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Abstract.

We consider three types of nonlinear partial differential equations of the polynomial form and obtain the explicit substitutions of dependent variables which transform the equations under study to linear equations. We also obtain some nonlinear second order partial differential equations which can be solved by the viscosity method.

Last years a great amount of papers concerning various methods of obtaining exact solutions of nonlinear partial differential equations appeared. Reading these papers (see, e.g., the recent papers 1-5 and references therein) one can see that although there are many different methods of finding exactly solvable equations, all of them consist in obtaining certain transformations which would reduce a nonlinear equation to a linear one or to another nonlinear equation with the known solutions. Therefore in ref. 6 the following problem was formulated: to describe the nonlinear equations which can be reduced to linear ones with the aid of certain substitutions of dependent and independent variables. Of course, this class of equations is very small in comparison with the class of all exactly solvable equations. None the less we know that this small class contains some physically interesting equations, for example, the Burgers-Hopf equation 7 . We hope that the methodical study of all possible substitutions of variables and the initial linear equations may lead to some new exactly solvable nonlinear equations being of physical interest.

In this paper which is continuation of paper 6 we investigate some nonlinear equations obtained from linear partial differential equations of the second order by substitutions of dependent variables.

Namely, considering the heat equation

$$\Psi_t = \Psi_{xx} \tag{1}$$

and replacing Ψ by $exp[W(\varphi, \varphi_x, \ldots)]$, where $W(\varphi, \varphi_x, \ldots)$ is an arbitrary differentiable function, we obtain the following nonlinear equation

$$W_t(\varphi, \varphi_x, \ldots) = W_x^2 + W_{xx} \tag{2}$$

Some explicit special cases of this equation are given by eqs. (3)-(5). For every equation we show the replacement of variables reducing it to the heat equation or show the equation relating solutions of the nonlinear equation with the solutions of the heat equation.

$$\gamma U_t + \alpha U_{xt} \pm 2\gamma^2 U U_x + 2\alpha\gamma(U_x^2 + U U_{xx}) +$$

$$+2\alpha^2 U_x U_{xx} + \gamma U_{xx} + \alpha U_{xxx}$$

$$U = \frac{\partial \varphi}{\partial x}; \quad \Psi = \exp(\alpha \varphi_x' + \gamma \varphi) \qquad (3)$$

$$\varphi = e^{-\frac{\gamma}{\alpha}x} \int e^{\frac{\gamma}{\alpha}x} \ln \Psi \, dx$$

$$\gamma U_t + \alpha U_{xxt} = 2\gamma^2 U U_x + 2\alpha\gamma (U_x U_{xx} +$$

$$+ U U_{xxx}) + 2\alpha^2 U_{xx} + 2\gamma U_x U_{xx} + \alpha U_{xxxx}$$

$$U = \frac{\partial \varphi}{\partial x}; \quad \Psi = \exp(\alpha \varphi_x'' + \gamma \varphi) \qquad (4)$$

$$\varphi = \frac{i}{2}\sqrt{\frac{\alpha}{\gamma}} \left(e^{i\sqrt{\frac{\gamma}{\alpha}}x} \int e^{-i\sqrt{\frac{\gamma}{\alpha}}x} \ln \Psi \, dx + e^{-i\sqrt{\frac{\gamma}{\alpha}}x} \int e^{i\sqrt{\frac{\gamma}{\alpha}}x} \ln \Psi \, dx \right)$$

$$\gamma U_t + 2(U_{xx}^2 + U_x U_{xxx}) + 2\gamma^2 U U_x +$$

$$+ 4\alpha\gamma (U_x^2 U_{xx} + U U_x U_{xxx}) + 8\alpha^2 (U_x U_{xx}^3 +$$

$$+ U_x^2 U_{xx} U_{xxx}) + \gamma U_{xx} + 2\alpha (3 U_{xx} U_{xxx} +$$

$$+ U_x U_{xxxx})$$

$$U = \frac{\partial \varphi}{\partial x}$$

$$\alpha^2 \varphi_{xx} + \gamma\varphi = \ln \Psi \tag{5}$$

Now let us consider a general linear partial differential equa on of the second order

