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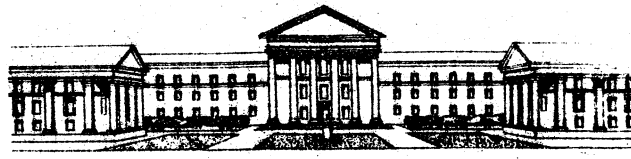

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ARBITRARINESS IN THE CHOICE OF ACTION AND QUANTIZATION OF THE GIVEN CLASSICAL EQUATIONS OF MOTION

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Abstract.

We analyzed the nonuniqueness of variational formulations (Lagrangians, Hamiltonians, actions) of a given system of classical equations of motion. We give a critical review of ideas and results on this matter up to this date. We show in a manifest way that this nonuniqueness leads to the existence of infinitely many different quantum systems corresponding to one and the same classical system. We suggest a new method of quantization based on the classical integrals of motion but free from Lagrangians and Hamiltonians. In the known cases our method is equivalent to the conventional ones. Finally we discuss possible criteria in the choice of the action, e.g. the additivity principle. Several examples (free particle, linear friction, particle in a magnetic field) are considered in detail.

1. Introduction.

In this paper we plan to discuss certain aspects of the uniqueness problem in the quantization of a given classical system, more precisely, to draw attention to the fact that due to the existence of various different variational formulations of classical mechanics one can construct infinitely many different quantum systems corresponding to one and the same classical system. This statement may seem quite trivial since the reconstruction of a quantum quantity $Q(\hbar)$ from its classical limit $Q(0)$ only is obviously nonunique. One reason for bringing this problem to light is that the quantization procedure has not been studied thoroughly from this angle. There are but a few papers on this matter and we have listed them at the end. From the other hand we hope that a detailed analysis in this spirit will provide a better understanding of the specific features of the usual quantum systems and, besides, will give us methods of quantization applicable to more complex systems.

First of all let us specify what we shall mean by "quantization". Several different methods of quantization are available: the canonical quantization [1], Feynman's path integrals method [2,3] (a good survey of Schrödinger's, Heisenberg's and Feynman's formulations of quantum mechanics is given in ref. [4]). Various generalizations of these methods have been the subject of a number of papers [5-13]. Schwinger [5] generalized Hamilton's action principle of classical mechanics to the operator case and introduced the "quantum

namical principle". General methods of quantization appli-
able both to systems with constraints and to systems in an
bitrary Riemanian manifold were elaborated in refs. [6-8].
teresting new approaches to the problem of quantization in
ich formally there is not any difference between classical
d quantum systems were considered in refs. [9,10]. Still
nother method was suggested in ref. [11]. Although all these
thods are different, their common feature is the "algebraic"
sis: they deal with such objects as commutators, Poisson's
Dirac's brackets, canonically conjugated coordinates and
menta, etc. Besides, in all these approaches the Hamiltonian
or Lagrangian) is believed to be given beforehand, and its
rigin is obscure. In addition there are methods which do not
equire the knowledge of Hamiltonian or Lagrangian, e.g. the
thod of differentiation with respect to the coupling const-
ant [12] or Yang-Feldman's method of solving Heisenberg's
perator equations [13].

We shall investigate the uniqueness of quantization on the
ample of Feynman's approach but in a slightly generalized
rsion. In classical mechanics to describe completely the
ystem means to indicate all its possible trajectories in the
oordinate space $x(t/x_0, t_0)$. Each trajectory determines
recisely the point x to which the system comes at the
oment of time t if it was in the point x_0 at the moment
t_0. In quantum mechanics we can deal only with the probabi-
lity of this transition. Therefore we assume that to quantize
ne classical system with the family of trajectories $x(t/x_0, t_0)$

means to indicate the complex function $G(x,t;\ x_0,t_0)$ which is the probability amplitude of the transition from any point x_0 to any point x. The function G (we shall call it hereafter the Green function or the transition amplitude) contains the complete information about the quantum system.

To calculate the transition amplitude two different but equivalent methods are usually applied. The first one is based on the fact that the function $G(x,t;\ x_0,t_0)$ is the Green function of the Schrödinger equation:

$$i\hbar\frac{\partial G}{\partial t} - \hat{H}\left(x, \frac{\partial}{\partial x}, t\right) G(x,t;x_0,t_0) = i\hbar\,\delta(t-t_0)\,\delta(x-x_0) \qquad (1.1)$$

This method corresponds to the most usual "canonical" quantization [1]; it can be applied provided the Hamiltonian of the classical system $H(x,p,t)$ is known (the quantities x and p are the N - dimensional vectors, N being the number of degrees of freedom of the system). The replacement of the C - function $H(x,p,t)$ by the operator $\hat{H}(\hat{x},\hat{p},t)$ is not trivial and unique problem itself, especially in curvilinear coordinates, because there are many ways to choose the operator $\hat{p}$ (not necessarily in the form $\hat{p} = -i\hbar\,\partial/\partial x$ and to order the operators $\hat{x}$ and $\hat{p}$. Let us however disregard the nonuniqueness of this type here. We shall consider a more fundamental nonuniqueness inherent already in classical mechanic and arising because of the existence of more than one classical Hamiltonians leading to ones and the same equations of motion.

In the second method the classical Lagrangian $L(x,\dot{x},t)$

is assumed known, and the transition amplitude is calculated by means of Feynman's path integral [2]:

$$G(x, t; x_0, t_0) = \int_{x_0}^{x} \mathcal{D}\{x(t)\} \exp\left[\frac{i}{\hbar} \int_{t_0}^{t} L(x, \dot{x}, t)\, dt\right] \qquad (1.2)$$

This calculation does not lead to the unique result even for the given function $L(x, \dot{x})$, because one can use various definitions of the measure $\mathcal{D}\{x(t)\}$ and various limit procedures; this "quantum" nonuniqueness is equivalent to different orderings of the operators $\hat{x}$ and $\hat{p}$ in the Hamiltonian operator $\hat{H}(\hat{x}, \hat{p})$. On the contrary we shall investigate the nonuniqueness of the choice of the classical Lagrangian.

Formula (1.2) shows distinctly that to quantize the classical system we need in fact not the Lagrangian nor the Hamiltonian but the classical action function

$$S(x, t; x_0, t_0) = \int_{x_0, t_0}^{x, t} L(x, \dot{x}, t)\, dt \qquad (1.2a)$$

related to the given system. If we know the function $S_{cl}(x, t; x_0, t_0)$ (the subscript "cl" means that the integral (1.2a) is calculated along the classical trajectory) then excluding from the known equations $p = \partial S_{cl}/\partial x$; $H = -\partial S_{cl}/\partial t$ the parameters x_0 and t_0 we can reconstruct the Hamiltonian $H(x, p, t)$ and use not only eq. (1.2) but also eq. (1.1). Moreover, we can construct the Green function in the form of the asymptotic expansion in Planck's constant using Van Vleck's [14] formula (see also ref. [15])

$$G(x,t;x_0,t_0)=\left[\det\left(\frac{i}{2\pi\hbar}\left\|\frac{\partial^2 S_{cl}}{\partial x\,\partial x_0}\right\|\right)\right]^{1/2}\exp\left\{\frac{i}{\hbar}S_{cl}+\dots\right\} \qquad (1.3)$$

this formula is obviously valid to an accuracy of the factor $\varphi(t,t_0,x,x_0)$ not containing Planck's constant and satisfying the relation $\varphi(t=t_0)=1$; this factor corresponds to different possible orderings of the operators $\hat{x}$ and $\hat{p}$ in the quantum Hamiltonian, i.e. proceeding directly from the classical action.

