \quad e_{i0} = -e_0 && n+1 \le i \le 2n \end{aligned} \tag{44}$$

The result is

$$\langle \exp K \rangle = \exp(-X) \tag{45}$$

where

$$X =$$

$$= \frac{1}{2(2\pi)^4}\int_{|K|\le K_0}\frac{d^4K}{K^4}\sum_{i,j=1}^{2n} e_i e_j\left(\frac{K^2(x_i-u_i,x_j-u_j)}{(K,x_i-u_i)(K,x_j-u_j)} - 1\right) \cdot \tag{46}$$

$$\cdot\left(e^{iKx_i} - e^{iKu_i}\right)\left(e^{-iKx_j} - e^{-iKu_j}\right)$$

This expression plays the role of the "infra-red factor" for the Green function (39). In p space we will obtain the following formula

$$\prod_{i=1}^{n}(-i\hat{p}_i+m)\prod_{j=1}^{n}(-ip_j-m)\prod_{s=1}^{K}D_{\mu_s\nu_s}(k_s)\cdot$$

$$\cdot M_{\nu_1\ldots\nu_s}(p_1,\ldots,p_{2n},q_1,\ldots,q_K)\,\delta\Big(\sum_{i=1}^{2n}p_i+\sum_{s=1}^{K}q_s\Big)\cdot \qquad (47)$$

$$\cdot\int_0^\infty\ldots\int_0^\infty\prod_i d\sigma_i\,\exp\Big(\sum_i(p_i^2+m^2)\sigma_i-Y\Big)$$

Here

$$Y=\frac{im^2e^2}{2\pi^2}\sum_i\sigma_i+$$

$$+\sum_{i,j=1}^{2n}\frac{e_ie_j}{2(2\pi)^4}\int\limits_{|k|\le k_0}\frac{d^4k}{k^4}\left(e^{2i\sigma_i(p_ik)}-1\right)\left(e^{-2i\sigma_j(p_jk)}-1\right)\left(\frac{k^2(p_ip_j)}{(kp_i)(kp_j)}-1\right) \qquad (48)$$

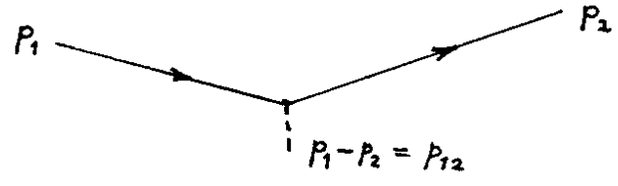
Expression (47) behaves like

$$\left(1+\frac{p_i^2}{m^2}\right)^{-1-\frac{3e^2}{8\pi^2}},$$

if $p_i^2\to -m^2$.

It is interesting that Eqs. (47) and (48) are valid not only in the infra-red region. Actually they describe some high energy processes as well. All the information is contained in (47) and (48) and can be extracted from these after a slight modification.

For instance, let us consider the electron scattering process with large momentum transfer



(49)

We will suppose that $|p_{12}^2| \gg |p_1^2|, |p_2^2| \sim m^2$. In this case it is natural to choose k_0 in the following way

$$p_1^2, p_2^2 \ll \kappa_0^2 \ll p_{12}^2 \tag{50}$$

Equations (47) and (48) (for $n=1$, $k=1$) give us the so-called double logarithmic asymptotics of an irreducible vertex part Γ_μ

$$\Gamma_\mu \approx e\gamma_\mu \exp\left(-\frac{e^2}{8\pi^2} \ln\frac{p_{12}^2}{p_1^2} \ln\frac{p_{12}^2}{p_2^2}\right) \tag{51}$$

In the conclusion, let me say a little about a theory with an infra-red behaviour is different from that in QED. This is a theory of massive isotopic fermions interacting with massive isotopic scalar fields. The corresponding action functional is

$$-\frac{1}{2}\int \partial_\mu \varphi^a \partial_\mu \varphi^a d^4x - \int \bar{\psi}(\gamma^\mu \partial_\mu - m + g\tau_a \varphi_a)\psi\, d^4x \tag{52}$$

In order to apply the above-mentioned formalism to this theory we need a formula for the Green function of a fermion in the "slow" part of the field φ_a. Such a formula is the following one

$$G(x,y\,|\,\varphi_a^{(0)}) \approx G(x,y)\; T\exp\left(g\int_x^y \tau_a \varphi_a^{(0)}(z)\, dS(z)\right) \tag{53}$$