$$a\Psi_t + b\Psi_{xt} + c\Psi_{tt} + d\Psi_{xx} = 0 \tag{6}$$

Making the substitution

$$\Psi = \exp(\alpha\varphi + \beta\varphi_x + \gamma\varphi_t) \tag{7}$$

We confine ourselves to the equations of the polynomial type, we obtain the following equation:

$$a\left(\alpha\frac{\partial\varphi}{\partial t} + \beta\frac{\partial^2\varphi}{\partial x\partial t} + \gamma\frac{\partial^2\varphi}{\partial t^2}\right) + b\left(\alpha^2\frac{\partial\varphi}{\partial t}\cdot\frac{\partial\varphi}{\partial x} +\right.$$

$$+\alpha\gamma\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{\partial\varphi}{\partial x} + \alpha\beta\frac{\partial\varphi}{\partial t}\cdot\frac{\partial^2\varphi}{\partial x^2} + \beta^2\frac{\partial^2\varphi}{\partial x\partial t} +$$

$$+\beta\gamma\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{\partial^2\varphi}{\partial x^2} + \gamma^2\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{\partial^2\varphi}{\partial t\partial x} + \alpha\gamma\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial x\partial t} +$$

$$+\beta\gamma\frac{\partial^2\varphi}{\partial x\partial t} + \alpha\frac{\partial^2\varphi}{\partial x\partial t} + \beta\frac{\partial^3\varphi}{\partial x^2\partial t} +$$

$$\left.+\gamma\frac{\partial^3\varphi}{\partial x\partial t^2}\right) + c\left[\alpha^2\left(\frac{\partial\varphi}{\partial t}\right)^2 + \beta^2\left(\frac{\partial^2\varphi}{\partial x\partial t}\right)^2 +\right.$$

$$+\gamma^2\left(\frac{\partial^2\varphi}{\partial t^2}\right)^2 + 2\alpha\beta\frac{\partial^2\varphi}{\partial x\partial t}\frac{\partial\varphi}{\partial t} + \tag{8}$$

$$+2\alpha\gamma\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{\partial\varphi}{\partial t}+2\beta\gamma\frac{\partial^2\varphi}{\partial x\partial t}\cdot\frac{\partial^2\varphi}{\partial t^2}+$$

$$+\alpha\frac{\partial^2\varphi}{\partial t^2}+\beta\frac{\partial^3\varphi}{\partial x\partial t^2}+\gamma\frac{\partial^3\varphi}{\partial t^3}\Bigg]+d\Bigg[\alpha^2\left(\frac{\partial\varphi}{\partial x}\right)^2+$$

$$+\beta^2\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2+\gamma^2\left(\frac{\partial^2\varphi}{\partial t\partial x}\right)^2+2\alpha\beta\frac{\partial^2\varphi}{\partial x^2}\frac{\partial\varphi}{\partial x}+$$

$$+2\alpha\gamma\frac{\partial^2\varphi}{\partial t\partial x}\cdot\frac{\partial\varphi}{\partial x}+2\beta\gamma\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{\partial^2\varphi}{\partial t\partial x}+\alpha\frac{\partial^2\varphi}{\partial x^2}+$$

$$+\beta\frac{\partial^3\varphi}{\partial x^3}+\gamma\frac{\partial^3\varphi}{\partial t\partial x^2}\Bigg]=0$$

Eq. (8) is linear with respect to the derivatives of the third order and nonlinear with respect to the derivatives of the second and the first order. One can check that equations of this kind can be reduced to three different types of equations of this kind can be reduced to three different types of equations by means of linear replacements of independent variables:

$$\varphi_{xxx}+L(\varphi_{xx},\varphi_{tt},\varphi_{xt},\varphi_x,\varphi_t,\dots)=0 \qquad (9)$$

$$\varphi_{xxt}+M(\varphi_{xx},\varphi_{tt},\varphi_{xt},\varphi_x,\varphi_t)=0 \qquad (10)$$

$$\varphi_{xxx}+\varphi_{xtt}+N(\varphi_{xx},\varphi_{tt},\varphi_{xt},\varphi_x,\varphi_t)=0 \qquad (11)$$

(and symmetrically $x \rightleftarrows t$).