Thus the problem of quantization of classical systems is closely related to the variational principles of classical mechanics, especially to the inverse problem of the variational calculus, i.e. the problem how to find a functional having as its extremals the given family of trajectories or leading to the given classical equations of motion. This problem was considered by many authors beginning from Sonin [16], Helmholtz [17], Volterra [18], and Darboux [19], during the whole century, but it is still far from its complete solution. The most significant results relating to the inverse problem were obtained by Douglas [20], Havas [21], and Santilli [22], who considered only functionals of the form $\int L\,dt$, and by Skarzhinsky [23], who proposed a more general approach to this problem. One of many essential results obtained in these papers which is the most important for the problem of quantization is that if a classical system possesses a certain Lagrangian or Hamiltonian, it can be described also by means of an infinite number of other Lagrangians or Hamiltonians. The problem of equivalent Lagran-

giane and Hamiltonians in classical mechanics was studied in detail in papers [24-29] in which the analysis of classical mechanics was given in terms of global differential geometry (see also refs.[4,30]). However the quantal aspects of this problem were studied almost nowhere, excepting few papers [31-32] in which only some special cases were consodered (a deeper physical analysis was given in ref. [33], but we think that the problem needs a more detailed investigation). In the present paper we consider the relations between the existence of many classicaly equivalent variational formulations and the problem of quantization from a more general point of view.

The plan of the paper is the following. In the next section we discuss some possible variational formulations leading to the given classical equations of motion. In Secs. 3-4 we give a brief review of the known results concerning the existence of equivalent Lagrangians and Hamiltonians for the given classical system. We consider also some examples, including two examples of the systems which can not be derived from any Lagrangians or Hamiltonians. In Sec. 5 we discuss the extension of the concept of the symmetry of equations of motion and symmetry transformations taking into account the existence of equivalent Lagrangians and Hamiltonians, and indicate some possible applications of this extension. On Sec. 6 we propose a new method of quantization based on the classical integrals of motion. This method is equivalent to others in the case of usual quantum systems, but sometimes it occurs more simple from mathematical point of view. In Sec. 7 we consider in detail several examples of different quan-

tum systems corresponding to the same classical ones. We show that only in the quantum case the physical (i.e. not only formally mathematical) differences take place since only in this case the action becomes the physical (i.e. independently measurable in experiments) quantity. In Sec. 8 we discuss the possible criteria of the correct choice of action functionals (including the additivity principle), draw the attention to the problem of measurements in quantum mechanics and its role for the correct quantization, also we outline some possible generalisations and new approaches which should be investigated.

We confine ourselves to the case of systems with finite number of degrees of freedom but the most of the results obtained can be obviously reproduced in the field theory.

2. Action functionals and trajectories.

The problem considered in this section is the following: for the given family of the classical trajectories described by the functions

$$x_j = x_j^{(*)}(t; x_1^{(0)}, \dots x_N^{(0)}; \dot{x}_1^{(0)}, \dots \dot{x}_N^{(0)}); \qquad j = 1, 2, \dots N \tag{2.1}$$

where $x_k^{(0)}$ and $\dot{x}_k^{(0)}$ - initial values of coordinates and velocities, to find a functional $S[x_j(t)]$ whose extremals would coincide with all functions $x_j^*(t)$. Such a putting of the problem is very general, so that one should impose certain additional requirements. First of all we suppose that trajectories (2.1) are solutions of the system of N second order

differential equations (i.e. the classical system obeys Newton's laws)

$$\ddot{x}_j = f_j(t, x, \dot{x}); \quad j = 1, 2, \ldots N \qquad (2.2)$$

It is known (see, e.g., refs. [34-37] that the variational principle for system (2.2) always exists, if one introduces N subsidiary variables y_j, $j = 1, 2, \ldots N$: for example, for the functional

$$S[x_j(t), y_k(t)] = \int \sum_{j=1}^{N} y_j [\ddot{x}_j - f_j(t, x, \dot{x})] \, dt \qquad (2.3)$$

the Euler equations $\delta S / \delta y_j(t)$ coincide with equations (2.2). However the introduction of additional variables is usually undesirable, so that we restrict ourselves hereafter to the functionals dependent only on the given variables x_j, $j = 1, 2, \ldots N$, and their first derivatives. (The specific features of the problem of quantization of the systems with Lagrangians dependent on higher derivatives were studied, e.g., in ref.[38]). Let us note that if equations (2.2) are local (i.e. the forces depend only on the values and at the given moment of time t) then the equations following from the variational principle

$$\frac{\delta S}{\delta x_j(t)}\Big|_{x(t) = x^*(t)} = 0; \quad j = 1, 2, \ldots N \qquad (2.4)$$

would equivalent to the initial equations (2.2) only provided the functional $S[x(t)]$ (we shall call it also the action functi-

onal or simply the action) is local:

$$S = S_L[x(t)] = \int L(t, x, \dot{x})\, dt \tag{2.5}$$

or quasilocal, i.e. it is a function of local functionals:

$$S = S_F[x(t)] = F(S_L[x(t)]) \tag{2.6}$$

For example, extremals of the quasilocal functional

$$S[x(t)] = \frac{1}{2}\int\int dt'\, dt''\, \dot{x}^2(t')\, \dot{x}^2(t'') \tag{2.7}$$

are the solutions of the equation of the free motion $\ddot{x} = 0$:

$$\frac{\delta S}{\delta x(t)} = -\ddot{x}(t)\int dt'\, \dot{x}^2(t') \tag{2.8}$$

This example shows also that the quasilocal functional cannot be reduced to any local one, since for a local functional the functional derivative $\delta S/\delta x_j(t)$ is a function of variables x , $\dot{x}$, t , while for the nonlocal functional this derivative is also a functional.

At first glance one is tempted to find the action imposing the natural condition

$$\frac{\delta S}{\delta x_j(t)} \equiv f_j(t, x, \dot{x}) - \ddot{x}_j(t) \tag{2.9}$$

which explicitly guarantees the equivalence of eqs. (2.2) and (2.4), and taking functional integrals from both sides of eq. (2.9). However eq. (2.9) holds only for a restricted class of forces $f_j(t, x, \dot{x})$ satisfying the relation

$$\frac{\delta f_j(t', x', \dot{x}')}{\delta x_i(t)} = \frac{\delta f_i(t, x, \dot{x})}{\delta x_j(t')}; \quad x' \equiv x(t') \qquad (2.10)$$

following from the evident identity

$$\frac{\delta^2 S}{\delta x_i(t)\,\delta x_j(t')} = \frac{\delta^2 S}{\delta x_j(t')\,\delta x_i(t)} \qquad (2.11)$$

