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TRANSVERSE IMPEDANCE OF A RESISTIVE CYLINDER OF FINITE |LENGTH

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Abstract

In this report we calculate the transverse impedance of a charged particle beam, displaced from the axis of a circular cylindrical beam pipe which has finite resistivity only over a finite length. For this purpose we replace it with a cavity filled with the resistive material and connected to the region surrounding the beam by a gap of the same length. Matching of the electro-magnetic fields on both sides of the gap leads to an infinite set of linear equations for the field expansion coefficients with terms given by infinite sums over integrals which are evaluated by summing over their residues. By truncating these sums one can determine the transverse impedance and evaluate it numerically. Keeping only the lowest terms gives simple approximate expressions which are often sufficient to estimate the values and parameter dependence of the impedance.

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1 Introduction

Nearly all of the numerous publications on the transverse resistive wall impedance in charged particle accelerators[1, 2, 3, 4, 5, 6] analyze walls of infinite length, which is usually a good approximation for the effect of structures whose length is large compared to the distance from the beam. Exceptions treating resistive walls of finite length are ref[7] which uses the simplified Leontovich boundary conditions which are not always good approximations for all frequency regions, and ref[8] which uses the same approximations and is limited to the longitudinal impedance. Here we derive exact expressions for the transverse impedance of a resistive wall of finite length using field matching.

Walls of infinite length are usually a good approximation for the effect of structures whose length is large compared to the distance from the beam to the vacuum chamber wall. In particular for the LHC collimators, whose transverse impedance was recently a source of much concern for beam stability as it can become quite large when the highly resistive graphite jaws approach the beam to a distance of only millimeters, the effect of their finite length was hoped to reduce their impedance[9]. However, the length of these collimators is half a meter or more; hence the ratio length to transverse distance is so large that the use of the infinite length approximation for the estimate of their impedances[10, 11, 12] appears well justified.

Nevertheless, this concern was the reason for starting the present analysis, the results of which have now been found to be more important for the calculation of the impedance of a number of kickers which were recently installed in the PS and SPS for CNGS ans LHC operation. They have larger openings and thus their walls do not come very close to the beam, but their lengths are subdivided into short sections for which the ratio of length to transverse distance becomes quite small and needs to be taken into account in the calculation of their impedance which should be low enough to permit injection of rather strong beam currents. A similar subdivision has recently also been proposed for the LHC collimators and its effect therefore needs to be considered[13].

Here we calculate the transverse impedance for a model geometry consisting of an infinitesimally thin, annular beam of radius a, surrounded by a perfectly conducting, axially aligned, semi-infinite, circular-cylindrical beam pipe of radius b which is widened to a radius R over a length g thus forming an annular cavity (see Fig1). Through the gap an annular cylinder of resistive material is exposed to the beam. Its outer radius is R, its length g, conductivity σ_c, permittivity $\varepsilon = \varepsilon_0\varepsilon_r$ and permeability $\mu = \mu_0\mu_r$. The whole structure is surrounded by a perfectly conducting cylinder with radius R and annular end plates at $z = 0$ and $z = g$; thus the outer region forms an annular cavity.

This model for an element of finite length and finite conductivity permits the calculation of the impedance contribution due to the finite wall resistivity. The geometric part of impedance, due to the cross-section variation of the beam pipe can be calculated with standard analytical or numerical methods and should be added to the contribution calculated here. The separation into two independent contributions not only simplifies the calculation but also permits evaluating the influence of each contribution independently. It is a good approximation for elements consisting of materials with sufficiently high conductivity such as metals, but may break down when it is too low.

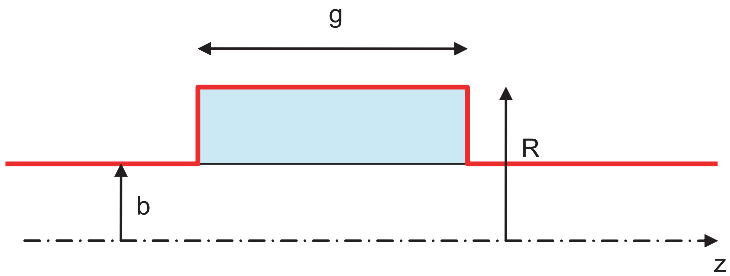

Figure 1: Geometry of beam model.

Our approach will be to use the frequency domain and solve the Helmholtz equations for the electric and magnetic fields in two separate uniform regions, obtain the expansion coefficients by field matching and finally obtain the transverse impedance as a function of frequency. Unlike the calculation of the longitudinal impedance, which needs only TM field components, for the transverse impedance both TM and TE field components are required in general[1].

2 Source Fields

In the time domain, a ring of radius a, traveling in the axial z direction, is assumed to have a charge distribution proportional to $\cos\theta$ which excites the dipole fields of interest here, traveling with velocity $v = \beta c$ in the axial z direction, is described by

$$\tilde{\rho}(r, \theta, z; t) = \frac{P}{\pi a^2} \cos\theta\, \delta(r-a)\, \delta(z - vt). \tag{2.1}$$

The dipole moment (in the x-direction) of this distribution is

$$\int\int r dr d\theta\, (r\cos\theta)\, \tilde{\rho}(r, \theta, z; t) = P\delta(z - vt). \tag{2.2}$$

We now proceed to the frequency domain by using the integral representation of the delta function

$$\delta(z - vt) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathrm{e}^{j\omega t} \left[\frac{\mathrm{e}^{-jkz}}{v}\right] \tag{2.3}$$

where $k = \omega/v$, and in the following drop the factor $\int \exp{(j\omega t)} d\omega/(2\pi)$.

We then can write the charge density in the frequency domain as

$$\rho(r, \theta, z; k) = \frac{P}{\pi a^2 v} \cos\theta\, \delta(r-a) \mathrm{e}^{-jkz}, \tag{2.4}$$

and the axial current density as

$$J_z(r, \theta, z; k) = \rho v = \frac{P}{\pi a^2} \cos\theta\, \delta(r-a) \mathrm{e}^{-jkz}. \tag{2.5}$$

The longitudinal electric field strength $E_z^{(s)}(r, \theta, z; k)$ due to the source terms in Eqs. (2.4) and (2.5) is given by the Helmholtz equation (i.e. the wave equation in the frequency domain)

$$\begin{aligned} \left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - k^2 + \frac{\omega^2}{c^2}\right] E_z^{(s)} &= j\omega\mu_0 J_z + \frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial z} \\ &= -\frac{jkP}{\pi\varepsilon_0 a^2 v \gamma^2} \cos\theta\, \delta(r-a) \mathrm{e}^{-jkz}\,, \end{aligned} \tag{2.6}$$

where

$$\gamma = [1 - v^2/c^2]^{-1/2}. \tag{2.7}$$

The same Helmholtz equation holds for the longitudinal magnetic field strength H_z, but with $curl_z J$ on the right hand side. However, since the source current in Eqs. (2.5) is purely longitudinal, the z-component of its curl is zero, hence also the driving term for the magnetic source field and thus $H_z^{(s)} \equiv 0$.

For dipole oscillations, the driving term for $E_z^{(s)}$ is proportional to $\cos\theta$ and hence also $E_z^{(s)}$ can be written

$$E_z^{(s)}(r, \theta, z; k) = \frac{jkP}{\pi\varepsilon_0 a v \gamma^2} \cos\theta\, \mathrm{e}^{-jkz} \left\{ \begin{array}{lcl} I_1(s)K_1(u) & , & r \geq a \\ I_1(u)K_1(s) & , & r \leq a \end{array} \right\}, \tag{2.8}$$

[1] The superscript TM refers to "transverse magnetic modes" with $H_z = 0$, while TE refers to "transverse electric modes" with $E_z = 0$. All TM and TE modes considered in this paper are dipole modes with azimuthal mode number one.

where $s = ka/\gamma$, $u = kr/\gamma$ and $I_1,\ K_1$ are modified Bessel functions of first order and both kinds.

From this point on, we shall only use the electro-magnetic fields *outside the particle beam*, i.e. for $r \geq a$, and thus

$$E_z^{(s)}(r, \theta, z; k) = j \cos\theta \, D \, K_1(u) \mathrm{e}^{-jkz} \, . \tag{2.9}$$

For the usual case $s << 1$ one may approximate $I_1(s)$ by $s/2 = ka/2\gamma$ and the constant D then is given by

$$D = \frac{k^2 \, P}{2\pi\varepsilon_0 v \gamma^3} = \frac{k^2 \, P \, Z_0}{2\pi\beta\gamma^3} \, . \tag{2.10}$$

The transverse field components can be obtained from $E_z^{(s)}$, using the Maxwell equations, e.g as described in Appendix B of ref.[12] or in section 8.2 of ref.[14] . They can be written

$$\begin{aligned}
E_\theta^{(s)} &= \gamma D \, \sin\theta \left[\frac{K_1(u)}{u} \right] \mathrm{e}^{-jkz} \, , \\
Z_0 H_\theta^{(s)} &= -\beta\gamma D \, \cos\theta K_1'(u) \mathrm{e}^{-jkz} \, , \\
E_r^{(s)} &= \frac{c}{v} Z_0 H_\theta^{(s)} = -\gamma D \, \cos\theta \, K_1'(u) \mathrm{e}^{-jkz} \, , \\
Z_0 H_r^{(s)} &= -\frac{v}{c} E_\theta^{(s)} = -\beta\gamma D \, \sin\theta \left[\frac{K_1(u)}{u} \right] \mathrm{e}^{-jkz} \, .
\end{aligned} \tag{2.11}$$

All source fields are purely TM in character when the driving current is purely longitudinal. There are no transverse currents in the model we have assumed and hence no TE source fields. However, TE modes are excited by the presence of conducting walls and have to be included in the analysis.