It is not difficult to show that the equation of type (9 is obtained from eq. (8) provided

$$b = c = \gamma = 0 \tag{12}$$

that is, we have the following equation

$$a\left(\alpha\frac{\partial\varphi}{\partial t} + \beta\frac{\partial^2\varphi}{\partial x\partial t}\right) + d\left[\alpha^2\left(\frac{\partial\varphi}{\partial x}\right)^2 + \beta^2\left(\frac{\partial^2\varphi}{\partial x\partial t}\right)^2 + 2\alpha\beta\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{\partial\varphi}{\partial x} + \alpha\frac{\partial^2\varphi}{\partial x^2} + \beta\frac{\partial^3\varphi}{\partial x^3}\right] = 0 \tag{13}$$

which is reduced to the equation

$$a\Psi_t + d\Psi_{xx} = 0 \tag{14}$$

by the replacement

$$\Psi = \exp\left(\alpha\varphi + \beta\varphi_x\right) \tag{15}$$

The equation of type (10) is obtained from eq. (8) provided either

a) $$d = \gamma = c = 0 \tag{16}$$

or

b) $$b = c = \beta = 0 \tag{17}$$

In the first case we obtain the equation

$$a\left(\alpha\frac{\partial\varphi}{\partial t} + \beta\frac{\partial^2\varphi}{\partial x\partial t}\right) + b\left(\alpha^2\frac{\partial\varphi}{\partial t}\cdot\frac{\partial\varphi}{\partial x} + \alpha\beta\frac{\partial^2\varphi}{\partial x\partial t}\cdot\frac{\partial\varphi}{\partial x} + \alpha\beta\frac{\partial\varphi}{\partial t}\cdot\frac{\partial^2\varphi}{\partial x^2} + \beta^2\frac{\partial^2\varphi}{\partial x\partial t}\cdot\frac{\partial^2\varphi}{\partial x^2} + \alpha\frac{\partial^2\varphi}{\partial x\partial t} + \beta\frac{\partial^3\varphi}{\partial x^2\partial t}\right) = 0 \tag{18}$$

which is reduced to the equation

$$a\Psi_t + b\Psi_{xt} = 0 \tag{19}$$

by the substitution

$$\Psi = \exp(\alpha\varphi + \beta\varphi_x) \tag{20}$$

In the second case we have the equation

$$a\left(\alpha\frac{\partial\varphi}{\partial t} + \gamma\frac{\partial^2\varphi}{\partial t^2}\right) + d\left[\alpha^2\left(\frac{\partial\varphi}{\partial x}\right)^2 + \gamma^2\left(\frac{\partial^2\varphi}{\partial t\partial x}\right)^2 + 2\alpha\gamma\frac{\partial^2\varphi}{\partial t\partial x}\cdot\frac{\partial\varphi}{\partial x} + \alpha\frac{\partial^2\varphi}{\partial x^2} + \gamma\frac{\partial^3\varphi}{\partial t\partial x^2}\right] = 0 \tag{21}$$

which is reduced to the equation

$$a\Psi_t + d\Psi_{xx} = 0 \tag{22}$$

by the replacement

$$\Psi = \exp(\alpha\varphi + \gamma\varphi_t) \tag{23}$$

The equation of type (11) is obtained from eq. (8) provided either

a) $$c = d, \quad \gamma = b = 0 \tag{24}$$

or

b) $$c = 0, \quad b = d, \quad \gamma + \beta = 0 \tag{25}$$

In the first case we obtain the equation

$$a\left(\alpha\frac{\partial\varphi}{\partial t}+\beta\frac{\partial^2\varphi}{\partial x\partial t}\right)+c\Big[\alpha^2\left(\frac{\partial\varphi}{\partial t}\right)^2+\beta^2\left(\frac{\partial^2\varphi}{\partial x\partial t}\right)^2+\alpha\frac{\partial^2\varphi}{\partial t^2}+$$