Here $T\exp$ means an ordering exponent along a straight line connecting two points x and y in Euclidean space-time. The problem of infra-red asymptotics reduces to averaging the expression

$$T\exp g\int_x^y \tau_a \varphi_a^{(0)}(z)\, dS(z) \tag{54}$$

with the weight factor

$$\exp\left(-\frac{1}{2}\int \partial_\mu \varphi_a^{(0)} \partial_\mu \varphi_a^{(0)}\, d^4x\right) \tag{55}$$

It turns out that after such an averaging, we will obtain the following factor

$$\left(1+\frac{2g^2}{\pi^2}\ln mr\right)^{-3/8} \tag{56}$$

instead of the exponential factor $\exp(3e^2/8\pi^2)\ln mr$ in QED. So here we have no "exponentiation" which is characteristic of QED.

Problem :

Obtain the double logarithm asymptotics (51) from (47) and (48).

L E C T U R E 8

EXTENDED OBJECTS

Now the theory of extended objects is itself a very extended object, and it is impossible to grasp it in one lecture. I should therefore dwell on one or two examples where such objects could arise.

The first example is the Goldstone model in (2+1) dimensions, where the extended objects are vortex-like excitations. Let us consider a Euclidean field theory of the complex scalar field with an action functional

$$S = -\int \left(\partial_\mu \bar{\psi} \partial_\mu \psi - \lambda |\psi|^2 + \frac{g}{4} |\psi|^4 \right) dx \qquad (1)$$

The coupling constant g is positive $(g>0)$. If $\lambda < 0$ it is not a very interesting case. Here we have an ordinary renormalizable theory of a self-interacting complex scalar field.

The $\lambda > 0$ case is much more interesting. If $g=0$, we have particles with negative mass square $m^2 < 0$ (tachyons). Goldstone noticed that the interaction term $g/4|\psi|^4$ changes the situation radically. The Bose condensation occurs and as a result we have two sorts of particles, namely the massless (Goldstone) particles and the massive ones.

Here I should like to speak mainly on vortex-like excitations in the model - which may be considered as a third sort of particles. Let us consider the classical equation $(\delta S = 0)$

$$-\Delta \psi - \lambda \psi + \frac{g}{2} \bar{\psi} \psi \psi = 0 \qquad (2)$$

There exists a trivial solution $\psi = 0$ of this equation and a non-trivial one $\psi_0 = \text{const}$, where

$$\rho_0 = |\psi_0|^2 = \frac{2\lambda}{g} \qquad (3)$$

This solution corresponds to the absolute maximum of the functional (1). The main contribution in the functional integral over the ψ, $\bar{\psi}$ fields is due to functions with $|\psi|^2$ near ρ_0. Such fields correspond to the superfluid state of the system, and ρ_0 is nothing but a density of the Bose condensate. In order to describe excitations in the superfluid state, it is convenient to go to polar co-ordinates of the ψ, $\bar{\psi}$ fields

$$\psi(x) = \sqrt{\rho(x)}\, e^{i\varphi(x)} \quad , \quad \bar{\psi}(x) = \sqrt{\rho(x)}\, e^{-i\varphi(x)} \tag{4}$$

and then to

$$\pi(x) = \rho(x) - \rho_0 \tag{5}$$

Let us rewrite the S functional in terms of the new variables

$$S = -\int\left((\rho_0+\pi)\,\partial_\mu\varphi\,\partial_\mu\varphi + \frac{\partial_\mu\pi\,\partial_\mu\pi}{4(\rho_0+\pi)} + \frac{g}{4}\pi^2\right)dx + \frac{\lambda^2}{g}\int dx \tag{6}$$

After a new transformation

$$\varphi \to \frac{1}{\sqrt{2\rho_0}}\varphi \quad , \quad \pi \to \sqrt{2\rho_0}\,\pi \tag{7}$$

we get

$$S = -\int\left(\left(1+\sqrt{\frac{2}{\rho_0}}\pi\right)\frac{\partial_\mu\varphi\,\partial_\mu\varphi}{2} + \frac{1}{2}\left(\frac{\partial_\mu\pi\,\partial_\mu\pi}{1+\sqrt{\frac{2}{\rho_0}}\pi} + 2\lambda\pi^2\right)\right)dx + \frac{\lambda^2}{g}\int dx \tag{8}$$