3 Fields for $a \leq r \leq b$: TM Part

In the infinitely long, hollow cylindrical region $a \leq r \leq b$ between the beam and the wall, the complete solution for the longitudinal electric dipole field strength consists of the source field to which one must add solutions of the homogeneous Helmholtz equation. We will express them as a sum or integral over product solutions for arbitrary values of the separation constant q which equals the *axial propagation constant*. The axial dependence is given by the exponential function e^{-jqz}, and the radial one by Bessel functions of first order $J_1(\kappa r)$, which we normalize by the value at the beam pipe radius $J_1(\kappa b)$.

The radial propagation constant κ is related to the axial one by

$$\kappa^2 = \beta^2 k^2 - q^2 \,. \tag{3.1}$$

Since q can take any value, we integrate the product solutions over q from $-\infty$ to $+\infty$, after multiplication with a yet unknown weight function $A(q)$ to get

$$\begin{aligned} E_z(r,\theta,z;k) = j\cos\theta \Big\{ & D\,F(u)\,\mathrm{e}^{-jkz} \\ & + \int_{-\infty}^{\infty} dq\,\mathrm{e}^{-jqz} A(q) \frac{J_1(\kappa r)}{J_1(\kappa b)} \Big\} \,. \end{aligned} \tag{3.2}$$

The weight function $A(q)$ will be determined by the matching conditions at the gap. In writing this equation we have changed the modified Bessel function $K_1(u)$ in the expression for the source field, Eq.(2.9), to the function $F(u)$ which is zero at the beam pipe radius. It is obtained by subtracting a term which is well behaved for $r \to 0$:

$$F(u) = K_1(u) - \frac{K_1(x)}{I_1(x)} I_1(u) \,, \tag{3.3}$$

where $x = kb/\gamma$. This choice makes $F(x) = 0$, and therefore the integral over $A(q)$ alone generates the gap field:

$$E_z(b,\theta,z;k) = j\cos\theta \int_{-\infty}^{\infty} dq\,\mathrm{e}^{-jqz} A(q) \,. \tag{3.4}$$

In evaluating the integral in Eq.(3.2), we have to avoid the apparent singularities due to the zeros of $J_1(u)$ which occur when $\kappa b = r_\ell$, i.e. when

$$qb = \pm \left(\beta^2 k^2 b^2 - r_\ell^2 \right)^{1/2} \tag{3.5}$$

where r_ℓ are the positive zeros of J_1. We take the integration contour in the complex q-plane to go *above* the poles on the positive real q-axis and *below* the poles on the negative real q-axis where their contributions vanish.

For $z > 0$, we may then close the contour in the *lower* half plane at $|q| = \infty$ where the factor $\exp(-jqz)$ goes to zero strongly, while for $z < 0$ we close it in the *upper* half plane where the factor $\exp(jq|z|)$ goes to zero strongly, and thus do not contribute to the integral.

As before, the transverse field components can be found from the Maxwell equations

$$E_\theta^{(\mathrm{TM})} = \frac{\sin\theta}{r} \left\{ \frac{\gamma^2}{k} D\,F(u)\mathrm{e}^{-jkz} - \int_{-\infty}^{\infty} \frac{q dq}{\kappa^2}\,\mathrm{e}^{-jqz} A(q) \frac{J_1(\kappa r)}{J_1(\kappa b)} \right\} , \tag{3.6}$$

$$Z_0 H_\theta^{(\mathrm{TM})} = \cos\theta \left\{ -\beta\gamma D\,F'(u)\mathrm{e}^{-jkz} + \beta k b \int_{-\infty}^{\infty} dq\,\mathrm{e}^{-jqz} A(q) \frac{J_1'(\kappa r)}{\kappa b\, J_1(\kappa b)} \right\} , \tag{3.7}$$

$$E_r^{(\mathrm{TM})} = \cos\theta \left\{ -\gamma D\,F'(u)\mathrm{e}^{-jkz} \int_{-\infty}^{\infty} \frac{q dq}{\kappa}\,\mathrm{e}^{-jqz} A(q) \frac{J_1'(\kappa r)}{J_1(\kappa b)} \right\} , \tag{3.8}$$

$$Z_0 H_r^{(\mathrm{TM})} = \frac{\sin\theta}{r} \left\{ \frac{\beta\gamma^2}{k} D \int_{-\infty}^{\infty} \frac{dq}{\kappa^2}\,\mathrm{e}^{-jqz} C(q) \frac{J_1'(\kappa r)}{J_1'(\kappa b)} \right\} . \tag{3.9}$$

4 Fields for $a \le r \le b$: TE Part

In order to satisfy all boundary conditions, we need to include TE field components. As pointed out at the end of Section **2**, there are no TE source fields, and hence we generate the TE fields for $r \le b$ from the solutions of the homogeneous Helmholtz equation alone

$$Z_0 H_z(r, \theta, z; k) = \sin\theta \int_{-\infty}^{\infty} dq\, \mathrm{e}^{-jqz} C(q) \frac{J_1(\kappa r)}{J_1'(\kappa b)}\,, \tag{4.1}$$

where the new weight function $C(q)$ is yet to be determined and we changed the normalization to $J_1'(\kappa b)$.

In evaluating the integral in Eq.(4.1) we avoid the singularities at the zeros of $J_1'(y)$, which occur when $\kappa b = p_\ell$, when p_ℓ are the positive zeros of $J_1'(y)$, i.e. when

$$qb = \pm \left(\beta^2 k^2 b^2 - p_\ell^2 \right)^{1/2} . \tag{4.2}$$

As before, we take the integration contour in the complex q-plane to go *above* the poles on the positive real q-axis and *below* the poles on the negative real q-axis. For $z > 0$, the contour will be closed in the *lower* half q-plane where its contribution vanishes for $|q| \to \infty$ as the factor $\exp(-jqz)$ goes to zero strongly. For $z < 0$, the contour should be closed in the *upper* half q-plane where for $q \to \infty$ the function $\exp(jq|z|)$ tends to zero strongly.

The transverse TE field components can be found from the Maxwell equations as before and are written

$$E_\theta^{(\mathrm{TE})} = j\beta k b \sin\theta \int_{-\infty}^{\infty} \frac{dq}{\kappa}\, \mathrm{e}^{-jqz} C(q) \frac{J_1'(\kappa r)}{J_1'(\kappa b)}\,, \tag{4.3}$$

$$Z_0 H_\theta^{(\mathrm{TE})} = -\frac{j\cos\theta}{r} \int_{-\infty}^{\infty} \frac{q dq}{\kappa^2}\, \mathrm{e}^{-jqz} C(q) \frac{J_1(\kappa r)}{J_1'(\kappa b)}\,, \tag{4.4}$$

$$E_r^{(\mathrm{TE})} = -\frac{j\beta k \cos\theta}{r} \int_{-\infty}^{\infty} dq\, \frac{1}{\kappa^2}\, \mathrm{e}^{-jqz} C(q) \frac{J_1(\kappa r)}{J_1'(\kappa b)}\,, \tag{4.5}$$

$$Z_0 H_r^{(\mathrm{TE})} = -j \sin\theta \int_{-\infty}^{\infty} \frac{q dq}{\kappa}\, \mathrm{e}^{-jqz} C(q) \tag{4.6}$$

5 Fields for $b \le r \le R$: TM Part

The region $b \le r \le R,\ 0 \le z \le g$ forms a cavity which we assume to be filled with resistive material with complex permittivity $\varepsilon' = \varepsilon - j\sigma/\omega$ and complex permeability $\mu' = \mu_0(\mu_r + j\tan\theta_M)$. Since the region is of finite length, the axial propagation constant can only take discrete values q_n, where $0 < n < \infty$ is called *axial mode number*. The boundary conditions for the electric field at the perfectly conducting annular end plates $b \le r \le R$ at $z = 0$ and $z = g$ require vanishing of the radial and azimuthal components, or simpler of the axial derivative of the longitudinal component i.e. $dE_z/dz = 0$. This can be achieved by an axial dependence $\propto \cos(q_n z)$ i.e. the axial propagation constants are then given by

$$q_n = \frac{n\pi}{g}\,. \tag{5.1}$$

The radial dependence is expressed by a combination of Bessel functions of first order and of both kinds with argument $\alpha_n r$, where the *radial propagation constants* α_n are related to the axial ones by the root in the 4-th quadrant of

$$\alpha_n^2 = \omega^2\mu'\varepsilon' - q_n^2\,. \tag{5.2}$$

The longitudinal electric field strength is obtained by multiplying the product solution with coefficients A_n and summing over all axial mode numbers

$$E_z(r,\theta,z;k) = j\cos\theta\sum_{n=0}^{\infty} A_n\cos(q_n z)\frac{S(\alpha_n r)}{S(\alpha_n b)}\,, \tag{5.3}$$

where the combinations of Bessel functions $S(w)$ and $S'(w)$ are defined by

$$\begin{aligned} S(w) &= Y_1(w)J_1(\alpha_n R) - J_1(w)Y_1(\alpha_n R)\,,\\ S'(w) &= Y_1'(w)J_1(\alpha_n R) - J_1'(w)Y_1(\alpha_n R)\,. \end{aligned} \tag{5.4}$$

The functions $S(w)$ and dS'/dw vanish at $w = \alpha_n R$, hence the boundary conditions $E_z = 0$ as well as $E_\theta = 0$ for the tangential electric field components at the outer wall $r = R$ are fulfilled automatically.