$$+2\alpha\beta\frac{\partial^2\varphi}{\partial x\partial t}\cdot\frac{\partial\varphi}{\partial t}+\alpha^2\left(\frac{\partial\varphi}{\partial x}\right)^2+\beta^2\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2+\alpha\frac{\partial^2\varphi}{\partial x^2}+ \tag{26}$$

$$+2\alpha\beta\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{\partial\varphi}{\partial x}+\beta\left(\frac{\partial^3\varphi}{\partial x^3}+\frac{\partial^3\varphi}{\partial x\partial t^2}\right)\Big]=0$$

which is reduced to the equation

$$a\Psi_t+c\left(\Psi_{xx}+\Psi_{tt}\right)=0 \tag{27}$$

by the replacement

$$\Psi=\exp\left(\alpha\varphi+\beta\varphi_x\right) \tag{28}$$

In the second case we obtain the equation

$$a\left[\frac{\partial\varphi}{\partial t}+\beta\left(\frac{\partial^2\varphi}{\partial x\partial t}-\frac{\partial^2\varphi}{\partial t^2}\right)\right]+$$

$$+b\Big[\alpha^2\left(\frac{\partial\varphi}{\partial t}\cdot\frac{\partial\varphi}{\partial x}+\left(\frac{\partial\varphi}{\partial x}\right)^2\right)+\alpha\beta\Big(\frac{\partial^2\varphi}{\partial x\partial t}\cdot\frac{\partial\varphi}{\partial x}-$$

$$-\frac{\partial^2\varphi}{\partial t^2}\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial\varphi}{\partial t}\cdot\frac{\partial^2\varphi}{\partial x\partial t}+2\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{\partial\varphi}{\partial x} \tag{29}$$

$$-2\frac{\partial^2\varphi}{\partial t\partial x}\cdot\frac{\partial\varphi}{\partial x}\Big)+\beta^2\left(\frac{\partial^2\varphi}{\partial x\partial t}\cdot\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{\partial^2\varphi}{\partial x^2}+\right.$$

$$+\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{\partial^2\varphi}{\partial t\partial x}-\frac{\partial^2\varphi}{\partial x\partial t}+\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2-\left(\frac{\partial^2\varphi}{\partial t\partial x}\right)^2$$

$$-2\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{\partial^2\varphi}{\partial t\partial x}\Big)+\alpha\left(\frac{\partial^2\varphi}{\partial x\partial t}\right)+\alpha\frac{\partial^2\varphi}{\partial x^2}+$$

$$+\beta\left(\frac{\partial^3\varphi}{\partial x^3}-\frac{\partial^3\varphi}{\partial x\partial t^2}\right)\Big]=0$$

which is reduced to the equation

$$a\Psi_t+b\left(\Psi_{xt}+\Psi_{xx}\right)=0 \tag{30}$$

by the replacement

$$\Psi=\exp\left(\alpha\varphi+\beta\varphi_x-\beta\varphi_t\right) \tag{31}$$

Let us consider in detail equation (13)

It is convenient to introduce a new notation

$$\begin{aligned} a\alpha &= L & \qquad \alpha\beta^2 &= K \\ a\beta &= N & \qquad \alpha\beta &= C \end{aligned} \tag{32}$$

From eq. (32) we find

$$\beta=\frac{K}{C}, \quad \alpha=\frac{C^2}{K}; \quad a=\frac{N\cdot C}{K}; \quad \alpha=L\frac{K}{N\cdot C} \tag{33}$$