The quadratic form of this functional

$$-\frac{1}{2}\int\left(\partial_\mu\varphi\,\partial_\mu\varphi + \partial_\mu\pi\,\partial_\mu\pi + 2\lambda\pi^2\right)dx \tag{9}$$

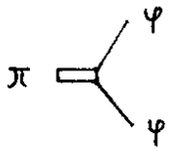
describes a system of two fields : a massless φ field and a massive π field. The corresponding masses are

$$m_\varphi^2 = 0 \quad , \quad m_\pi^2 = 2\lambda > 0 \tag{10}$$

The interacting terms in (8) can be obtained from (9) by the change

$$\partial_\mu \varphi \partial_\mu \varphi \to \left(1+\sqrt{\frac{2}{\rho_0}}\,\pi\right) \partial_\mu \varphi \partial_\mu \varphi \quad , \quad \partial_\mu \pi \partial_\mu \pi \to \left(1+\sqrt{\frac{2}{\rho_0}}\,\pi\right)^{-1} \partial_\mu \pi \partial_\mu \pi \tag{11}$$

Due to the interaction, the π particle becomes unstable. It may decay into two φ particles



(12)

Let us now consider vortex-like solutions of Eq. (2). The simplest one is a function of x_1, x_2 of the form

$$\psi = e^{i\theta} f(r) \quad , \quad r = \sqrt{x_1^2 + x_2^2} \quad , \quad \theta = \operatorname{arctg} \frac{x_2}{x_1} \tag{13}$$

The non-linear equation

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2} + \lambda f - \frac{g}{2} f^3 = 0 \tag{14}$$

can be written down in dimensionless variables

$$x = \lambda^{1/2} r \quad , \quad h = \sqrt{\frac{g}{2\lambda}}\, f \tag{15}$$

in the following way

$$h_{xx} + x^{-1} h_x - x^{-2} h + h - h^3 = 0 \tag{16}$$

This equation coincides with that of the theory of a non-ideal Bose gas. It can be shown that there exists a solution of (16) which is proportional to x for $x \to 0$ and goes to one for $x \to \infty$.

This solution describes a quantum vortex in the (x_1, x_2) plane. We can consider the straight line $x_1 = x_2 = 0$ as a "world-line" of the vortex. At this line we have $\psi = \bar{\psi} = 0$ and the φ variable is not defined. It is clear that every straight line in three-dimensional space may be a world line for a solution which can be obtained from that corresponding to $x_1 = x_2 = 0$, by a linear co-ordinate transformation.

Now, let us consider the situation with many vortices. We will suppose a distance between the vortex world lines to be much greater than the characteristic length $\lambda^{-\frac{1}{2}}$. Because of the interaction, the world lines now are not straight but are slightly curved. Let us confine every vortex line to a tube with radius r_0, such that the following inequality would be valid

$$\lambda^{-1/2} \ll r_0 \ll R \tag{17}$$

where R is a distance between vortices. We can write down an action functional in the following way

$$-\int' \left(\frac{1}{2}\left(1+\sqrt{\frac{2}{\rho_0}}\pi\right)\partial_\mu\varphi\,\partial_\mu\varphi + \frac{1}{2}\left(\frac{\partial_\mu\pi\,\partial_\mu\pi}{1+\sqrt{\frac{2}{\rho_0}}\pi} + 2\lambda\pi^2\right)\right)dx - \tag{18}$$

$$-\sum_i m(r_0)\int ds_i + \frac{\lambda^2}{g}\int dx$$

Here the first term is an integral over the space outside the vortex tubes. The value $m(r_0)$ [energy (mass) of a vortex inside a tube] :

$$m(r_0) = \int_{r \le r_0} d^2x \left(\partial_i\bar{\psi}\,\partial_i\psi + \frac{g}{4}\left(|\psi|^2-\rho_0\right)^2\right) \approx 2\pi\rho_0 \ln\frac{r_0}{a} \tag{19}$$

depends logarithmically on r_0. The value a in (19) is of the order of $\lambda^{-\frac{1}{2}}$. The symbol ds_i in (18) is a differential of the length of the i^{th} vortex world line.

In the presence of vortices, the φ variable (7) is not one-valued. It gets an addendum

$$\delta\varphi = \pm 2\pi\sqrt{2\rho_0} = \pm q \tag{20}$$

if we go around a vortex line.