The transverse field components can again be found from the Maxwell equations and become

$$E_\theta^{(\mathrm{TM})}(r,\theta,z;k) = \frac{j\sin\theta}{r}\sum_{n=1}^{\infty}\frac{q_n}{\alpha_n^2}A_n\sin(q_n z)\frac{S(\alpha_n r)}{S(\alpha_n b)}\,, \tag{5.5}$$

$$E_r^{(\mathrm{TM})}(r,\theta,z;k) = -j\cos\theta\sum_{n=1}^{\infty}\frac{q_n}{\alpha_n}A_n\sin(q_n z)\frac{S'(\alpha_n r)}{S(\alpha_n b)}\,, \tag{5.6}$$

$$Z_0H_\theta^{(\mathrm{TM})}(r,\theta,z;k) = \beta k\cos\theta\frac{\varepsilon'}{\varepsilon_0}\sum_{n=0}^{\infty}\frac{q_n}{\alpha_n}A_n\cos(q_n z)\frac{S'(\alpha_n r)}{S(\alpha_n b)}\,, \tag{5.7}$$

and

$$Z_0H_r^{(\mathrm{TM})}(r,\theta,z;k) = \frac{\beta k\sin\theta}{r}\frac{\varepsilon'}{\varepsilon_0}\sum_{n=0}^{\infty}\frac{A_n}{\alpha_n^2}\cos(q_n z)\frac{S(\alpha_n r)}{S(\alpha_n b)} \tag{5.8}$$

which fulfill the boundary conditions for a perfectly conducting wall at $r = R$, i.e. all tangential electric field components vanish there: $E_z(R) = E_\theta(R) = 0$.

6 Fields for $b \leq r \leq R$: TE Part

The expression for the longitudinal magnetic field H_z is similar to that for E_z, the longitudinal electric one, but with $cos(q_n z)$ replaced by $sin(q_n z)$ so that the normal component of the magnetic fiel$H_z = 0$, vanishes at both end plates $z = 0$ and $z = g$. Furthermore we replace the functions $S(w)$ and $S'(w)$ by $T(w)$ and $T'(w)$ which has a vanishing derivative at $r = R$. Hence we write H_z as

$$Z_0 H_z(r, \theta, z; k) = \sin\theta \sum_{n=1}^{\infty} C_n \sin(q_n z) \frac{\alpha_n b\, T(\alpha_n r)}{T'(\alpha_n b)} \,. \tag{6.1}$$

The radial derivative of the tangential magnetic field component is zero at the outer wall and fulfills the boundary condition for a perfect conducting outer wall.

The functions $T(w)$ and $T'(w)$ are defined by

$$\begin{aligned} T(w) &= Y_1(w) J_1'(\alpha_n R) - J_1(w) Y_1'(\alpha_n R) \,, \\ T'(w) &= Y_1'(w) J_1'(\alpha_n R) - J_1'(w) Y_1'(\alpha_n R) \,. \end{aligned} \tag{6.2}$$

The function $T(w)$ has a vanishing radial derivative at $r = R$, hence $dH_z/dr = 0$ as required at a perfectly conducting wall. The coefficients C_n will be determined by field matching. The other transverse field components are again found from the Maxwell equations

$$E_\theta^{(\mathrm{TE})}(r, \theta, z; k) = j \frac{\mu}{\mu_0} \beta k b \sin\theta \sum_{n=1}^{\infty} C_n \sin(q_n z) \frac{T'(\alpha_n r)}{T'(\alpha_n b)} \,, \tag{6.3}$$

$$E_r^{(\mathrm{TE})}(r, \theta, z; k) = -j \frac{\mu}{\mu_0} \frac{\beta k b^2 \cos\theta}{r} \sum_{n=1}^{\infty} C_n \sin(q_n z) \frac{T(\alpha_n r)}{\alpha_n b\, T'(\alpha_n b)} \,, \tag{6.4}$$

$$Z_0 H_\theta^{(\mathrm{TE})}(r, \theta, z; k) = \frac{b \cos\theta}{r} \sum_{n=1}^{\infty} q_n\, C_n \cos(q_n z) \frac{T(\alpha_n r)}{\alpha_n b\, T'(\alpha_n b)} \,, \tag{6.5}$$

$$Z_0 H_r^{(\mathrm{TE})}(r, \theta, z; k) = b \sin\theta \sum_{n=1}^{\infty} q_n C_n \cos(q_n z) \frac{T'(\alpha_n r)}{T'(\alpha_n b)} \tag{6.6}$$

which fulfill all boundary conditions at $r = R$.

7 Field Matching at the gap

7.1 Longitudinal electric field E_z:

From Eq. (3.2) and with $F(x) = 0$ we obtain the axial electric field just inside the tube radius

$$E_z(b_-, \theta, z; k) = \left\{ \begin{array}{ccc} j\cos\theta \int_{-\infty}^{\infty} dq\, \mathrm{e}^{-jqz} A(q) & , & 0 < z < g \\ 0 & , & z < 0 \text{ and } z > g \end{array} \right\} . \tag{7.1}$$

The inverse Fourier transform of this expression leads to

$$2\pi j \cos\theta\, A(q) = \int_0^g dz\, \mathrm{e}^{jqz} E_z(b_-, \theta, z; k). \tag{7.2}$$

The axial electric field just outside the gap is found from Eq. (5.3)

$$E_z(b_+, \theta, z; k) = j\cos\theta \sum_{n=0}^{\infty} A_n \cos(q_n z)\,. \tag{7.3}$$

Since the axial electric field must be equal on both sides of the gap, substituting Eq. (7.3) into (7.2) yields an expression for the weight function $A(q)$

$$A(q) = g \sum_{n=0}^{\infty} A_n G_n(q) \tag{7.4}$$

where

$$G_n(q) \equiv \frac{1}{2\pi g} \int_0^g dz\, \mathrm{e}^{jqz} \cos(q_n z) = \frac{qg}{2\pi j} \left[\frac{(-1)^n \mathrm{e}^{jqg} - 1}{q^2 g^2 - n^2 \pi^2} \right] . \tag{7.5}$$

7.2 Azimuthal electric field E_θ:

Also the tangential electric field component E_θ must be continuous across the gap at $r = b$, $0 < z < g$, and should be zero at the perfectly conducting beam pipes at $r = b$, $z < 0$, and $z > g$.

From Eqs. (3.6) and (4.3) we get

$$\begin{aligned} E_\theta(b_-, \theta, z; k) = -\frac{\sin\theta}{b} \int_{-\infty}^{\infty} \frac{qdq}{\kappa^2}\, \mathrm{e}^{-jqz} A(q) + \\ j\beta kb\, \sin\theta \int_{-\infty}^{\infty} \frac{dq}{\kappa}\, \mathrm{e}^{-jqz} C(q)\,. \end{aligned} \tag{7.6}$$

while

$$E_\theta(b_+, \theta, z; k) = j\sin\theta \sum_{n=1}^{\infty} D_n \sin(q_n z) \tag{7.7}$$

where we define the coefficients D_n as

$$D_n \equiv \frac{q_n}{\alpha_n^2 b} A_n + \frac{\mu\beta kb}{\mu_0} C_n\,. \tag{7.8}$$

Dropping the common factor $j\sin\theta$ leads to

$$\begin{aligned} \frac{j}{b} \int_{-\infty}^{\infty} \frac{qdq}{\kappa^2}\, \mathrm{e}^{-jqz} A(q) + \beta k \int_{-\infty}^{\infty} \frac{dq}{\kappa}\, \mathrm{e}^{-jqz} C(q) = \\ \left\{ \begin{array}{ccc} \sum_{n=1}^{\infty} D_n \sin(q_n z) & , & 0 < z < g \\ 0 & , & z < 0,\ z > g \end{array} \right\} . \end{aligned} \tag{7.9}$$

Taking the inverse Fourier transform of this equation leads to

$$\frac{jqA(q)}{b\kappa^2} + \frac{\beta k C(q)}{\kappa} = \frac{j\pi}{q} \sum_{n=0}^{\infty} D_n G_n(q)\,. \tag{7.10}$$

Using Eqs. (7.5) and (7.10), we find

$$\frac{\beta k}{\kappa}\, C(q) = \frac{j\pi}{q} \sum_{n=1}^{\infty} G_n(q) \left[nD_n - \frac{gq^2}{\pi b \kappa^2}\, A_n \right]\,. \tag{7.11}$$

7.3 Longitudinal magnetic field H_z:

We also require continuity of the longitudinal magnetic field component across the gap at $r = b$, $0 < z < g$. Using Eqs. (4.1) and (6.1) one finds

$$\int_{-\infty}^{\infty} dq\, \mathrm{e}^{-jqz} C(q)\, \frac{J_1(\kappa b)}{J_1'(\kappa b)} = \sum_{n=1}^{\infty} C_n \sin(q_n z) \frac{\alpha_n b\, T(\alpha_n b)}{T'(\alpha_n b)}\,. \tag{7.12}$$

Integrating over z leads to

$$C_m\, \frac{\alpha_m b\, T(\alpha_m b)}{T'(\alpha_m b)} = -4\pi^2 mj \int_{-\infty}^{\infty} dq\, \frac{1}{qg} \frac{J_1(\kappa b)}{J_1'(\kappa b)} G_m^*(q) C(q)\,. \tag{7.13}$$

We now use the expression for $C(q)$ in Eq. (7.11) to obtain

$$\begin{aligned} C_m\, \frac{\alpha_m b\, T(\alpha_m b)}{T'(\alpha_m b)} &= \frac{4\pi^3 m}{\beta k b g} \int_{-\infty}^{\infty} dq\, \frac{1}{q^2} \frac{\kappa b\, J_1(\kappa b)}{J_1'(\kappa b)} G_m^*(q) G_n(q) \sum_{n=1}^{\infty} \left[nD_n - \frac{gq^2}{\pi b\kappa^2}\, A_n \right] \\ &= \frac{4\pi}{\beta k b} \sum_{n=1}^{\infty} D_n Q_{nm} - \frac{4m}{\beta k g} \sum_{n=1}^{\infty} A_n P_{nm}\,, \end{aligned} \tag{7.14}$$

for $m \geq 1$, and where

$$P_{nm} \equiv \pi^2 g \int_{-\infty}^{\infty} dq\, \frac{J_1(\kappa b)}{\kappa b\, J_1'(\kappa b)} G_m^*(q) G_n(q) \tag{7.15}$$

and

$$Q_{nm} \equiv \frac{\pi^2 nm}{g} \int_{-\infty}^{\infty}, \frac{dq}{q^2} \frac{\kappa b\, J_1(\kappa b)}{J_1'(\kappa b)} G_m^*(q) G_n(q)\,. \tag{7.16}$$

It may appear that $P_{nm} = P_{nm}^*$ and $Q_{nm} = Q_{nm}^*$ but this is *not* correct, since the integration contours in Eqs. (7.15) and (7.16) must extend into the complex plane to avoid the singularities of $J_1'(\kappa b)$ at $\kappa b = \pm p_\ell$.