Relation (2.10) does not hold, e.g., in the following cases:

a) The unidimensional motion with friction

$$\ddot{x} = -\kappa \dot{x}(t); \quad f(t, x, \dot{x}) = -\kappa \dot{x}(t);$$

$$\frac{\delta \dot{x}(t)}{\delta x(t')} = \frac{d}{dt}\delta(t-t') \neq \frac{\delta \dot{x}(t')}{\delta x(t)} = \frac{d}{dt'}\delta(t-t') = -\frac{d}{dt}\delta(t-t')$$

b) The twodimensional "anisotropic oscillator"

$$\ddot{x}_i = \omega^2 \varepsilon_{ij} x_j; \quad f_i(t, x, \dot{x}) = \omega^2 \varepsilon_{ik} x_k; \quad \varepsilon_{ij} = -\varepsilon_{ji}; \quad i, j = 1, 2$$

$$\frac{\delta \varepsilon_{ik} x_k(t)}{\delta x_j(t')} = \varepsilon_{ij}\delta(t-t') \neq \frac{\delta \varepsilon_{jk} x_k(t')}{\delta x_i(t)} = \varepsilon_{ji}\delta(t-t')$$

c) The motion of a charged particle in the field of a magnetic monopole

$$\ddot{x}_i = \frac{e}{c}\varepsilon_{ijk} H_k(x) \dot{x}_j; \quad H_k(x) = g\frac{x_k}{|x|^3} = -g\nabla_i\left(\frac{1}{|x|}\right)$$

In this case the compatibility condition (2.10) is not fulfilled only in the singular point $x = 0$. Note that if one adds the string-like singular magnetic field ending in the point $x = 0$ (then the field becomes solenoidal: $H_k = \mathrm{rot}_k A(x)$) then eq. (2.10) will be satisfied everywhere.

Thus we see that the requirement (2.9) is too hard since it makes impossible to give the variational formulation of many Physically interesting problems. A more wide sufficient (but not necessary) condition of the equivalence of eqs. (2.2) and (2.4) consists in the introduction of the system of "integrating factors" $\mu_{ij}(t, x, \dot{x})$ so that

$$\frac{\delta S}{\delta x_i(t)} = \mu_{ij}(t, x, \dot{x})\left[f_j(t, x, \dot{x}) - \ddot{x}_j\right]; \quad \det\|\mu_{ij}\| \neq 0 \qquad (2.12)$$

(to ensure the minimum of the action for classical trajectories one should demand in addition the positive definiteness of the matrix $\|\mu_{ij}\|$. Then the compatibility condition (2.10) leads to the following equations for the functions $\mu_{ij}(t, x, \dot{x})$:

$$(\alpha)\ \mu_{ij} = \mu_{ji}; \quad (\beta)\ \frac{\partial \mu_{ij}}{\partial \dot{x}_k} = \frac{\partial \mu_{jk}}{\partial \dot{x}_i} = \frac{\partial \mu_{ki}}{\partial \dot{x}_j};$$

$$(\gamma)\ \mathcal{D}\mu_{ij} + \frac{1}{2}\left(\mu_{ik}\frac{\partial f_k}{\partial \dot{x}_j} + \mu_{jk}\frac{\partial f_k}{\partial \dot{x}_i}\right) = 0 \qquad (2.13)$$

$$(\delta)\ \mathcal{D}\left(\mu_{ik}\frac{\partial f_k}{\partial \dot{x}_j} - \mu_{jk}\frac{\partial f_k}{\partial \dot{x}_i}\right) - 2\left(\mu_{ik}\frac{\partial f_k}{\partial \dot{x}_j} - \mu_{jk}\frac{\partial f_k}{\partial x_i}\right) = 0$$

$$\mathcal{D} \equiv \frac{\partial}{\partial t} + \dot{x}_k\frac{\partial}{\partial x_k} + f_k(t, x, \dot{x})\frac{\partial}{\partial \dot{x}_k}$$

These equations are known in literature as Helmholtz's conditions [17]. We emphasize that eqs. (2.12) are only the simplest sufficient conditions of the existence of the action functional. For example, one could consider the integrating factors not only as functions but also as functionals - then the action S would be nonlocal (strictly speaking, quasilocal). The most general

necessary and sufficient condition is the existence of the functionals

$$\frac{\delta S}{\delta x_i(t)} = F_i\left(x(t),\ \dot{x}(t),\ \ddot{x}(t);\ [x(t)]\right)$$

satisfying the compatibility equations

$$\frac{\delta F_i(x(t),\dots)}{\delta x_j(t')} = \frac{\delta F_j(x(t'),\dots)}{\delta x_i(t)}$$

and turning into zero for all functions (2,1) (and only for them). The simplest example of such a functional is eq. (2,8).

However to quantize the systems with nonlocal functionals by means of usual methods described in the previous section is impossible because of the nonadditivity of such functionals (see an example in Sec. 8 and refs.[3,31]). Therefore we consider hereafter only local functionals.

3. Lagrangians and equations of motion.

For the local action functional (2.5) the equations of extremals have the well-known Euler-Lagrange form

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) \equiv$$

$$\frac{\partial L}{\partial x_i} - \frac{\partial^2 L}{\partial t\,\partial \dot{x}_i} - \frac{\partial^2 L}{\partial x_j\,\partial \dot{x}_i}\dot{x}_j - \frac{\partial^2 L}{\partial \dot{x}_j\,\partial \dot{x}_i}\ddot{x}_j = 0 \qquad (3,1)$$

To the family of solutions of eq. (3,1) include all solutions of the given equation (2.2) it is necessary that the substitution $\ddot{x}_j \to f_j(t, x, \dot{x})$ would transform eq. (3,1) to the identity

$$\frac{\partial L}{\partial x_i} - \frac{\partial^2 L}{\partial t\,\partial \dot{x}_i} - \frac{\partial^2 L}{\partial x_j \partial \dot{x}_i}\dot{x}_j - \frac{\partial^2 L}{\partial \dot{x}_j \partial \dot{x}_i} f_j(t,x,\dot{x}) = 0 \quad (3.2)$$

This identity can be considered as the partial differential equation for the unknown function $L(t,x,\dot{x})$. It is completelyequivalent to the system of equations (2.13) for the functions $\mu_{ij}(t,x,\dot{x}) = \partial^2 L/\partial \dot{x}_i \partial \dot{x}_j$. The requirement $\det \|\mu_{ij}\| \neq 0$ (or a more hard condition of the positive definiteness of the matrix $\|\mu_{ij}\|$) ensures the absense of other solutions of eq. (3.2) or (2.13) than the trajectories of eq. (2.2).

Equations (3.2) or (2.13) were studied during last century by many authors. The majority of them, however, investigated only the solutions of these equations (and of their generalizations to the cases of higher order equations and the field theory) satisfying the condition $\mu_{ij} = \delta_{ij}$. We call such Lagrangians "identity Lagrangians". The necessary and sufficient conditions of the existence of the identity Lagrangians were found in the papers of ref. [39]. The methods of constructing identity Lagrangians for the given equations of motion were considered in refs. [40-43]. However examples of the previous section show that identity Lagrangian exist not for all systems of physical interest.