Then equation (13) assumes the following form

$$L\frac{\partial\varphi}{\partial t}+N\frac{\partial^2\varphi}{\partial x\partial t}+\frac{L^2K}{N^2}\left(\frac{\partial\varphi}{\partial x}\right)^2+K\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2+$$
$$+2\frac{L\cdot K}{N}\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{\partial\varphi}{\partial x}+\frac{L\cdot C}{N}\frac{\partial^2\varphi}{\partial x^2}+C\frac{\partial^3\varphi}{\partial x^3}=0 \tag{34}$$

Apparently, using the substitution $X'=\alpha X$ one can always equate the coefficients L and N. Then the equation (34 assumes the form

$$C\left(\varphi_{xxx}+\varphi_{xx}\right)+2K\varphi_{xx}\cdot\varphi_x+K\varphi_{xx}^2+K\varphi_x^2+N\left(\varphi_t+\varphi_{xt}\right)=0 \tag{35}$$

Accordingly, equations (15) and (14) rewritten as follows

$$N\psi_t+C\psi_{xx}=0 \tag{14a}$$

$$\psi=\exp\left[\frac{K}{C}\left(\varphi+\varphi_x\right)\right] \tag{15a}$$

Now let us note that equation (35) can be used to find the solutions of the equation

$$2K\varphi_{xx}\varphi_x+K\varphi_{xx}^2+K\varphi_x^2+N\left(\varphi_t+\varphi_{xt}\right)=0 \tag{36}$$

Indeed, if φ_c is a solution of equation (35) then the function

$$\varphi=\lim_{c\to 0}\varphi_c \tag{37}$$

can be considered as a formal solution of eq. (36) (compare th formula with Hopf's method of solving the Burgers-Hopf equation 7). Such a method of solving equations is often called "the viscosity method". Let us write the explicit forms of some other equations which can be solved by this method.

Making the substitutions

$$
\begin{aligned}
a\alpha &= L_1 & b\beta^2 &= K_1 \\
a\beta &= N_1 & b\beta &= C_1
\end{aligned}
\tag{38}
$$

one can rewrite eq. (18), as follows,

$$
\begin{aligned}
&L_1\frac{\partial\varphi}{\partial t}+N_1\frac{\partial^2\varphi}{\partial x\partial t}+\frac{L_1^2\cdot K_1}{N_1^2}\frac{\partial\varphi}{\partial t}\cdot\frac{\partial\varphi}{\partial x}+\frac{L_1\cdot K_1}{N_1}\left(\frac{\partial^2\varphi}{\partial x\partial t}\cdot\frac{\partial\varphi}{\partial x}+\right.\\
&\left.+\frac{\partial\varphi}{\partial t}\cdot\frac{\partial^2\varphi}{\partial x^2}\right)+K_1\frac{\partial^2\varphi}{\partial x\partial t}\cdot\frac{\partial^2\varphi}{\partial x^2}+\frac{C_1 L_1}{N_1}\frac{\partial^2\varphi}{\partial x\partial t}+C_1\frac{\partial^3\varphi}{\partial x^2\partial t}=0
\end{aligned}
\tag{39}
$$

Equating L_1 and N_1 one obtains the equation

$$
\begin{aligned}
&C_1\left(\varphi_{xxt}+\varphi_{xt}\right)+K_1\varphi_{xt}\varphi_{xx}+K_1\left(\varphi_{xt}\varphi_x+\varphi_t\varphi_{xx}\right)+\\
&+K_1\varphi_t\varphi_x+N_1\left(\varphi_t+\varphi_{xt}\right)=0
\end{aligned}
\tag{40}
$$

which can be used to find the solutions of the equation

$$
K_1\left(\varphi_{xt}\varphi_{xx}+\varphi_{xt}\varphi_x+\varphi_t\varphi_{xx}+\varphi_t\varphi_x\right)+N_1\left(\varphi_t+\varphi_{xt}\right)=0 \tag{41}
$$