The terms in the integrands of Eqs. (7.15), (7.16), as well as in the later equations for T_{nm} and R_{nm} are all *even* functions of q. As discussed following Eqs. (3.4) and (4.2), the contour in the q-plane should go *above* the poles on the *positive* real q-axis and *below* the poles on the *negative* real q-axis. We can therefore change the sign of q in the third term within the bracket in Eq. (7.14), so that it can be replaced by

$$G_m^*(q) G_n(q) = \frac{q^2 g^2 \left[1 + (-1)^{m+n}\right] \left[1 - (-1)^n \mathrm{e}^{jqg}\right]}{4\pi^2\, (q^2 g^2 - m^2\pi^2)(q^2 g^2 - n^2\pi^2)}\,. \tag{7.17}$$

The dimensionless matrices Q_{nm} and P_{nm} are evaluated in Appendices A and B.

7.4 Azimuthal magnetic field H_θ:

Finally we turn to the continuity of $H_\theta(r,\theta,z;k)$ across the gap at $r=b$, $0<z<g$. Using Eqs. (3.7), (4.4), (5.7) and (6.5) we get

$$\begin{aligned} &-\beta\gamma D\, F'(x)\, \mathrm{e}^{-jkz} + \\ \beta kb \int_{-\infty}^{\infty} dq\, \mathrm{e}^{-jqz} A(q) \frac{J_1'(\kappa b)}{\kappa b\, J_1(\kappa b)} - \frac{j}{b}\int_{-\infty}^{\infty} \frac{q dq}{\kappa^2} \mathrm{e}^{-jqz} C(q) \frac{J_1(\kappa b)}{J_1'(\kappa b)} &= \\ \beta kb\, \frac{\varepsilon'}{\varepsilon_0} \frac{S'(\alpha_n b)}{\alpha_n b\, S(\alpha_n b)} + \sum_{n=1}^{\infty} \frac{q_n b}{g}\, C_n \cos(qz) \frac{T(\alpha_n r)}{\alpha_n b\, T'(\alpha_n b)}\,. \end{aligned} \tag{7.18}$$

We now multiply each side of this equation by $\cos m\pi z/g$ and integrate over the length of the gap. With the integrals

$$\begin{aligned} \frac{1}{g}\int_0^g dz\, \cos\left(\frac{m\pi z}{g}\right) \mathrm{e}^{-jkz} &= -jkg\left[\frac{(-1)^m \mathrm{e}^{-jkg}-1}{m^2\pi^2 - k^2 g^2}\right], \\ \frac{1}{g}\int_0^g dz\, \cos\left(\frac{m\pi z}{g}\right) \mathrm{e}^{-jqz} &= jqg\left[\frac{(-1)^m \mathrm{e}^{-jqg}-1}{q^2 g^2 - m^2\pi^2}\right] = 2\pi G_m^*(q)\,, \\ \frac{1}{g}\int_0^g dz\, \cos\left(\frac{m\pi z}{g}\right) \cos(q_n z) &= \frac{\delta_{mn}(1+\delta_{m0})}{2}\,, \end{aligned} \tag{7.19}$$

this leads to

$$\begin{aligned} &j\beta\gamma kg\, D\, F'(x)\left[\frac{(-1)^m \mathrm{e}^{-jkg}-1}{m^2\pi^2 - k^2 g^2}\right] + \\ + 2\pi\beta kb \int_{-\infty}^{\infty} dq\, A(q) \frac{J_1'(\kappa b)}{\kappa b\, J_1(\kappa b)} G_m^*(q) - 2\pi j \int_{-\infty}^{\infty} dq\, \frac{q}{\kappa} C(q) \frac{J_1(\kappa b)}{\kappa b\, J_1'(\kappa b)} G_m^*(q) &= \\ = \beta kb\, \frac{\varepsilon'}{\varepsilon_0} A_m \left(\frac{1+\delta_{m0}}{2}\right) \frac{S'(\alpha_m b)}{\alpha_m b\, S(\alpha_m b)} + \frac{m\pi b}{2g} C_m \frac{T(\alpha_m b)}{\alpha_m b\, T'(\alpha_m b)}\,. \end{aligned} \tag{7.20}$$

Using the Wronskian relation

$$K_1'(x)\, I_1(x) - I_1'(x)\, K_1(x) = -1/x\,, \tag{7.21}$$

we obtain from Eq. (3.3)

$$F'(x) = K_1'(x) - \frac{K_1(x)}{I_1(x)}\, I_1'(x) = -\frac{1}{x\, I_1(x)}\,. \tag{7.22}$$

7.5 Combining the Matching Equations

We now use Eq. (7.4) for $A(q)$, (7.11) for $C(q)$, and (3.3) for $F(u)$ to rewrite Eq. (7.20) as

$$\begin{aligned} & -\frac{j\beta\gamma kg\,D}{x\,I_1(x)}\left[\frac{(-1)^m \mathrm{e}^{-jkg}-1}{m^2\pi^2-k^2g^2}\right] \\ & + 2\pi\beta kbg\sum_{n=0}^{\infty}A_n\int_{-\infty}^{\infty}dq\,\frac{J_1'(\kappa b)}{\kappa b\,J_1(\kappa b)}G_n(q)G_m^*(q) \\ & + \frac{2\pi^2}{\beta k}\sum_{n=0}^{\infty}nD_n\int_{-\infty}^{\infty}dq\,\frac{J_1(\kappa b)}{\kappa b\,J_1'(\kappa b)}G_n(q)G_m^*(q) \\ & - \frac{2\pi g}{\beta kb}\sum_{n=0}^{\infty}A_n\int_{-\infty}^{\infty}dq\,\frac{q^2}{\kappa^2}\frac{J_1(\kappa b)}{\kappa b\,J_1'(\kappa b)}G_n(q)G_m^*(q) \\ = {} & \beta kb\,\frac{\varepsilon'}{\varepsilon_0}A_m\left(\frac{1+\delta_{m0}}{2}\right)\frac{S'(\alpha_m b)}{\alpha_m b\,S(\alpha_m b)}+\frac{m\pi b}{2g}C_m\frac{T(\alpha_m b)}{\alpha_m b\,T'(\alpha_m b)}\,, \end{aligned} \tag{7.23}$$

where

$$\begin{aligned} T_{nm} \equiv {} & \frac{4\pi^2g^3}{b^2}\int_{-\infty}^{\infty}\frac{q^2dq}{\kappa^2}\frac{J_1(\kappa b)}{\kappa b\,J_1'(\kappa b)}G_n(q)G_m^*(q) \\ = {} & g^5\left[1+(-1)^{m+n}\right]\int_{-\infty}^{\infty}dq\,\frac{q^4}{\kappa^2b^2}g\frac{J_1(\kappa b)}{\kappa b\,J_1'(\kappa b)}\frac{[1-(-1)^n\mathrm{e}^{jqg}]}{(q^2g^2-m^2\pi^2)(q^2g^2-n^2\pi^2)}\,. \end{aligned} \tag{7.24}$$

and

$$\begin{aligned} R_{nm} = 4\pi^2g\int_{-\infty}^{\infty}dq\,\frac{J_1'(\kappa b)}{\kappa b\,J_1(\kappa b)}G_n(q)G_m^*(q) = g^3\left[1+(-1)^{m+n}\right]\int_{-\infty}^{\infty}dq\,q^2\frac{J_1'(\kappa b)}{\kappa b\,J_1(\kappa b)}\times \\ \frac{[1-(-1)^n\mathrm{e}^{jqg}]}{(q^2g^2-m^2\pi^2)(q^2g^2-n^2\pi^2)} \end{aligned} \tag{7.25}$$

Both T_{nm} in Eq. (7.24) and R_{nm} in Eq. (7.25) have been arrived at by changing the sign of q in the third term of the bracket [ ] in Eq. (7.20) to reach the form in Eq. (7.23). The matrix T_{nm} is evaluated explicitly in Appendix C and matrix R_{nm} in Appendix D.