We can go back to the one-valued function in the following way. Let me explain it, neglecting the φ–π interaction term and considering only the functional

$$\frac{1}{2}\int \partial_i\varphi\,\partial_i\varphi\; d^3x \tag{21}$$

We can do a shift transformation

$$\varphi(x) \to \varphi(x) + \varphi_0(x) \tag{22}$$

where $\varphi_0(x)$ is a many-valued function which just gets the addendum (20) after going around a world line. Moreover, let φ_0 obey the Laplace equation

$$\Delta\varphi_0(x) = 0 \tag{23}$$

After such a shift we will have

$$\frac{1}{2}\int \partial_i\varphi\,\partial_i\varphi\; d^3x \to \frac{1}{2}\int \partial_i\varphi\,\partial_i\varphi\; d^3x + \frac{1}{2}\int \partial_i\varphi_0\,\partial_i\varphi_0\; d^3x \tag{24}$$

Now, a new φ variable [in the right-hand side of (22)] is one-valued.

How can we now obtain an explicit expression for $\varphi_0(x)$? It turns out that the vector field

$$\vec{h} = \vec{\partial}\varphi_0(x) \tag{25}$$

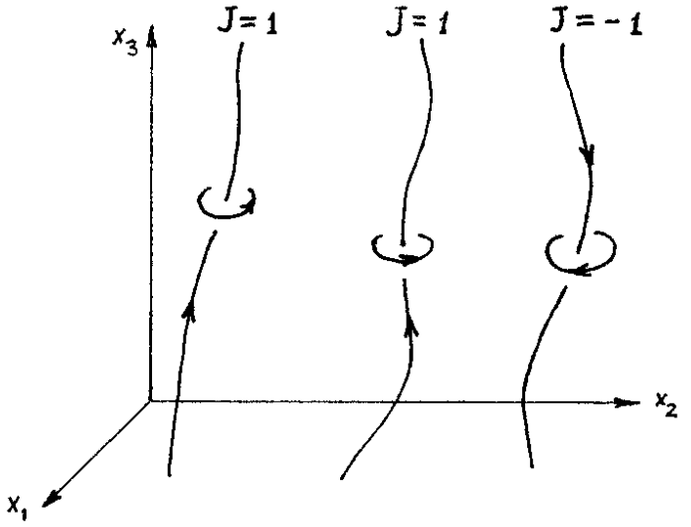
is nothing but a solution of the following magnetostatic problem in a three-dimensional space

$$\mathrm{rot}\,\vec{h} = q\vec{j}\;, \quad \mathrm{div}\,\vec{h} = \partial_i h_i = 0 \tag{26}$$

where

$$\vec{j} = \sum_i \vec{j}_i \tag{27}$$

is a sum of linear currents ($J = \pm 1$) along the vortex world lines.



We can solve this problem with the help of a vector potential $\vec{a}$ such that

$$\vec{h} = \mathrm{rot}\,\vec{a} \quad , \quad \mathrm{div}\,\vec{a} = 0 \tag{28}$$

So we obtain

$$\vec{a} = \frac{q}{4\pi} \int \frac{\vec{j}(y)\, d^3y}{|\vec{x} - \vec{y}|} \quad ,$$

$$\frac{1}{2}\int \partial_i \varphi_0 \partial_i \varphi_0 \, d^3x = \frac{1}{2}\int h^2 d^3x = \frac{q}{2}\int \vec{j}\,\vec{a}\, d^3x = \tag{29}$$

$$= \frac{q^2}{8\pi}\iint \frac{(\vec{j}(x), \vec{j}(y))}{|\vec{x} - \vec{y}|} d^3x\, d^3y = \pi\rho_0 \sum_{i\,j} \iint_{|\vec{x}-\vec{y}|>r_0} \frac{(d\vec{S}(x), d\vec{S}(y))}{|\vec{x} - \vec{y}|}$$

We impose the condition $|\vec{x}-\vec{y}| > r_0$, because we have to take into account only the field outside the vortex tubes.