The possibility of a more general Lagrangian formulation of eqs. (2.2) in the case when the functions $\mu_{ij} \neq \delta_{ij}$ are allowed (let us call the corresponding Lagrangians "the equivalent Lagrangians") was noticed for the first time by Sonin [16]. He considered the case of the singular equation and proved that an

arbitrary equation (2.2) in the case $N=1$ always possesses an infinite set of equivalent Lagrangians, and had obtained the explicit formula for all these Lagrangians. Later the same result was obtained also in ref. [19], The existence of equivalent Lagrangians for systems of N equations was studied in ref. [44], but the complete investigation of eqs. (2.13) in the general case was not performed. The systems of two equations were studied in detail by Douglas [20] who gave the complete classification of all possible solutions of equations (2.13) in the case $N=2$. One of the numerous examples of non-Lagrangian systems is the system of two coupled damped oscillators with different frequencies and friction coefficients described by the equations

$$\ddot{x}_1 + 2\gamma_1 \dot{x}_1 + \omega_1^2 x_1 - \xi x_2 = 0;$$
$$\ddot{x}_2 + 2\gamma_2 \dot{x}_2 + \omega_2^2 x_2 - \eta x_1 = 0; \qquad (3.3)$$

provided $\gamma_i, \omega_i, \xi, \eta = const$ $\xi\eta(\gamma_1-\gamma_2)(\omega_1^2-\gamma_1^2-\omega_2^2+\gamma_2^2) \neq 0$. System (3.3) relates to the type IV in Douglas classification. An even more simple example is the system (type III by Douglas)

$$\ddot{x} + \dot{y} = 0$$
$$\ddot{y} + y = 0 \qquad (3.4)$$

Indeed, in this case equation (γ) from the system of Helmholtz's conditions (2.13) yields for $i=j=1$: $D\mu_{11}=0$, while equation (δ) with $i=1$, $j=2$ yields $D\mu_{11}=2\mu_{12}$. Consequently, $\mu_{12}=0$. From another hand, equation (γ) with $i=1$, $j=2$ yields $2D\mu_{12}=\mu_{11}$; therefore we obtain also $\mu_{11}=0$. Consequently, no Lagrangian for system (3.4) can exist. The detailed investigation of equations (2.13) in the N - dimensional

case was carried out by Havas [21], but he considered only the case when the matrix $\|\mu_{ij}\|$ was diagonal. Last years the problem of equivalent Lagrangians and Hamiltonians in classical mechanics attracted the attention of many authors. The most deep investigations of this problem were performed in refs. [22-30]. Some aspects of this problem were stidied also in refs. [45-48]. The inverse problem of the variational calculus for systems of partial differential equations was considered in ref. [22].

After this brief historical survey let us consider possible solutions of eq. (3.2) provided the forces $f_i(t, x, \dot{x})$ are given Evidently, if a function $L(t, x, \dot{x})$ satisfies eq. (3.2), then the function

$$L' = cL + \frac{d}{dt}\psi(t, x) \tag{3.5}$$

also satisfies the same equation for an arbitrary constant number c and an arbitrary function ψ - the trivial solution of eq. (3.2). This arbitrariness is well known. However the nonuniqueness of solutions to eq. (3.2) is by no means exhausted with transformation (3.5). As a rule, the solutions to eq. (3.2) possess also a certain functional arbitrariness, i.e. they depend on arbitrary functions of some definite combinations of the variables t, x, $\dot{x}$ - the integrals of motion.

One can say that nontrivial solutions to eq. (3.2) form a certain set of functions $L(t, x, \dot{x})$ - a class of equivalent Lagrangians $L\{f\}$ corresponding to the given equations of motion (2.2). The set $L\{f\}$ is in the one-to-one correspondence with the set $S\{f\}$ of local action functionals (2.5). In classi-

cal mechanics all elements of the set $S\{f\}$ are physically equivalent, Since they lead to the same equations of motion. Further we shall show (this is just the aim of this paper) that in qunatum mechanics different action functionals corresponding to the same functions $f_j(t, x, \dot{x})$ are not equivalent since they lead to different quantum systems.

Let us consider some examples of equivalent Lagrangians.

I) One-dimensional problemes.

In this case the general solution the equivalent Lagrangians can be obtained for an arbitrary function $f(t, x, \dot{x})$, since system (2.13) is reduced to the only equation for the integration factor $\mu = \partial^2 L / \partial \dot{x}^2$

$$\frac{\partial \mu}{\partial t} + \frac{\partial \mu}{\partial x}\dot{x} + \frac{\partial \mu}{\partial \dot{x}} f(t, x, \dot{x}) + \mu \frac{\partial f}{\partial \dot{x}} = 0 \qquad (3.6)$$

The general solution of eq. (3.9) has the form [16,19]. $\nu(c_1, c_2, c_3) = 0$ where ν is an arbitrary function, and c_j are the first integrals of the system of differential equations

$$dt = \frac{dx}{\dot{x}} = \frac{d\dot{x}}{f} = \frac{d\mu}{\mu \frac{\partial f}{\partial \dot{x}}} \qquad (3.7)$$

If $\mu(t, x, \dot{x})$ is found, then the Lagrengian can be constructed as follows [4,16,19,21,23]:

$$L(t, x, \dot{x}) = \int_{x_0}^{x} d\eta\, f(t, \eta, x_0)\mu(t, \eta, x_0) + \int_{\dot{x}_0}^{\dot{x}} d\xi (\dot{x} - \xi)\mu(t, x, \xi) + \frac{d\psi}{dt} \qquad (3.8)$$

Iα) Motion with linear friction: $\ddot{x} = -2\gamma\dot{x}$.

The first integrals of system (3.7) are (we use here dimen-

sionless variables

$$C_1 = \ln \mu - 2\gamma t; \quad C_2 = \dot{x} + 2\gamma x; \quad C_3 = \dot{x} e^{2\gamma t} \tag{3.9}$$

Then $\mu = e^{2\gamma t} \chi(\dot{x} + 2\gamma x, \dot{x} e^{2\gamma t})$ where χ is an arbitrary function, and

$$L = -2\gamma \dot{x}_0 e^{2\gamma t} \int_{x_0}^{x} d\eta \, \chi(\dot{x}_0 + 2\gamma\eta, \dot{x}_0 e^{2\gamma t}) + e^{2\gamma t} \int_{\dot{x}}^{x} d\xi (\dot{x} - \xi) \chi(\xi + 2\gamma x, \xi e^{2\gamma t}) \tag{3.10}$$

Particular cases:

$$\chi_1 = m; \quad \mu_1 = m e^{2\gamma t}; \quad L_1 = \frac{m}{2} \dot{x}^2 e^{2\gamma t} \tag{3.11}$$

$$\chi_2 = \frac{a}{\dot{x}} e^{-2\gamma t}; \quad \mu_2 = \frac{a}{\dot{x}}; \quad L_2 = a\left[\dot{x} \ln \frac{\dot{x}}{\dot{x}_*} - 2\gamma x\right] \quad (\dot{x}_* = const) \tag{3.12}$$

The Lagrangian L_1 was found for the first time by Bateman [36]; the Lagrangian L_2 was obtained in refs. [45,46]. It can be proved that if the force $f(x, \dot{x})$ does not depend on time (in the one-dimensional case) then the equivalent Lagrangian can be also chosen in the form not depending explicitly on time [21,46]. The Lagrangian L_2 illustrates this statement. It does not turn into the usual Lagrangian of a free motion $\frac{m}{2}\dot{x}^2$ when $\gamma \to 0$. But we can construct another time-independent Lagrangian:

$$L_3 = m\left\{\frac{\dot{x}^2}{2} + 2\gamma x \dot{x} \ln \frac{\dot{x}}{\dot{x}_*} - 2\gamma^2 x^2\right\}; \quad \mu_3 = m(1 + 2\gamma x/\dot{x}) \tag{3.13}$$

which coincides with $\frac{m}{2}\dot{x}^2$ when $\gamma \to 0$.

Some other examples of equivalent Lagrangians in the one-dimensional case were given in refs. [4,19,21-33,45-48].

II*a*) Two-dimensional "anisotropic oscillator": $\ddot{x}_i = \omega^2 \varepsilon_{ij} x_j$, $\varepsilon_{ij} = -\varepsilon_{ji} = 1$.