Next we use Eqs. (7.15), (7.16), (7.24) and (7.25) to rewrite these expressions as

$$\begin{aligned} & -\frac{j\beta\gamma kg\,D}{x\,I_1(x)}\left[\frac{(-1)^m\mathrm{e}^{-jkg}-1}{m^2\pi^2-k^2g^2}\right]+\frac{\beta kb}{2\pi}\sum_{n=0}^{\infty}A_nR_{nm}+\frac{2}{\beta kg}\sum_{n=1}^{\infty}nD_nP_{nm} \\ & -\frac{b}{2\pi\beta kg^2}\sum_{n=0}^{\infty}A_nT_{nm}=\beta kb\,\frac{\varepsilon'}{\varepsilon_0}\left(\frac{1+\delta_{m0}}{2}\right)\frac{S'(\alpha_m b)}{\alpha_m b\,S(\alpha_m b)}A_m \\ & +\frac{m\pi}{2\beta kg}\frac{\mu_0}{\mu}\frac{T(\alpha_m b)}{\alpha_m b\,T'(\alpha_m b)}D_m-\frac{m^2\pi^2}{2\beta kg^2\alpha_m^2 b}\frac{\mu_0}{\mu}\frac{T(\alpha_m b)}{\alpha_m b\,T'(\alpha_m b)}A_m\,. \end{aligned} \tag{7.26}$$

We now use Eq. (7.8) to eliminate the coefficients C_n. This leads to

$$\begin{aligned} & \frac{\alpha_m b\,T(\alpha_m b)}{T'(\alpha_m b)}\frac{\mu_0}{\mu\beta kb}\left[D_m-\frac{m\pi}{bg\alpha_m^2}A_m\right]= \\ & \frac{4\pi}{\beta kb}\sum_{n=1}^{\infty}D_nQ_{nm}-\frac{4m}{\beta kg}\sum_{n=0}^{\infty}A_nP_{nm}\,, \end{aligned} \tag{7.27}$$

which is valid for $m \geq 1$. We also rewrite Eq. (7.23) as

$$\begin{aligned} & \frac{2m}{\beta kg}\sum_{n=0}^{\infty}A_nP_{nm}-\frac{m\pi}{2\beta kg}\frac{\mu_0}{\mu}\frac{T(\alpha_m b)}{\alpha_m b\,T'(\alpha_m b)}A_m \\ & =\frac{2\pi}{\beta kb}\sum_{n=1}^{\infty}D_nQ_{nm}-\frac{\mu_0}{2\beta kb\mu}\frac{\alpha_m b\,T(\alpha_m b)}{T'(\alpha_m b)}D_m\,, \end{aligned} \tag{7.28}$$

valid for $m \geq 1$. We also rewrite Eq. (7.26) as

$$\begin{aligned} &\frac{2}{\beta k g}\sum_{n=1}^{\infty} n D_n P_{nm} - \frac{m\pi}{2\beta k g}\frac{\mu_0}{\mu}\frac{T(\alpha_m b)}{\alpha_m b\, T'(\alpha_m b)} D_m \\ &+ \frac{\beta k b}{2\pi}\sum_{n=0}^{\infty} A_n R_{nm} - \frac{b}{2\pi\beta k g^2}\sum_{n=0}^{\infty} A_n T_{nm} - \beta k b\,\frac{\varepsilon'}{\varepsilon_0}\left(\frac{1+\delta_{m0}}{2}\right)\frac{S'(\alpha_m b)}{\alpha_m b\, S(\alpha_m b)} A_m \\ &+ \frac{m^2\pi^2}{2\beta k g^2 \alpha_m^2 b}\frac{\mu_0}{\mu}\frac{T(\alpha_m b)}{\alpha_m b\, T'(\alpha_m b)} A_m = \frac{j\beta\gamma k g\, D}{x\, I_1(x)}\left[\frac{(-1)^m e^{-jkg} - 1}{m^2\pi^2 - k^2 g^2}\right], \end{aligned} \tag{7.29}$$

which is valid for $m \geq 0$.

Finally, we renormalize A_n and D_n to remove the factor outside the bracket on the right hand side of Eq. (7.29). Specifically, we write

$$A_n = \frac{j\beta\gamma k g\, D}{x\, I_1(x)} a_n \quad \text{and} \quad D_n = \frac{j\beta\gamma k g\, D}{x\, I_1(x)} d_n\,. \tag{7.30}$$

Eqs. (7.28) and (7.29) are then replaced by

$$\begin{aligned} &\frac{2m}{\beta k g}\sum_{n=0}^{\infty} a_n P_{nm} - \frac{m\pi}{2\beta k g}\frac{\mu_0}{\mu}\frac{T(\alpha_m b)}{\alpha_m b\, T'(\alpha_m b)} a_m \\ &-\frac{2\pi}{\beta k b}\sum_{n=1}^{\infty} d_n Q_{nm} + \frac{\mu_0}{2\beta k b \mu}\frac{\alpha_m b\, T(\alpha_m b)}{T'(\alpha_m b)} d_m = 0 \end{aligned} \tag{7.31}$$

for $m \geq 1$, and

$$\begin{aligned} &\frac{2}{\beta k g}\sum_{n=1}^{\infty} n\, d_n P_{nm} - \frac{m\pi}{2\beta k g}\frac{\mu_0}{\mu}\frac{T(\alpha_m b)}{\alpha_m b\, T'(\alpha_m b)} d_m \\ &+ \frac{\beta k b}{2\pi}\sum_{n=0}^{\infty} a_n R_{nm} - \frac{b}{2\pi\beta k g^2}\sum_{n=0}^{\infty} a_n T_{nm} - \beta k b\,\frac{\varepsilon'}{\varepsilon_0}\left(\frac{1+\delta_{m0}}{2}\right)\frac{S'(\alpha_m b)}{\alpha_m b\, S(\alpha_m b)} a_m \\ &+ \frac{m^2\pi^2}{2\beta k g^2 \alpha_m^2 b}\frac{\mu_0}{\mu}\frac{T(\alpha_m b)}{\alpha_m b\, T'(\alpha_m b)} a_m = \left[\frac{(-1)^m e^{-jkg} - 1}{m^2\pi^2 - k^2 g^2}\right] \end{aligned} \tag{7.32}$$

for $m \geq 0$.

Eqs. (7.31) and (7.32) are two linear matrix equations for the unknown coefficient vectors a_n and d_n. The absence of terms proportional to powers of γ suggests that the solutions for the coefficients a_n and d_n will remain finite in the limit $\gamma \to \infty$, $\beta \to 1$. As we shall see in the next section, this also implies that the transverse impedance will reach a finite value as $\gamma \to \infty$.

8 Calculation of the Transverse Impedance

It is possible to write the transverse impedance in terms of P, the dipole moment of the driving current, defined in Eq. (2.2) and the value of $E_z(r,\theta,z)$ at $r=a$, the radius of the driving current ring:

$$Z_x(k) = -\frac{1}{\pi k a P}\int_0^{\mathcal{L}} dz \int_0^{2\pi} d\theta\, E_z(a,\theta,z;k)\cos\theta\, \mathrm{e}^{jkz}\,, \tag{8.1}$$

where $\mathcal{L}$ is the circumference of the accelerating ring. We can later check our normalization by requiring that the transverse impedance contain the correct space charge contribution.

From Eq. (3.2), we find at $r=a$, that

$$E_z(a,\theta,z;k) = j\cos\theta \left\{ D\,F\left(\frac{ka}{\gamma}\right)\mathrm{e}^{-jkz} + \int_{-\infty}^{\infty} dq\,\mathrm{e}^{-jqz} A(q)\frac{J_1(\kappa a)}{J_1(\kappa b)}\right\}. \tag{8.2}$$

The transverse impedance in Eq. (8.1) then can be written as

$$Z_x(k) = -\frac{j}{kaP}\left\{ D\,\mathcal{L}\,F\left(\frac{ka}{\gamma}\right) + 2\pi A(k)\left.\frac{J_1(\kappa a)}{J_1(\kappa b)}\right|_{q=k}\right\}, \tag{8.3}$$

where we have used the large $\mathcal{L}$ limit by writing

$$\int_0^{\mathcal{L}} dz\,\mathrm{e}^{-jqz}\mathrm{e}^{jkz} \cong 2\pi\delta(q-k)\,. \tag{8.4}$$

Since, from Eq. (3.1)

$$\kappa^2 \equiv \beta^2 k^2 - q^2\,, \tag{8.5}$$

we have for $q=k$

$$\left.\kappa^2\right|_{q=k} = -k^2/\gamma^2\,, \tag{8.6}$$

so that

$$Z_x(k) = -\frac{j}{ka}\left\{\frac{D}{P}\,\mathcal{L}\,F(ka/\gamma) + 2\pi\frac{A(k)}{P}\left.\frac{J_1(ka/\gamma)}{J_1(kb/\gamma)}\right|_{q=k}\right\}. \tag{8.7}$$

Assuming that $ka/\gamma << 1$, we have, from Eq. (3.3),

$$F(ka/\gamma) \cong \frac{\gamma}{ka}\left(1-\frac{a^2}{b^2}\right). \tag{8.8}$$

Using the value of D in Eq. (2.10), the contribution of the first term inside the brackets in Eq. (8.7) is

$$Z_x^{(SC)}(k) = -\frac{j\,\mathcal{L}\,Z_0}{2\pi\beta\gamma^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right), \tag{8.9}$$

which agrees with the well known expression for the transverse *space charge impedance*.

The contribution of the resistance of the tube of finite length is therefore given by the second term in the brackets of Eq. (8.7):

$$Z_x^{(RT)}(k) = -\frac{j\pi\,A(k)}{\gamma P\,J_1(kb/\gamma)}\,. \tag{8.10}$$

From Eqs. (7.4) and (7.5), we have

$$A(k) = \frac{jkg}{2\pi}\sum_{n=0}^{\infty} A_n\,\frac{[(-1)^n\mathrm{e}^{jkg}-1]}{n^2\pi^2-k^2g^2} \tag{8.11}$$

and the transverse impedance can be written

$$Z_x^{(RT)}(k) = \frac{kg^2}{2\gamma P\, J_1(kb/\gamma)} \sum_{n=0}^{\infty} A_n \, \frac{[(-1)^n \mathrm{e}^{jkg} - 1]}{n^2\pi^2 - k^2 g^2} \, . \tag{8.12}$$

Using the renormalization of A_n and D_n to a_n and d_n, defined in Eq. (7.30), we can rewrite the impedance as

$$Z_x^{(RT)}(k) = \frac{j\beta\gamma k^2 g^3 D}{2\gamma\, x I_1^2(x)\, P} \sum_{n=0}^{\infty} \frac{a_n \, [(-1)^n \mathrm{e}^{jkg} - 1]}{n^2\pi^2 - k^2 g^2} \, . \tag{8.13}$$

Using the expression for D in Eq. (2.10), this can be rewritten as

$$\frac{Z_x^{(RT)}(k)}{Z_0} = \frac{jk}{\pi} \frac{g^3}{b^3} \left(\frac{kb/2\gamma}{J_1(kb/\gamma)} \right)^2 \sum_{n=0}^{\infty} \frac{a_n \, [(-1)^n \mathrm{e}^{jkg} - 1]}{n^2\pi^2 - k^2 g^2} \, . \tag{8.14}$$

Since a_n and d_n remain finite as $\gamma \to \infty$, as discussed at the end of Section 7, also the transverse coupling impedance, which correctly has the dimension Ω/m, remains finite in this limit.