So we come to the expression

$$\tilde{S} = -m(r_0)\sum_i \int dS_i - \pi\rho_0 \sum_{i,j} \iint_{|\vec{x}-\vec{y}|>r_0} \frac{(d\vec{S}(x), d\vec{S}(y))}{|\vec{x} - \vec{y}|} \tag{30}$$

The second term in the right-hand side is due to the vortex interaction. This non-local interaction may be written down in a local form. Namely, we have the following formula

$$\exp \tilde{S} = \frac{\int \exp\left(-m\varepsilon_0 \sum_i \int ds_i - \frac{1}{2}\int (\mathrm{rot}\,\vec{A})^2 d^3x - iq\int \vec{j}\,\vec{A}\,d^3x\right) \prod_x \delta(\partial_i A_i)\, dA}{\int \exp\left(-\frac{1}{2}\int (\mathrm{rot}\,\vec{A})^2 d^3x\right) \prod_x \delta(\partial_i A_i)\, dA} \qquad (31)$$

where $A_\mu(x)$ is the "slow" electromagnetic field

$$A_\mu(x) = \int_{|K|<\varepsilon_0^{-1}} e^{iKx}\, \tilde{A}_\mu(K)\, d^3K \qquad (32)$$

We can prove this formula by the shift transformation : $A_\mu \to A_\mu + A_\mu^{(0)}$ which cancels the linear functional $iq\int \vec{j}\vec{A}\, d^3x$.

So we have introduced an electromagnetic vector field (which obeys a gauge condition $\partial_i A_i = 0$) responsible for the long range interaction between quantum vortices. It seems to be an attractive feature of the theory that here the gauge field A_μ arises dynamically. We have introduced it instead of the many valued φ field, and we can say that we deal with some non-trivial description of the φ field. [Let us note that the number of physical degrees of freedom (one) is the same for both φ and A fields.] The value q has the meaning of a coupling constant (change of vortex). It is interesting that q is inversely proportional to the coupling constant $\sqrt{2/\rho_0} = q'$ in π-φ and π-π interaction terms. Namely, we have

$$q\,q' = 4\pi \qquad (33)$$

If q' is small, q is large and we have strongly interacting vortices. Such a situation is typical for extended objects. That is why we can hope to obtain a strong interaction theory starting from a theory with a small coupling constant.

We thus have obtained three sorts of excitations in the Goldstone model :

1) massless particles ("photons")
2) massive particles ("pions")
3) quantum vortices ("protons and antiprotons").

We can regard this model as the simplest one for strong and electromagnetic interactions in (2+1) space-time. There are the following reasons for such an analogy.

a) The interaction between vortices is due to mass particles of the φ field at large distances and to both φ and π particles at small ones.

b) There is a conservation law for a number of particles minus a number of antiparticles. It is the analogue of a conservation law for an electric charge which coincides with a baryonic charge in this simple model.

c) The $\pi \to 2\varphi$ decay is an analogue of $\pi^0 \to 2\gamma$ in pion physics.

Let me now speak about some analogues of this model in a four-dimensional space-time. The simplest one is a model of a three-component real scalar field with an action function

$$S = -\frac{1}{2}\int d^4x \left(\sum_a \left(\partial_\mu \varphi_a \partial_\mu \varphi_a - \lambda \varphi_a \varphi_a \right) + \frac{g}{2} \left(\sum_a \varphi_a \varphi_a \right)^2 \right) \qquad (34)$$

The equation $\delta S = 0$

$$-\Delta \varphi_a - \lambda \varphi_a + g \left(\sum_b \varphi_b \varphi_b \right) \varphi_a = 0 \qquad (35)$$

has a non-trivial solution $\varphi_a = \text{const}$ with

$$\sum_a \varphi_a^2 = \frac{\lambda}{g} \qquad (36)$$

We can look for a non-trivial vortex-like solution of the form

$$\varphi_a = \frac{x_a}{r} f(r) \qquad (37)$$

The equation for f(r)

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{2f}{r^2} + \lambda f - g f^3 = 0 \qquad (38)$$

actually has a solution which behaves like $\sim r$, if $r \to 0$, and goes to $\sqrt{\lambda/g}$, if $r \to \infty$. But the energy in a sphere $r < r_0$

$$\frac{1}{2}\int_{r<r_0}\left(\partial_i \varphi_a \partial_i \varphi_a + \frac{g}{2}\left(\sum_a \varphi_a \varphi_a - \frac{\lambda}{g}\right)^2\right) d^3x \qquad (39)$$

is linear at r_0 for $r_0 \gg \lambda^{-\frac{1}{2}}$. This is why we cannot speak about a new particle in this case.