One can easily check that the following different Lagrangians lead to the given equations of motion:

$$L_1 = \frac{1}{2}(\dot{x}_1^2 - \dot{x}_2^2) + \omega^2 x_1 x_2 ;$$
$$L_2 = \dot{x}_1 \dot{x}_2 - \frac{\omega^2}{2}(x_1^2 - x_2^2) \qquad (3.14)$$

II*b*) The charged particle in a uniform magnetic field (two-dimensional motion across the field): $\ddot{x} - \omega \dot{y} = 0$; $\ddot{y} + \omega \dot{x} = 0$.

Besides the usual Lagrangian

$$L_1 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m\omega}{2}(x\dot{y} - \dot{x}y) \qquad (3.15)$$

there exists, e.g., the following equivalent Lagrangian [49]:

$$L_2 = \frac{m}{2}\cos\omega t\,(\dot{x}^2 - \dot{y}^2) - m \sin\omega t\, \dot{x}\dot{y};$$
$$\| \mu_{ij} \| = \begin{Vmatrix} \cos\omega t & -\sin\omega t \\ -\sin\omega t & -\cos\omega t \end{Vmatrix} \qquad (3.16)$$

III) Let us find the class of forces $f_j(t, x, \dot{x})$ for which the identity Lagrangians exist. The condition $\mu_{ij} = \delta_{ij}$ immediately leads to the expression

$$L = \frac{1}{2}\dot{x}_i^2 + \dot{x}_i A_i(t, x) - U(t, x)$$

Then from eq. (3.2) one has

$$f_i(t, x, \dot{x}) = -\frac{\partial U}{\partial x_i} - \frac{\partial A_i}{\partial t} + \dot{x}_\kappa\left(\frac{\partial A_\kappa}{\partial x_i} - \frac{\partial A_i}{\partial x_\kappa}\right) =$$

$$E_i + \varepsilon_{ijk}\,\dot{x}_j\,H_\kappa\,; \qquad (3.17)$$

$$E_i = -\nabla_i U - \frac{\partial A_i}{\partial t}\,; \quad H_\kappa = rot_\kappa \vec{A}$$

Thus the only force for which an identity Lagrangian exists is Lorentz's force [23,42].

IIc) The examples of the systems which have no Lagrangians were given above (eqs. (3.3)-(3.4)).

Note that for the system

$$\ddot{x} + \dot{y} = 0$$
$$\ddot{y} + y + \varepsilon\dot{x} = 0 \qquad (3.18)$$

which only slightly differs from system (3.4) the equivalent Lagrangians exist. For example, if one multiplies the first equation by $-\varepsilon$, then system (3.18) can be derived from the Lagrangian

$$L_3 = \frac{1}{2}\left[\dot{y}^2 - \varepsilon\dot{x}^2 - y^2 - \varepsilon(\dot{x}y - \dot{y}x)\right] \qquad (3.19)$$

However in the limit case $\varepsilon = 0$ the variables x and $\dot{x}$ disappear from this Lagrangian, so that it leads in this case only to the second equation from system (3.4); i.e. Lagrangian (3.19) becomes singular.

4. Hamiltonians and equations of motion.

In the Hamiltonian formulation of classical mechanics the action functional has the following form:

$$S[x(t),\ p(t)] = \int [p_i \dot{x}_i - H(t, x, p)]\, dt \qquad (4.1)$$

The Euler equations for this functional are just Hamilton's equations:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}\ ; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad (4.2)$$

In the Hamiltonian formulation the notion of the "identity Hamiltonian may be used only in the sense that such Hamiltonians are quadratic in momenta p_i . However any Hamiltonian $H(p)$, for example, leads to the equation $\ddot{x}_i = 0$ without a integration factor while the corresponding Lagrangian $L(\dot{x})$ has the factor $\mu = L''(\dot{x})$. Therefore we shall say about the equivalent Hamiltonians. The equation for the Hamiltonian can be obtained, if one differentiates the first equation from system (4.2) with respect to t and substitutes $f_j(t, x, \dot{x})$ instead of $\ddot{x}_j$:

$$\frac{\partial^2 H}{\partial t \partial p_i} + \frac{\partial^2 H}{\partial x_j \partial p_i}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial p_j \partial p_i}\frac{\partial H}{\partial x_j} = f_j\left(t, x, \frac{\partial H}{\partial p}\right) \qquad (4.3)$$

This equation unlike eq. (3.2), is nonlinear. That is why it is very difficalt to solve it in the general case and, as far as we know, this problem was not considered anywhere. Of course, if an equivalent Lagrangian is known then the Hamiltonian can be reconstructed by means of the standard procedure: $H = p_i \dot{x}_i - L(t, x, \dot{x})$.

$p_i = \partial L / \partial \dot{x}_i$; however in the present section the direct construction of the Hamiltonian is what we have in mind. But in certain simple cases the rather general solutions to eq. (4.3) can be easily found. We consider the only example the unidimensional motion with a linear friction: $f(t, x, \dot{x}) = -2\gamma\dot{x}$.

1) Let $H = H(t, p)$. Then eq. (4.3) is reduced to $\frac{\partial^2 H}{\partial t \partial p} + 2\gamma \frac{\partial H}{\partial p} = 0$
Therefore $H = \varphi(p)\exp(-2\gamma t)$, φ being an arbitrary function.

2) Let $H = H_1(p) + H_2(x)$. Then $H_1''(p) H_2'(x) = 2\gamma H_1'(p)$,
whence $H_2'(x) = 2\gamma p_* = const$; $p_* H_1''(p) = H_1'(p)$;

$$H = E_0 \exp(p/p_*) + 2\gamma p_* x; \quad E_0 = const \qquad (4.4)$$

3) $H = H(t, xp) \equiv H(t, \beta)$.

$$\frac{\partial^2 H}{\partial t \partial \beta} + \left(\frac{\partial H}{\partial \beta}\right)^2 + 2\gamma \frac{\partial H}{\partial \beta} = 0,$$

$$H(t, xp) = \int_{\beta_0}^{xp} \frac{2\gamma d\beta}{\exp[2\gamma\{t + \varphi(\beta)\}] - 1}$$

where $\varphi(\beta)$ is an arbitrary function.

Lagrange's and Hamilton's formulations of the least action principle are equivalent provoded the equations $p_i = \partial L / \partial \dot{x}_i$; $\dot{x}_i = \partial H / \partial p_i$ can be resolved with respect to $\dot{x}_i$ and p_i . At the same time it is well-known that Hamilton's approach in certain cases is more flexible than Lagrange's one. Unfortunately, a more flexibility of the usual Hamilton's approach cannot help us in the cases when equivalent Lagrangians do not exist. Indeed, in such cases Hamiltonians can be only singular (other-

wise we could find nonsingular Lagrangians), i.e. the system of equations $\dot{x}_i = \partial H / \partial p_i$ cannot be resolved with respect to the momenta p_i , $i = 1, 2, \ldots N$. This means that if we excluded the unknown variables p_i from this system we would arrive after all at the equation which would not contain the variables p_i at all: $\varphi(x_1, \ldots x_N; \dot{x}_1, \ldots \dot{x}_N) = 0$. Consequently, singular Hamiltonians can describe only a part of solutions of eq. (2.2) satisfying certain constraints.