9 Variational Form for the Impedance

From Eqs. (7.31) and (7.32), the equations for the normalized coefficient vectors a_n and d_n are given by

$$m \sum_{n=0}^{\infty} a_n \tilde{p}_{nm} + \sum_{n=1}^{\infty} d_n \tilde{q}_{nm} = 0\,, \qquad \text{for} \quad m \geq 1 \tag{9.1}$$

and

$$\sum_{n=1}^{\infty} n\, d_n \tilde{p}_{nm} + \sum_{n=0}^{\infty} a_n \tilde{r}_{nm} = \tilde{f}_m\,, \qquad \text{for} \quad m \geq 0\,. \tag{9.2}$$

Here the coefficients $\tilde{p}_{nm}$ etc are the conjugates of p_{nm}, i.e. the indices are inverted. We particularly note the close relation between the coefficient of a_n in Eq. (9.1) and the coefficient of nd_n in Eq. (9.2). Specifically,

$$\tilde{p}_{nm} = \frac{2}{\beta k g} P_{nm} - \frac{\pi}{2\beta k g} \frac{\mu_0}{\mu} \frac{T(\alpha_m b)}{\alpha_m b\, T'(\alpha_m b)} \delta_{nm}\,. \tag{9.3}$$

We now multiply Eq. (9.1) by $\sum_{m=1}^{\infty} d_m$ to obtain

$$\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} m\, a_n d_m \tilde{p}_{nm} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} d_n d_m \tilde{q}_{nm} = 0\,. \tag{9.4}$$

Next, we multiply Eq. (9.2) by $\sum_{m=0}^{\infty} a_m$ to obtain

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} n\, a_m d_n \tilde{p}_{nm} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m a_n \tilde{r}_{nm} = \sum_{m=0}^{\infty} a_m \tilde{f}_m\,. \tag{9.5}$$

We then construct

$$\begin{aligned} W = {} & 2 \sum_{m=0}^{\infty} a_m \tilde{f}_m - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} d_n d_m \tilde{q}_{nm} \\ & - 2 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_m d_n \tilde{p}_{nm} - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_m a_n \tilde{r}_{nm} \end{aligned} \tag{9.6}$$

and evaluate δW, the change in W which occurs when d_m changes by δd_m and a_m changes by δa_m. We find

$$\begin{aligned} \delta W = {} & -2 \sum_{m=1}^{\infty} \delta d_m \left[\sum_{n=1}^{\infty} d_n \tilde{q}_{nm} + m \sum_{n=0}^{\infty} a_n \tilde{p}_{nm} \right] \\ & - 2 \sum_{m=0}^{\infty} \delta a_m \left[\sum_{n=1}^{\infty} n\, d_n \tilde{p}_{nm} + \sum_{n=0}^{\infty} a_n \tilde{r}_{nm} - \tilde{f}_m \right]. \end{aligned} \tag{9.7}$$

Requiring $\delta W = 0$ for arbitrary δd_m and δa_m, we duplicate Eqs. (7.31) and (7.32). Thus W, defined in Eq. (9.7), is an invariant under arbitrary variation of d_m and a_m. Using Eq. (7.31) and (7.32), the extreme value of W turns out to be given by

$$W_{extreme} = \sum_{m=1}^{\infty} a_m \tilde{f}_m \tag{9.8}$$

which agrees with Eq. (8.9). As a consequence, the value of W varies only *quadratically* with any errors in a_n and d_n. We may therefore truncate the infinite sums over m and n and still obtain a good approximation to W which varies only quadratically in the discarded terms.

Therefore one may truncate the infinite sums over n in Eq. (7.31) and (7.32) at a value N without large error. There a total of $2N+1$ unknowns ($N+1$ values of a_n for $0 \leq n \leq N$ and and N values of d_n where $1 \leq n \leq N$), and just as many equations (N values of m in Eq. (7.31) and $N+1$ values of m in Eq.(7.32))' Thus these equations by can be solved by conventional matrix techniques, confident that the result for the impedance will be *quadratic* in any error caused by the truncation. This can easily be tested by performing the calculation for several values of N. A simple extrapolation technique can then be used to obtain more accurate values of the impedance.

The most sweeping approximation one can make is to choose $N = 0$, so that only the coefficient a_0 remains in the equations ($a_n = 0$, $d_n = 0$ for $n \geq 1$). This is discussed in Appendix E and is equivalent to using a constant trial function for $E_z(b)$.

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Appendix A: Evaluation of Q_{nm}

We start with Q_{nm} defined in Eq. (7.16) and use Eq. (7.17) to rewrite it as

$$Q_{nm} = \frac{gnm}{4}\left[1+(-1)^{m+n}\right]\int_{-\infty}^{\infty} dq\, \frac{\kappa b\, J_1(\kappa b)}{J_1'(\kappa b)} \frac{\left[1-(-1)^n \mathrm{e}^{jqg}\right]}{(q^2g^2-m^2\pi^2)(q^2g^2-n^2\pi^2)}\,. \tag{A.1}$$

Examining Eqs. (7.12) and (7.14), we see that there is no singularity at either $qg=\pm m\pi$ or $qg=\pm n\pi$. The change in the sign of q in the third term within the square brackets in Eq. (7.17) has introduced singularities at $qg=\pm m\pi$, but not at $qg=\pm n\pi$.

We now close the contour in Eq. (A.1) at $q=\infty$ in the *upper* half q-plane. As discussed before, the contour goes *above* the poles on the *positive* real q-axis and *below* the poles on the *negative* real q-axis. Only the poles on the negative real q-axis are then enclosed. Since from Eq.(3.1)

$$\kappa^2 = \beta^2 k^2 - q^2\,, \tag{A.2}$$

the factor $\kappa b\, J_1(\kappa b)/J_1'(\kappa b)$ in Eq. (A.1) is an *even* function of q.

Closing the contour of integration at $q=\infty$ in the *upper* half q-plane, we get

$$Q_{nm} = \frac{gnm}{4}\left[1+(-1)^{m+n}\right]\oint dq\, \frac{\kappa b\, J_1(\kappa b)}{J_1'(\kappa b)} \frac{\left[1-\exp(j(qg+n\pi))\right]}{(q^2g^2-m^2\pi^2)(q^2g^2-n^2\pi^2)}\,, \tag{A.3}$$

where the symbol $\oint$ implies a contour which goes counter-clockwise around all poles on the *negative* real q-axis and on the *positive* imaginary q-axis. To avoid a second-order pole when $m=n$, we consider n to be a continuous variable in the integrand. Since the integrand in Eq. (7.16) has no pole at $qg=-n\pi$, the only poles enclosed by the contour are those at $qg=-m\pi$ and at the zeros of $J_1'(\kappa b)$.

The contribution from the pole at $qg=-m\pi$, denoted by $Q_{nm}^{(A)}$, is then given by

$$Q_{nm}^{(A)} = \frac{jn\left[1+(-1)^{n+m}\right]}{4\pi(n+m)} \frac{\kappa_m b\, J_1(\kappa_m b)}{J_1'(\kappa_m b)} \frac{\mathrm{e}^{-j(m-n)\pi}-1}{(m-n)\pi}\,, \tag{A.4}$$

where

$$\kappa_m = \left\{ \begin{array}{ll} \left(\beta^2k^2g^2-m^2\pi^2\right)^{1/2}/g, & ,m\pi\le\beta kg \\ \pm j\left(m^2\pi^2-\beta^2k^2g^2\right)^{1/2}/g & ,m\pi\ge\beta kg \end{array}\right\}. \tag{A.5}$$

We now return to considering n to be an integer. The factor $[1+(-1)^{n+m}]$ requires both $m+n$ and $m-n$ to be even, so that

$$\lim_{n\to\text{integer}} \frac{\mathrm{e}^{-j(m-n)\pi}-1}{(m-n)\pi} = \left\{ \begin{array}{lll} -j & , \text{ if} & n=m \\ 0 & , \text{ if} & n\to\text{integer}\ne m \end{array}\right\}. \tag{A.6}$$

Thus,

$$Q_{nm}^{(A)} = \frac{1}{4\pi}\frac{\kappa_n b\, J_1(\kappa_n b)}{J_1'(\kappa_n b)}\delta_{nm}$$

$$= \frac{\delta_{nm}}{4\pi}\frac{b}{g}\left\{ \begin{array}{ll} \dfrac{\left(\beta^2k^2g^2-n^2\pi^2\right)^{1/2} J_1\left[\left(\beta^2k^2g^2-n^2\pi^2\right)^{1/2} b/g\right]}{J_1'\left[\left(\beta^2k^2g^2-n^2\pi^2\right)^{1/2} b/g\right]} & ,\ n\pi<\beta kg \\ -\dfrac{\left(n^2\pi^2-\beta^2k^2g^2\right)^{1/2} I_1\left[\left(n^2\pi^2-\beta^2k^2g^2\right)^{1/2} b/g\right]}{I_1'\left[\left(n^2\pi^2-\beta^2k^2g^2\right)^{1/2} b/g\right]} & ,\ n\pi>\beta kg \end{array}\right\}. \tag{A.7}$$