Re- marking the coefficients in eq. (21)

$$a\alpha = L_2 \qquad d\gamma^2 = K_2$$
$$a\gamma = N_2 \qquad d\gamma = C_2 \tag{42}$$

we obtain the equation

$$L_2\frac{\partial\varphi}{\partial t} + N_2\frac{\partial^2\varphi}{\partial t^2} + \frac{L_2^2\cdot K_2}{N_2^2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + K_2\left(\frac{\partial^2\varphi}{\partial t\partial x}\right)^2 +$$
$$+2\frac{L_2\cdot K_2}{N_2}\frac{\partial^2\varphi}{\partial t\partial x}\cdot\frac{\partial\varphi}{\partial x} + \frac{C_2 L_2}{N_2}\frac{\partial^2\varphi}{\partial x^2} + C_2\frac{\partial^3\varphi}{\partial t\partial x^2} = 0 \tag{43}$$

Equating L_2 and N_2 , we have

$$C_2\left(\varphi_{txx} + \varphi_{xx}\right) + 2K_2\varphi_{tx}\varphi_x + K_2\varphi_{tx}^2 +$$
$$K_2\varphi_x^2 + N_2\left(\varphi_t + \varphi_{tt}\right) = 0 \tag{44}$$

Supposing C_2 to be a little quantity we see that the equatio

$$K_2\left(2\varphi_{tx}\varphi_x + \varphi_{tx}^2 + \varphi_x^2\right) + N_2\left(\varphi_t + \varphi_{tt}\right) = 0 \tag{45}$$

can be solved by the viscosity method as well.

If we re-mark the coefficients in equation (26)

$$a\alpha = L_3 \qquad c\beta^2 = K_3 \tag{46}$$
$$a\beta = N_3 \qquad c\beta = C_3$$

the following equation will be obtained

$$L_3 \frac{\partial \varphi}{\partial t} + N_3 \frac{\partial^2 \varphi}{\partial X \partial t} + \frac{L_3^2 \cdot K_3}{N_3^2}\left(\frac{\partial \varphi}{\partial t}\right)^2 +$$
$$+ K_3\left(\frac{\partial^2 \varphi}{\partial X \partial t}\right)^2 + 2\frac{L_3 K_3}{N_3}\cdot\frac{\partial^2 \varphi}{\partial X \partial t}\cdot\frac{\partial \varphi}{\partial t} + \frac{C_3 L_3}{N_3}\frac{\partial^2 \varphi}{\partial t^2} +$$
$$+ \frac{L_3^2 \cdot K_3}{N_3^2}\left(\frac{\partial \varphi}{\partial X}\right)^2 + K_3\left(\frac{\partial^2 \varphi}{\partial X^2}\right)^2 + 2\frac{L_3 K_3}{N_3}\cdot\frac{\partial^2 \varphi}{\partial X^2}\cdot\frac{\partial \varphi}{\partial X} + \tag{47}$$
$$+ \frac{C_3 L_3}{N_3}\cdot\frac{\partial^2 \varphi}{\partial X^2} + C_3\left(\frac{\partial^3 \varphi}{\partial X \partial t^2} + \frac{\partial^3 \varphi}{\partial X^3}\right) = 0$$

Equating the coefficients and eq. (47) we have:

$$C_3\left(\varphi_{XXX} + \varphi_{Xtt} + \varphi_{XX} + \varphi_{tt}\right) + K_3\left(2\varphi_{XX}\varphi_X +\right.$$
$$\left. + \varphi_{XX}^2 + \varphi_X^2 + 2\varphi_{Xt}\varphi_t + \varphi_{Xt}^2 + \varphi_t^2\right) + \tag{48}$$
$$+ N_3\left(\varphi_t + \varphi_{Xt}\right) = 0$$