In this connection it would be very interesting to obtain an answer to the following question: whether there exists such a generalizatuin of the usual Hamilton's action functional (4.1) that would lead to equations of the type (3.4) having no equivalent Lagrangians. May be nontrivial results can be obtained if one considers the functionals of the type

$$\tilde{S}[x(t), p(t)] = \int f(p_i \dot{x}_i - H(t, x, p))\, dt$$

(in the special case $f(z) = \mu(t) z$ such functionals leading to quasihamiltonian equations were considered in ref. [50]).

5. Variational principles and symmetries of equations of motion.

The nonuniqueness of the variational formulations of equations of motion of a classical system can be exploited in investigating the symmetry transformations of the system. We define the symmetry transformation of the given classical system as a transformation transforming any trajectory of the system to some another trajectory of the same system (i.e. any solution of eq.(2.2)

to another solution of the same equation; see ref.[51]). Such a definition does not necessarily mean that a symmetry transformation must leave the action unchanged, as it is usually assumed, but sooner that it is a sort of automorphism within the class of equivalent action functionals corresponding to the given equations of motion [23]. So, if one uses the Lagrangian approach, then the symmetry transformation should be understood in our sense as any transformation under which a given Lagrangian $L(t, x, \dot{x})$ goes to some another Lagrangian from the class of equivalent Lagrangians $L\{f\}$ (of course, the identity transformation $L \to L$ is also admitted). In the Hamiltonian approach the transformations of the canonical variables are, analogously, the symmetry transformations if they keep the Hamiltonians within the class of equivalent Hamiltonians $H\{f\}$. Note that this definition of the symmetry transformation in terms of $L\{f\}$ and $H\{f\}$ is a direct application of the general concept of the symmetry of equations [51] to the concrete equations for the Lagrangian (3.2) or Hamiltonian (4.3).

The extension of the variational formulations given above can be used for studying the symmetry properties of classical systems at least in two directions.

Firstly, considering the changes of the values S, L, H under the symmetry transformations one can generalize Noether's theorem concerning conservation laws. Moreover, considering variations which do not keep the Lagrangian invariant one can extend the variational principles to obtain the equations for nonconserved values (such an approach was used for solving certain problems of the quantum (and classical) field theory in ref. [52]).

Secondly, one can study the problem of determining the invariant functions S , L , H **from the classes of equivalentness** $\{f\}$ **under the given symmetry transformation. To do this one should solve equations for the functions** S , L , **or** H **together with the conditions of the invariance.**

Various aspects of the formulated problems and some other problems relating to the symmetry transformations of equivalent Lagrangians and Hamiltonians were studied in detail in refs. [23-29,53]. Here we consider only several examples

1) The scaling invariance (i.e. the invariance with respect to the infinitesimal transformation $x \to x + \delta x = x + \varepsilon x$) of the three-dimensional equation $\ddot{x}_i = 0$.

We confine ourselves to the Lagrangians of the form $L(t, \dot{x}^2, x\dot{x}, x^2)$. Then the scaling invariance condition is $x_i \partial L/\partial x_i + \dot{x}_i \partial L/\partial \dot{x}_i = 0$. This condition together with eq. (3.2) lead to the expression

$$L(t, \dot{x}^2, x\dot{x}, x^2) = \int_{\xi_0}^{\dot{x}^2} \frac{d\xi}{[x^2\xi - (x\dot{x})^2]^{1/2}} \varphi\Big(\xi^{-1}[x^2\xi - (x\dot{x})^2]^{1/2} + i\xi^{-1}(x\dot{x} - \xi t)\Big) - 2\int_{\eta_0}^{x\dot{x}} d\eta \frac{\eta + i[x^2\xi_0 - \eta^2]^{1/2}}{x^2[x^2\xi_0 - \eta^2]^{1/2}} \varphi\Big(i\xi_0^{-1}(\eta - \xi_0 t) + \xi_0^{-1}[x^2\xi_0 - \eta^2]^{1/2}\Big) + \int_{\xi_0}^{x^2} d\xi \frac{2\eta_0^2 - \xi\xi_0 + 2i\eta_0[\xi\xi_0 - \eta_0^2]^{1/2}}{\xi^2[\xi\xi_0 - \eta_0^2]^{1/2}} \times \varphi\Big(\xi_0^{-1}[\xi\xi_0 - \eta_0^2]^{1/2} + i\xi_0^{-1}(\eta_0 - \xi_0 t)\Big)$$

where $\varphi(z)$ is an arbitrary analitical function. Noether's theorem yields the following conservation law:

$$x_i \frac{\partial L_i}{\partial \dot{x}_i} = -2i\varphi\left(\frac{[x^2\dot{x}^2-(x\dot{x})^2]^{1/2}}{\dot{x}^2} + i\frac{(x\dot{x})-\dot{x}^2 t}{\dot{x}^2}\right) = const$$

or

$$\frac{[x^2\dot{x}^2-(x\dot{x})^2]^{1/2}}{\dot{x}^2} = const_1, \qquad \frac{x\dot{x}-\dot{x}^2 t}{\dot{x}^2} = const$$

Let us note that the scaling invariant lagrangians $L(\dot{x}^2, x\dot{x})$ independent on t have the following form:

$$L = \frac{c}{\dot{x}^2}\left[x^2\dot{x}^2-(x\dot{x})^2\right]^{1/2}, \quad c = const \qquad (5.1)$$

However the corresponding conserved quantity $x_i \partial L/\partial \dot{x}_i$ is identically equal to zero due to the singular nature of this lagrangian *).

2) The two-dimensional "anisotropic oscillator":

$$\ddot{x}_i = \omega^2 \varepsilon_{ij} x_j \; ; \quad \varepsilon_{ij} = -\varepsilon_{ji} = 1$$

One can easily verify that the given equations are invariant with respect to the rotation

$$x \to x + \delta x = x + \varepsilon y, \quad y \to y + \delta y = y - \varepsilon x \qquad (5.2)$$

However lagrangians (3.14) are not invariants of this transformation: they are transformed as the components of the tensor

$$\|L_{ij}\| = \begin{Vmatrix} L_1 & L_2 \\ L_2 & -L_1 \end{Vmatrix}$$

$$L_{ij} = -\delta_{ij}\frac{\dot{x}^2}{2} + \dot{x}_i\dot{x}_j + \frac{1}{2}(x_i\varepsilon_{jk}x_k + x_j\varepsilon_{ik}x_k)$$

*) Our attention was paid to lagrangian (5.1) by Yu.D.Usachev.

$$x_i \frac{\partial L_1}{\partial \dot{x}_i} = -2i\varphi\left(\frac{\left[x^2\dot{x}^2-(x\dot{x})^2\right]^{1/2}}{\dot{x}^2}+i\,\frac{(x\dot{x})-\dot{x}^2 t}{\dot{x}^2}\right)=const$$

or

$$\frac{\left[x^2\dot{x}^2-(x\dot{x})^2\right]^{1/2}}{\dot{x}^2}=const,\qquad \frac{x\dot{x}-\dot{x}^2 t}{\dot{x}^2}=const$$

Let us note that the scaling invariant Lagrangians $L(\dot{x}^2, x\dot{x}, x^2)$ independent on t have the following form:

$$L=\frac{c}{\dot{x}^2}\left[x^2\dot{x}^2-(x\dot{x})^2\right]^{1/2},\quad c=const \qquad (5.1)$$

However the corresponding conserved quantity $x_i \partial L_1/\partial \dot{x}_i$ is identically equal to zero due to the singular nature of this Lagrangian *).