The remaining singularities in the integrand in Eq. (A.3) are located at the zeros of $J_1'(\kappa b)$. Specifically, these occur at $\kappa b=\pm p_\ell$, where p_ℓ are the zeros of $J_1'(y)$, in increasing order. The corresponding values of q are given by

$$q_\ell b = -\left(\beta^2k^2b^2-p_\ell^2\right)^{1/2} \quad , \quad 1\le\ell\le L \tag{A.8}$$

along the *negative* real q-axis, and

$$q_\ell b = j\phi_\ell \tag{A.9}$$

along the *positive* imaginary q-axis, where

$$\phi_\ell \equiv \left(p_\ell^2 - \beta^2 k^2 b^2\right)^{1/2} \quad , \quad \ell \geq L+1\,, \tag{A.10}$$

as discussed following Eq. (A.3). The integer L is chosen so that

$$p_L < \beta k b < p_{L+1}\,. \tag{A.11}$$

Near $\kappa b = \pm p_\ell$, the zeros of $J_1'(y)$, we have

$$\frac{\kappa b\, J_1(\kappa b)}{J_1'(\kappa b)} = \frac{\pm p_\ell\, J_1(\pm p_\ell)}{J_1''(\pm p_\ell)\,(\kappa b \mp p_\ell)}\,. \tag{A.12}$$

The differential equation satisfied by $J_1(\pm p_\ell)$, with $J_1'(\pm p_\ell) = 0$, leads to

$$\frac{J_1(\pm p_\ell)}{J_1''(\pm p_\ell)} = -\frac{p_\ell^2}{p_\ell^2 - 1} \tag{A.13}$$

so that Eq. (A.12) can be rewritten, near $\kappa b = \pm p_\ell$, as

$$\frac{\kappa b\, J_1(\kappa b)}{J_1'(\kappa b)} \cong \frac{p_\ell^3}{p_\ell^2 - 1}\frac{(p_\ell \pm \kappa b)}{(p_\ell^2 - \kappa^2 b^2)} \cong \frac{2p_\ell^4/(p_\ell^2 - 1)}{(q^2 b^2 - \beta^2 k^2 b^2 + p_\ell^2)}\,. \tag{A.14}$$

The contribution to Q_{nm} from the zeros of $J_1'(\kappa b)$, which are located within the contour at

$$q_\ell b = \left\{ \begin{array}{ccc} -\left(\beta^2 k^2 b^2 - p_\ell^2\right)^{1/2} & , & 1 \leq \ell \leq L \\ j\phi_\ell & , & \ell \geq L+1 \end{array} \right\}, \tag{A.15}$$

can be written as

$$Q_{nm}^{(B)} = \frac{gnm}{4}\left[1 + (-1)^{m+n}\right]\sum_{\ell=1}^{\infty}\frac{2p_\ell^4/(p_\ell^2 - 1)\ \left[1 - (-1)^n \exp(jq_\ell g)\right]}{(m^2\pi^2 - q_\ell^2 g^2)(n^2\pi^2 - q_\ell^2 g^2)} \times \int \frac{dq}{q^2 b^2 + p_\ell^2 - \beta^2 k^2 b^2}\,. \tag{A.16}$$

Using Eq. (A.15), $Q_{nm}^{(B)}$ can be written as

$$Q_{nm}^{(B)} = \frac{\pi g n m}{b} \times \left\{ -j\sum_{\ell=1}^{L}\frac{\left[1 - (-1)^n \exp\left(-jg\left(\beta^2 k^2 b^2 - p_\ell^2\right)^{1/2}/b\right)\right] p_\ell^4/(p_\ell^2 - 1)}{\left(\beta^2 k^2 b^2 - p_\ell^2\right)^{1/2}(m^2\pi^2 - \beta^2 k^2 g^2 + p_\ell^2 g^2/b^2)(n^2\pi^2 - \beta^2 k^2 g^2 + p_\ell^2 g^2/b^2)} + \sum_{\ell=L+1}^{\infty}\frac{\left[1 - (-1)^n \exp\left(-\phi_\ell g/b\right)\right] p_\ell^4/(p_\ell^2 - 1)}{\phi_\ell\,(m^2\pi^2 + \phi_\ell^2 g^2/b^2)(n^2\pi^2 + \phi_\ell^2 g^2/b^2)} \right\} \tag{A.17}$$

for $m+n$ even, while $Q_{nm}^{(B)} = 0$ for $m+n$ odd.

Our final result for Q_{nm} is then the sum of Eqs. (A7) and (A17) for $m+n$ even, while $Q_{nm} = 0$ for $m+n$ odd.

Appendix B: Evaluation of P_{nm}

We start with P_{nm} in Eq. (7.15), using the expression for $G_m^*(q)G_n(q)$ as given in Eq. (7.17),

$$P_{nm} = \frac{g^3}{4}\left[1+(-1)^{m+n}\right]\int_{-\infty}^{\infty} dq\, q^2\, \frac{J_1(\kappa b)}{\kappa b\, J_1'(\kappa b)}\frac{\left[1-(-1)^n \mathrm{e}^{jqg}\right]}{(q^2g^2-m^2\pi^2)(q^2g^2-n^2\pi^2)}\,, \tag{B.1}$$

and proceed with the calculation for $m \geq 1$ as P_{n0} is not needed in the infinite sum . Since $J_1(\kappa b) \to \kappa b/2$ for small κb there is no singularity at $\kappa b = 0$.

As before, the integration contour in the q-plane is taken *above* the poles on the *positive* real q-axis and *below* the poles on the *negative* real q-axis, and closed at $q = \infty$ in the *upper* half$q$-plane. Thus only the poles at $qg = -m\pi$ and at the zeros of $J_1'(\kappa b)$ are enclosed.

We first consider the contribution from the pole at $qg = -m\pi$, where we rewrite the bracket in Eq. (7.17) as

$$\left[1-(-1)^n \mathrm{e}^{jqg}\right] \to \left[1-\exp\left[j(qg+n\pi)\right)\right] \tag{B.2}$$

This leads to

$$P_{nm}^{(A)} = \frac{\pi}{4}\frac{J_1(\kappa_n b)}{\kappa_n b\, J_1'(\kappa_n b)}\,\delta_{nm}\,, \tag{B.3}$$

where κ_n is given in Eq. (A.7) separately for $n\pi < \beta k g$ and $n\pi > \beta k g$. We now evaluate the contribution of the poles at $\kappa b = \pm p_\ell$. Near these poles, we see, from Eq. (A.14), that

$$\frac{J_1(\kappa b)}{\kappa b\, J_1'(\kappa b)} \cong \frac{2p_\ell^2/(p_\ell^2-1)}{(q^2b^2-\beta^2k^2b^2+p_\ell^2)}\,, \tag{B.4}$$

where the contour integral encloses the pole identified in Eq. (A.15) at

$$q_\ell b = \left\{ \begin{array}{ccc} -\left(\beta^2k^2b^2-p_\ell^2\right)^{1/2} & , & 1 \leq \ell \leq L \\ j\phi_\ell & , & \ell \geq L+1 \end{array} \right\}. \tag{B.5}$$

Our final result for $P_{nm}^{(B)}$ is then

$$P_{nm}^{(B)} = \frac{\pi g^3}{b^3} \times \left\{ -j\sum_{\ell=1}^{L} \frac{p_\ell^2\left(\beta^2k^2b^2-p_\ell^2\right)^{1/2}\left[1-(-1)^n\exp\left(-jg\left(\beta^2k^2b^2-p_\ell^2\right)^{1/2}/b\right)\right]}{(p_\ell^2-1)(m^2\pi^2-\beta^2k^2g^2+p_\ell^2g^2/b^2)(n^2\pi^2-\beta^2k^2g^2+p_\ell^2g^2/b^2)} - \sum_{\ell=L+1}^{\infty} \frac{\phi_\ell p_\ell^2\left[1-(-1)^n\exp\left(-\phi_\ell g/b\right)\right]}{(p_\ell^2-1)(m^2\pi^2+\phi_\ell^2g^2/b^2)(n^2\pi^2+\phi_\ell^2g^2/b^2)} \right\} \tag{B.6}$$

for $m+n$ even , while $P_{nm}^{(B)} = 0$ when $m+n$ is odd . Once again

$$\phi_\ell \equiv \left(p_\ell^2-\beta^2k^2b^2\right)^{1/2} \quad , \quad \ell \geq L+1\,. \tag{B.7}$$

The matrix $P_{nm} = P_{nm}^{(A)} + P_{nm}^{(B)}$ is the sum of Eq. (B.3) and Eq. (B.6) for $m+n$ even, while it is zero for $m+n$ odd. Clearly, P_{nm} is symmetric in the interchange $m \leftarrow n$. Therefore $\tilde{p}_{nm} = p_{nm} = p_{nm}$. As a consequence, the first terms on the left side of Eqs. (9.4) and (9.5) are equal. Similarly, $\tilde{q}_{nm} = q_{nm}q_{nm}$ and $\tilde{r}_{nm} = r_{nm}$. Our final result for $P_{nm} = P_{nm}^{(A)} + P_{nm}^{(B)}$ then is the sum of Eqs. (B.3) and (B.6) for $m+n$ even, while $P_{nm} = 0$ for $m+n$ odd.

Appendix C: Evaluation of T_{nm}

We start with Eq. (7.24) for T_{nm}:

$$T_{nm} = g^5 \left[1 + (-1)^{m+n}\right] \int_{-\infty}^{\infty} dq \, \frac{q^4}{\kappa^2 b^2} \frac{J_1(\kappa b)}{\kappa b \, J_1'(\kappa b)} \times \frac{\left[1 - (-1)^n e^{jqg}\right]}{(q^2 g^2 - m^2\pi^2)(q^2 g^2 - n^2 \pi^2)} \, . \tag{C.1}$$

The contour in the complex q-plane is again chosen as for the coefficients P and Q, Then only the poles on the *negative* real q-axis and on the *positive* imaginary q-axis are within the contour. We first consider the poles at $qg = -m\pi$, but there is no pole at $qg = -n\pi$ in the integrand of Eq. (7.24).