As you know, more complicated solutions do exist if we add to (34) the Yang-Mills action functional and change $\partial_\mu \to \nabla_\mu = \partial_\mu - \epsilon B_\mu$. Namely, for the action functional

$$-\frac{1}{2}\int d^4x \left(\sum_{a,\mu}\left(\partial_\mu \varphi_a + \varepsilon \varepsilon_{abc} b_\mu^b \varphi_c\right)^2 - \lambda \sum_a \varphi_a^2 + \frac{g}{2}\left(\sum_a \varphi_a^2\right)^2\right) -$$

$$-\frac{1}{2}\int d^4x \sum_{a,\mu,\nu}\left(\partial_\mu b_\nu^a - \partial_\nu b_\mu^a + \varepsilon \varepsilon_{abc} b_\mu^b b_\nu^c\right)^2 \qquad (40)$$

we have the 't Hooft-Polyakov monopole solution

$$\varphi_a(x) = x_a \frac{u(r)}{r}, \quad b_\mu^a(x) = \varepsilon_{\mu a b} x_b \left(a(r) - \frac{1}{\varepsilon r^2}\right) \qquad (41)$$

The energy (mass) of this excitation is equal to

$$M = \frac{A\lambda}{g\varepsilon^2} \qquad (42)$$

where $A \sim 1$. So M is inversely proportional to $g\epsilon^2$ and we can obtain a heavy particle in a theory with small coupling constants g, ϵ.

Many examples are known of extended objects in the so-called exactly integrable systems in (1+1) space-time. Classical equations for such systems can be solved by the inverse scattering problem method. For instance, this is the case for a relativistic Sine-Gordon model with the following action potential

$$S = \frac{1}{\gamma}\int dx\,dt\left(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - (1-\cos u)\right) \tag{43}$$

(in units with $\hbar = m = 1$). It was shown that here we have a rich spectrum of excitations :

a) initial particles of unity mass

b) particles of mass $M = 8/\gamma$ (solitons)

c) bound states of solitons with

$$M = M_n = \frac{16}{\gamma}\sin\theta_n, \quad \theta_n = \frac{\pi}{4N}(2n+1), \quad n = 0,\ldots,N,$$

$$N = \left[\frac{8\pi}{\gamma}\right]$$

We can obtain the S matrix for soliton scattering by semi-classical methods. The result is

$$S(s) = \exp\left(\frac{16i}{\gamma}\int_1^{\xi}\frac{dx}{x}\ln\frac{x-1}{x+1}\right), \quad \xi = \frac{s-2M^2+\sqrt{s(s-4M^2)}}{2M^2}, \tag{44}$$

$$M \doteq \frac{8}{\gamma}.$$

Quantum corrections to the soliton mass and the scattering phase may be obtained in the functional integral approach. These corrections turn out to be some power series on the γ parameter. So they are small for small γ comparatively to the quasi-classical values which are of the order of γ^{-1}.

The idea that some of the particles are nothing but vortex-like excitations (or solitons) is very attractive because it allows us to reduce a number of fundamental fields. Classical solutions can be labelled by some quantum numbers which have a topological nature. We can interpret these numbers as some charges. In theories with small coupling constants, we can develop a new perturbative scheme in functional integral formalism in order

to take into account small quantum fluctuations around classical solutions. Extended objects themselves interact strongly with each other as we have seen for both Goldstone and Sine-Gordon models.

Nowadays the search for more realistic examples of field theories with extended objects is a very actual problem. It may be that, namely in this way, we can find a key to strong interactions.

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The outlined scheme of integration over "fast" and "slow" variables may be applied to many problems of quantum field theory and statistical physics [see Ref. 5)].

Formula (56) - Lecture 7 - is a result of our discussions with Professor T.T. Wu.

The Goldstone model was suggested in

18) J. Goldstone, Nuovo Cimento 19, 154-164 (1961).

Quantum vortices in the Goldstone model are considered in Ref. 5). The trick of introducing a gauge field $A_\mu(x)$ in order to describe an interaction between vortices was applied to some models of superfluids in

19) V.N. Popov, JETP 64, 672 (1973) - [English translation in JETP 37, 341 (1973)].

Monopole solutions were found independently in

20) G. 't Hooft, Nuclear Phys. B79, 276 (1974).

21) A.M. Polyakov, JETP Letters 20, 194 (1974).

One can find soliton quantization methods in functional integral formalism in

22) V.E. Korepin, P.P. Kulish and L.D. Faddeev, JETP Letters 21, 60 (1975).

23) L.D. Faddeev, JETP Letters 21, 62 (1975).

24) L.D. Faddeev and V.E. Korepin, Teor.i Mat.Fiz. 25, 147 (1975).

25) L.D. Faddeev and V.E. Korepin, Quantization of solitons, Physics Reports (to be published).