7) The two-dimensional "anisotropic oscillator":

$$\ddot{x}_i=\omega^2\varepsilon_{ij}x_j\ ;\quad \varepsilon_{ij}=-\varepsilon_{ji}=1$$

One can easily verify that the given equations are invariant with respect to the rotation

$$x\to x,\ \delta x=\varepsilon y;\quad y\to y+\delta y=y-\varepsilon x \qquad (5.2)$$

However Lagrangians (3.14) are not invariants of this transformation: they are transformed as the components of the tensor

$$\|L_{ij}\|=\begin{Vmatrix} L_1 & L_2 \\ L_2 & -L_1 \end{Vmatrix}$$

$$L_{ij}=-\delta_{ij}\frac{\dot{x}^2}{2}+\dot{x}_i\dot{x}_j+\frac{1}{2}\left(x_i\varepsilon_{jk}x_k+x_j\varepsilon_{ik}x_k\right)$$

*) Our attention was paid to Lagrangian (5.1) by Yu.D.Usachev.

Since both linear invariants of this tensor namely, L_{ii} and $\varepsilon_{ij} L_{ij}$, are equal to zero, there exists no quadratic in respect of $\dot{x}$ Lagrangian which would be invariant under the transformation (5.2).

At the same time the quasilocal functional

$$S' = \sqrt{S_1^2 + S_2^2}\,; \qquad S_j = \int L_j\, dt\,; \quad j = 1, 2$$

is explicity invariant under this transformation and leads to the given equations of the motion.

6. Quatization with the aid of integrals of motion.

The existence of different equivalent Lagrangians or Hamiltonians describing the same classical system immediately leads to the conclusion formulated in the beginning of the paper that one can construct many different quantum systems corresponding to the given classical one. The detailed examples and analysis of this statement are given in the following section. Before this we would like to considere the problem whether it is possible to invent such a method of quantisation in which neither Lagrangian nor Hamiltonian do not appear explicitly. It appears that such a method indeed exists. It is based on the equations relating the Green function $G(x_2, t_2; x_1, t_1)$ to the quantum integrals of the motion $\hat{X}_0$ and $\hat{P}_0$ having the sense of the operators of the initial points in the phase space of average values of coordinates and momenta:

$$\hat{X}_0\left(x_2, \frac{\partial}{\partial x_2}, t_2\right) G(x_2, t_2; x_1, t_1) = x_1 G(x_2, t_2, x_1, t_1),$$

$$\hat{P}_0\left(x_2, \frac{\partial}{\partial x_2}, t_2\right) G(x_2, t_2; x_1 t_1) = i\hbar \frac{\partial}{\partial x_1} G(x_2, t_2, x_1, t_1) \quad (6.1)$$

$$G(x_2, t_2, x_1, t_1) = \delta(x_2 - x_1), \quad (6.2)$$

$$\hat{X}_0(t_2 = t_1) = \hat{x}; \quad \hat{P}_0(t_2 = t_1) = \hat{p}$$

These equations were discussed in detail in ref. [54]. Similar equations were also used in ref. [55] (note that analogous equations were applied by Dirac [56] and Kennard [57] already in the first papers on quantum mechanics). If the Hamiltonian operator $\hat{H}$ is known, then the integrals of motion $\hat{I}(t)$ can be found with the aid of the equation

$$i\hbar \frac{\partial \hat{I}}{\partial t} - [\hat{H}, \hat{I}] = 0 \quad (6.3)$$

and the initial conditions (6.2). Then equations (6.1) determine the Green function to an accuracy of an arbitrary time-dependent factor $K(t_2, t_1) = \exp[i\varphi(t_2, t_1)]$. To calculate this factor one should take into account the important nonlinear condition for the evolution operator $\hat{U}(t_2, t_1)$ [3] (the Green function $G(x_2, t_2; x_1, t_1)$ is the kernel of this operator in the coordinate representation):

$$\hat{U}(t_3, t_1) = \hat{U}(t_3, t_2)\hat{U}(t_2, t_1), \quad t_3 > t_2 > t_1 \quad (6.4)$$

$$\hat{U}(t, t) = \hat{E}$$

($\hat{E}$ is the unit operator). This condition leads to the following restriction for the function $\varphi(t_2, t_1)$: $\varphi(t_3, t_1) = \varphi(t_3, t_2) + \varphi(t_2, t_1)$; $\varphi(t, t) = 0$. Therefore

$$K(t_2, t_1) = \exp\left[i \int_{t_1}^{t_2} \chi(\tau)\, d\tau\right] \qquad (6.5)$$

where $\chi(\tau)$ is an arbitrary function. If the Hamiltonian is known, then this function can be determined uniquely from eq. (1.1).

Now let us suppose that we have found somehow classical integrals of motion $x_0(x, p, t)$ and $p_0(x, p, t)$ (they can be obtained directly from the equations of motion) Then constructing from them quantum operators $\hat{X}_0$ and $\hat{P}_0$ and substituting these operators into eqs. (6.1) we can find the Green function (i.e. quantize the classical system) without introducing the Hamiltonian operator or calculating Feynman's path integral. In certain cases this method of quantization can occur more simple and effective than other ones.

However from the point of view of principle the new method seems completely equivalent to the usual methods of quantization. Indeed, if we suppose that the evolution operator $\hat{U}(t_2, t_1)$ is differentiable with respect to t_2, then to an accuracy of the terms of the order $O(\Delta t^2)$ one can write

$$\hat{U}(t + \Delta t, t) = \hat{E} - \frac{i}{\hbar} \Delta t\, \hat{H}(t);$$
$$\hat{H}(t) = i\hbar \frac{\partial}{\partial t_2} U(t_2, t_1)\Big|_{t_2 = t_1 = t} \qquad (6.6)$$

Then assuming in eq. (6.4) $t_3 = t_2 + \Delta t$ and taking into account eq. (6.6) one can easily obtain the following differential equation for the operator $\hat{U}(t_2, t_1)$:

$$i\hbar \frac{\partial \hat{U}}{\partial t_2} = \hat{H}(t_2)\,\hat{U}(t_2, t_1) \qquad (6.7)$$

which is nothing but the Schrödinger equation.

Therefore the new method could give the possibility to extend the frames of the usual formalism only if the quantum Hamiltonian (6.6) $\hat{H}(t)$ (generally speaking, it is an integral operator with a certain complicated kernel) would not have the classical limit in the form of a function $H(x, p, t)$. However we have no example of such a classical system and do not know whether such a exemple exists at all.

The method of quantization with the aid of integrals of the motion possesses the same ambiguities as other ones. Firstly, the procedure of transforming the classical integrals of the motion x_0 and p_0 to the operators $\hat{X}_0$ and $\hat{P}_0$ is nonunique in the general case because of the problem of ordering the operators $\hat{x}$ and $\hat{p}$. Secondly, the Green function can be determined to an accuracy of the factor (6.5). This nonuniqueness is explained by the fact that the gauge transformation $H \to H + \chi(t)$ does not change equations of motion. However these ambiguities can be called trivial. The nontrivial nonuniqueness (completely equivalent to the nonuniqueness of quantization with the aid of usual methods due to the existense of different equivalent Lagrangians and Hamiltonians) arises from the fact that solving the equations of motion (2.2) one finds the integrals of motion $x_0(x, \dot{x}, t)$ and $\dot{x}_0(x, \dot{x}, t)$ corresponding to the initial values of coordinates and velocities,

while in quantum mechanics we need the operators of the coordinates $\hat{x}$ and the generalized momenta $\hat{p}$. Consequently, choosing various possible dependences $p(x, \dot{x}, t)$ one can obtain different operators $\hat{X}_0(\hat{x}, \hat{p}, t)$ and $\hat{P}_0(\hat{x}, \hat{p}, t)$, and therefore different Green functions. Note that the nonuniqueness discussed is analogous to a certain extent to the nonuniqueness of the Yang-Feldman method [13] of solving Heisenberg's operator equations when different solutions of the same equations can be obtained depending on the choice of the commutation relations between the operators at the initial moment.