The contribution to T_{nm} from these poles is

$$T_{nm}^{(A)} = \frac{n^2 \pi^3}{\kappa_n^2 b^2} \frac{J_1(\kappa_n b)}{\kappa_n b \, J_1'(\kappa_n b)} \, \delta_{nm} \, , \tag{C.2}$$

where

$$\kappa_n b = \left\{ \begin{array}{ccc} (\beta^2 k^2 g^2 - m^2 \pi^2)^{1/2} \, b/g & , & m\pi \le \beta k g \\ \pm j \, (m^2\pi^2 - \beta^2 k^2 g^2)^{1/2} \, b/g & , & m\pi \ge \beta k g \end{array} \right\} . \tag{C.3}$$

The remaining part of T_{nm} comes from the zeros of $J_1'(\kappa b)$ at $\kappa b = \pm p_\ell$, including $\ell = 0$, where p_0 is defined as $p_0 \equiv 0$. We find

$$T_{nm}^{(B)} = \frac{4\pi g^5}{b^5} \times \left\{ -j \sum_{\ell=0}^{L} \frac{(\beta^2 k^2 b^2 - p_\ell^2)^{3/2} \left[1 - (-1)^n \exp\left(-jg \, (\beta^2 k^2 b^2 - p_\ell^2)^{1/2} \, /b\right)\right]}{(p_\ell^2 - 1)(m^2\pi^2 - \beta^2 k^2 g^2 + p_\ell^2 g^2/b^2)(n^2\pi^2 - \beta^2 k^2 g^2 + p_\ell^2 g^2/b^2)} - \sum_{\ell=L+1}^{\infty} \frac{\phi_\ell^3 \, \left[1 - (-1)^n \exp\left(-\phi_\ell g/b\right)\right]}{(p_\ell^2 - 1)(m^2\pi^2 + \phi_\ell^2 g^2/b^2)(n^2\pi^2 + \phi_\ell^2 g^2/b^2)} \right\} , \tag{C.4}$$

where $m + n$ an even integer, while $T_{nm}^{(B)} = 0$ for $m + n$ odd. Here

$$\phi_\ell \equiv \left(p_\ell^2 - \beta^2 k^2 b^2\right)^{1/2} , \tag{C.5}$$

with L chosen so that

$$p_L < \beta k b < p_{L+1} \, . \tag{C.6}$$

Our final result for T_{nm} is then the sum of $T_{nm}^{(A)}$ given in Eq. (C.2) and $T_{nm}^{(B)}$ in Eqs. (C.4). Specifically

$$T_{nm} = T_{nm}^{(A)} + T_{nm}^{(B)} \tag{C.7}$$

for $m + n$ *even*, while it is zero for $m + n$ *odd*.

Appendix D: Evaluation of R_{nm}

We start with Eq. (7.25)

$$R_{nm} = g^3 \left[1 + (-1)^{m+n}\right] \int_{-\infty}^{\infty} dq\, q^2 \frac{J_1'(\kappa b)}{\kappa b\, J_1(\kappa b)} \frac{[1 - (-1)^n \mathrm{e}^{jqg}]}{(q^2g^2 - m^2\pi^2)(q^2g^2 - n^2\pi^2)} \tag{D.1}$$

choosing the same contour as for the coefficients P and Q. We first consider the poles at $qg = -m\pi$, but there is no pole at $qg = -n\pi$ in the integrand of Eq. (D.1). The contribution to R_{nm} from these poles is

$$R^{(A)}_{nm} = \pi \frac{J_1'(\kappa_n b)}{\kappa_n b\, J_1(\kappa_n b)}\, \delta_{nm}\,, \tag{D.2}$$

where κ_n was given in Eq. (C.3).

The remaining part of R_{nm} comes from the zeros of $J_1(\kappa b)$ at $\kappa b = \pm r_\ell$, where r_ℓ are the positive zeros of $J_1(t)$, including $r_0 \equiv 0$. These occur within the closed contour in the q-plane at

$$qb = \left\{ \begin{array}{ccc} -q_\ell b & , & 1 \le \ell \le \bar{L} \\ j\psi_\ell & , & \ell \ge \bar{L}+1 \end{array} \right\} \tag{D.3}$$

where

$$q_\ell b = \left(\beta^2k^2b^2 - r_\ell^2\right)^{1/2} > 0 \quad , \quad 0 \le \ell \le \bar{L} \tag{D.4}$$

and where

$$\psi_\ell \equiv \left(r_\ell^2 - \beta^2k^2b^2\right)^{1/2} > 0 \quad , \quad \ell \ge \bar{L}\,. \tag{D.5}$$

The parameter $\bar{L}$ is defined such that

$$r_{\bar{L}} \le \beta kb \le r_{\bar{L}+1}\,. \tag{D.6}$$

Near the zeros of $J_1(\kappa b)$, we find

$$\frac{J_1'(\kappa b)}{\kappa b\, J_1(\kappa b)} - \frac{\kappa b \pm r_\ell}{\pm r_\ell\, (r_\ell^2 - \kappa^2 b^2)} - \frac{2}{q^2b^2 - \beta^2k^2b^2 + r_\ell^2}\,. \tag{D.7}$$

Specifically, for $0 \le \ell \le \bar{L}$

$$\frac{J_1'(\kappa b)}{\kappa b\, J_1(\kappa b)} \cong \frac{1}{q_\ell b(qb + q_\ell b)}\,, \tag{D.8}$$

and for $\ell \ge \bar{L}+1$

$$\frac{J_1'(\kappa b)}{\kappa b\, J_1(\kappa b)} \cong -\frac{1}{j\psi_\ell(qb - j\psi_\ell)}\,. \tag{D.9}$$

Using these expressions we find

$$R^{(B)}_{nm} = \frac{4\pi g^3}{b^3} \times \left\{ j \sum_{\ell=0}^{\bar{L}} \frac{(\beta^2k^2b^2 - r_\ell^2)^{1/2}\left[1 - (-1)^n \exp\left(-jg\,(\beta^2k^2b^2 - r_\ell^2)^{1/2}/b\right)\right]}{(m^2\pi^2 - \beta^2k^2g^2 + r_\ell^2g^2/b^2)(n^2\pi^2 - \beta^2k^2g^2 + r_\ell^2g^2/b^2)} + \sum_{\ell=\bar{L}+1}^{\infty} \frac{\psi_\ell\,[1 - (-1)^n \exp(-\psi_\ell g/b)]}{(m^2\pi^2 + \psi_\ell^2g^2/b^2)(n^2\pi^2 + \psi_\ell^2g^2/b^2)} \right\} \tag{D.10}$$

when $m+n$ is an even integer, and $R^{(B)}_{nm} = 0$ for $m+n$ odd.

Our final result for R_{nm} is then the sum of $R^{(A)}_{nm}$ in Eqs. (D.2) and $R^{(B)}_{nm}$ in Eqs. (D.10). Specifically

$$R_{nm} = R^{(A)}_{nm} + R^{(B)}_{nm} \tag{D.11}$$

for $m+n$ even, while it is zero fro $m+n$ odd.

Appendix E: Analytic Approximation to the Transverse Impedance

We start by choosing $N = 0$, so that only a_0 remains in the expression for $Z_x^{RT}(k)$ in Eq. (8.14) since $a_n = 0$ and $d_n = 0$ for $n \geq 1$ are all zero. The set of equations then reduces to a single one

$$a_0 S_{00} = \frac{1 - e^{-jkg}}{k^2 g^2}\,, \tag{E.1}$$

where

$$S_{00} = \frac{\beta kb}{2\pi} R_{00} - \frac{b}{2\pi\beta k g^2} T_{00} - \beta kb \frac{\varepsilon'}{\varepsilon_0} \frac{S'(\alpha_0 b)}{\alpha_0 b\, S(\alpha_0 b)}\,. \tag{E.2}$$

From Eq. (7.24) we get

$$T_{00} = 2g \int_{-\infty}^{\infty} dq\, \frac{J_1(\kappa b)(1 - e^{jqg})}{\kappa^3 b^3\, J_1'(\kappa b)}\,. \tag{E.3}$$

According to Eq. (7.24) $T_{00}^{(A)} = 0$, hence $T_{00} = T_{00}^{(B)}$ and from Eq. (D.10)

$$R_{00} = \frac{2}{g} \int_{-\infty}^{\infty} dq\, \frac{1}{q^2} \frac{J_1'(\kappa b)(1 - e^{jqg})}{\kappa b\, J_1(\kappa b)}\,. \tag{E.4}$$

From Eq. (5.2) we see that

$$\alpha_0 b = \beta kb \left(\frac{\mu\varepsilon'}{\mu_0\varepsilon_0}\right)^{1/2} \tag{E.5}$$

and the quantities $S(\alpha_0 b)$ and $S'(\alpha_0 b)$ can be obtained from Eqs. (5.4) with $n = 0$.

In the approximation containing only the single $n = 0$ term of the infinite sum over n, the impedance can be found from Eq. (8.14)

$$\frac{a_0 \left[1 - e^{jkg}\right]}{k^2 g^2} = \frac{(1 - e^{jkg})(1 - e^{-jkg})}{k^2 g^2 S_{00}} = 2\frac{1 - \cos(kg)}{k^2 g^2 S_{00}}\,. \tag{E.6}$$

With S_{00} from Eq.(E.2) we finally get a simple approximate expression for the transverse impedance of a resistive wall of finite length

$$\frac{Z_x^{(RT)}(k)}{Z_0} = \frac{2j}{\pi k^3 b^3 g} \left[\frac{kb/2\gamma}{I_1(kb/\gamma)}\right]^2 \frac{(1 - \cos(kg))}{S_{00}}\,. \tag{E.7}$$