Consequently, the equation

$$K_3\left(2\varphi_{XX}\varphi_X + \varphi_{XX}^2 + \varphi_X^2 + 2\varphi_{Xt}\varphi_t + \varphi_{Xt}^2 +\right.$$
$$\left. + \varphi_t^2\right) + N_3\left(\varphi_t + \varphi_{Xt}\right) = 0 \tag{49}$$

can be also solved by the viscosity method Re-marking the coefficients in eq. (29)

$$a\alpha = L_4 \qquad b\beta^2 = K_4$$
$$a\beta = N_4 \qquad b\beta = C_4 \tag{50}$$

we have

$$L_4\frac{\partial\varphi}{\partial t}+N_4\left(\frac{\partial^2\varphi}{\partial x\partial t}-\frac{\partial^2\varphi}{\partial t^2}\right)+\frac{K_4L_4^2}{N_4^2}\left[\frac{\partial\varphi}{\partial t}\cdot\frac{\partial\varphi}{\partial x}+\right.$$
$$+\left(\frac{\partial\varphi}{\partial x}\right)^2+\frac{K_4L_4}{N_4}\left(\frac{\partial^2\varphi}{\partial x\partial t}\cdot\frac{\partial\varphi}{\partial x}-\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{\partial\varphi}{\partial x}\right)+$$
$$+\frac{K_4L_4}{N_4}\left(\frac{\partial\varphi}{\partial t}\cdot\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial\varphi}{\partial t}\,\frac{\partial^2\varphi}{\partial x\partial t}+2\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{\partial\varphi}{\partial x}-\right.$$
$$\left.-2\frac{\partial^2\varphi}{\partial t\partial x}\cdot\frac{\partial\varphi}{\partial x}\right)+K_4\left[\frac{\partial^2\varphi}{\partial x\partial t}\cdot\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{\partial^2\varphi}{\partial x^2}+\right. \tag{51}$$
$$+\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{\partial^2\varphi}{\partial t\partial x}-\frac{\partial^2\varphi}{\partial x\partial t}+\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2+\left(\frac{\partial^2\varphi}{\partial t\partial x}\right)^2-$$
$$\left.-2\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{\partial^2\varphi}{\partial t\partial x}\right]+\frac{C_4L_4}{N_4}\left(\frac{\partial^2\varphi}{\partial x\partial t}+\frac{\partial^2\varphi}{\partial x^2}\right)+$$
$$+C_4\left(\frac{\partial^3\varphi}{\partial x^3}-\frac{\partial^3\varphi}{\partial x\partial t^2}\right)=0$$

Equating L_4 and N_4 we arrive at the equation

$$C_4\left(\varphi_{xxx}-\varphi_{xtt}+\varphi_{xx}+\varphi_{xt}\right)+K_4\left(\varphi_t\varphi_x+\right.$$
$$+2\varphi_x^2-\varphi_{tt}\varphi_x+\varphi_t\varphi_{xx}-\varphi_t\varphi_{xt}+2\varphi_{xx}\varphi_x-$$
$$-\varphi_{tx}\varphi_x-\varphi_{tt}\varphi_{xx}+\varphi_{tt}\varphi_{tx}-\varphi_{xt}+\varphi_{tx}^2- \tag{52}$$
$$\left.-\varphi_{xx}\varphi_{tx}\right)+N_4\left(\varphi_t+\varphi_{xt}-\varphi_{tt}\right)=0$$

supposing C_4 to be a little quantity we obtain just one more equation which can be solved by the viscosity method:

$$K_4\left(\varphi_t\varphi_x+2\varphi_x^2-\varphi_{tt}\varphi_x+\varphi_t\varphi_{xx}-\varphi_t\varphi_{xt}+\right.$$
$$+2\varphi_{xx}\varphi_x-\varphi_{tx}\varphi_x-\varphi_{tt}\varphi_{xx}+\varphi_{tt}\varphi_{tx}-\varphi_{xt}+ \tag{53}$$
$$\left.+\varphi_{tx}^2-\varphi_{xx}\varphi_{tx}\right)+N_4\left(\varphi_t+\varphi_{xt}-\varphi_{tt}\right)=0$$

In conclusion we note that we have considered only the one-dimensional case. There exists also a possibility to obtain new solutions of nonlinear equations on the basis of the Schrödinger - type equations with multidimensional Hamiltonians. It will be discussed in another paper. Another (very important) problem which was not considered in this paper is the problem

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