It is important that the dependence $p(x, \dot{x}, t)$ cannot be chosen quite arbitrarily, since the compatibility conditions for equations (6.1) require that the operators $\hat{X}_0$ and $\hat{P}_0$ satisfy the relation

$$(\hat{X}_0 \hat{P}_0 - \hat{P}_0 \hat{X}_0) G = i\hbar G \qquad (6.8)$$

in the space of solutions of eq. (6.1). The sufficient condition is the operator identity

$$\hat{X}_0 \hat{P}_0 - \hat{P}_0 \hat{X}_0 = i\hbar \hat{E} \qquad (6.9)$$

If the evolution operator $\hat{U}$ and its inverse one $\hat{U}^{-1}$ exist, then the solutions of the operator equations (6.2), (6.3) are

$$\hat{X}_0 = \hat{U} \hat{x} \hat{U}^{-1}; \quad \hat{P}_0 = \hat{U} \hat{p} \hat{U}^{-1} \qquad (6.10)$$

so that relation (6.9) is fulfilled. However it is not evident that proceeding from arbitrary classical integrals of motion $x_0(x, \dot{x}, t)$ and $\dot{x}_0(x, \dot{x}, t)$ and making the substitution

$\dot{x} = \dot{x}(x, p, t)$ one can always construct the new integrals of motion $x_0(x, p, t)$ and $p_0(x, p, t)$ which would satisfy relation (6.9) after the replacement x and p by the operators $\hat{x}$ and $\hat{p}$. If the substitution $x, p \to \hat{x}, \hat{p}$ is made in a usual manner then the functions x_0 and p_0 must satisfy the identity

$$\left\{ p_0^{(i)}, x_0^{(k)} \right\} \equiv \sum_j \left(\frac{\partial p_0^{(i)}}{\partial p_j} \frac{\partial x_0^{(k)}}{\partial x_j} - \frac{\partial p_0^{(i)}}{\partial x_j} \frac{\partial x_0^{(k)}}{\partial p_j} \right) = \delta_{ik} \quad (6.11)$$

It seems the most probable (although we have not a strict proof) that for systems like (3.4) which have no equivalent Lagrangians it is impossible to find the functions $\dot{x}_i(x, p, t)$ which would transform integrals of motion x_0 and $\dot{x}_0$ to the functions satisfying eq. (6.11) (see also in this connection Sec.8). Therefore the method discussed would hardly enable to quantize systems like (3.4). However this problem requires further detailed investigations.

In the conclusion of this section to illustrate the effectiveness of the integrals of motion method let us consider the quantum system with N degrees of freedom described by an arbitrary quadratic Hamiltonian of the type [54]

$$\hat{H}(t) = \frac{1}{2} \hat{Q} B(t) \hat{Q} + C(t) \hat{Q}$$

$$Q = \begin{pmatrix} p \\ x \end{pmatrix}; \quad B = \begin{Vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{Vmatrix}; \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (6.12)$$

Q and C are $2N$- dimensional vectors, $B(t)$- a symmetrical matrix of the dimension $2N \times 2N$. Hamiltonian (6.12) leads to the following equations of motion:

$$\ddot{x} + \Gamma \dot{x} + \Omega x + F = 0 \qquad (6.13)$$

where the $N \times N$ - matrices Γ and Ω and the N- dimensional vector F have the form

$$\begin{aligned} \Gamma &= (b_1 b_3 - \dot{b}_1) b_1^{-1} - b_2 , \\ \Omega &= b_1 b_4 + (\dot{b}_1 - b_1 b_3) b_1^{-1} b_2 , \\ F &= -\dot{c}_1 + b_1 c_2 + (\dot{b}_1 - b_1 b_3) b_1^{-1} c_1 \end{aligned} \qquad (6.14)$$

(The inverse problem of constructing quadratic Lagrangians or Hamiltonians for a given linear system (6.13) was considered in refs. [47,49] ; from the previous sections we know that this problem has solutions only for a restricted class of matrices Γ and Ω).

The integrals of motion x_0 and p_0 (they have the same form both in classical and quantum cases due to the linearity of the equations of motion) can be expressed as follows,

$$Q_0(Q, t_2, t_1) = \Lambda(t_2, t_1) Q + \delta(t_2, t_1) ; \quad Q_0 = \begin{pmatrix} p_0 \\ x_0 \end{pmatrix}$$

$$\frac{d\Lambda}{dt_2} = \Lambda \Sigma B(t_2) ; \quad \frac{d\delta}{dt_2} = \Lambda \Sigma C(t_2) ; \quad \Sigma = \begin{Vmatrix} 0 & E_N \\ -E_N & 0 \end{Vmatrix} \qquad (6.15)$$

$$\Lambda(t, t) = E_{2N} ; \quad \delta(t, t) = 0 , \quad \Lambda = \begin{Vmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{Vmatrix} ; \quad \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

E_N is the $N \times N$ unit matrix.

Then solving eqs. (6.1) together with eq. (1.1.) (to determine the time-dependent factor (6.5)) one can obtain the following formula [54] (provided $\det \lambda_3 \neq 0$):

$$G(x_2,t_2;\,x_1,t_1)=\left[\det(-2\pi i\hbar\lambda_3)\right]^{-1/2}\exp\left\{-\frac{i}{2\hbar}\left[x_2\lambda_3^{-1}\lambda_4 x_2-\right.\right.$$

$$2x_2\lambda_3^{-1}x_1+x_1\lambda_1\lambda_3^{-1}x_1+2x_2\lambda_3^{-1}\delta_2+ \qquad (6.16)$$

$$\left.\left.2x_1(\delta_1-\lambda_1\lambda_3^{-1}\delta_2)+\delta_2\lambda_1\lambda_3^{-1}\delta_2-2\int_{t_1}^{t_2}\dot\delta_1\delta_2\right]\right\}$$

This formula will be used in the next section. Note that it coincides exactly with Van-Vleck's formula (1.3) because of the quadraticity of Hamiltonian (6.12). Let us make also the following simple but important note: in all the cases when integrals of motion x_0 and p_0 have the form (6.15) with a symplectic matrix $\Lambda(t)$ (i.e. $\Lambda\Sigma\tilde\Lambda=\Sigma$, $\tilde\Lambda$ being the transposed matrix) the Green function has inevitably form (6.16) to an accuracy of the factor (6.5), and the Hamiltonian has the form (6.12) with the following matrix B and vector C :

$$B=-\tilde\Lambda\Sigma\dot\Lambda\,,\qquad C=-\tilde\Lambda\Sigma\dot\delta \qquad (6.17)$$

The symplecticity of the matrix $\Lambda(t)$ is equivalent for linear integrals of motion to the compatibility condition (6.9).

7. Examples of different quantum systems corresponding to the same classical equations of motion.

A) Let us consider as the first example the problem of quantisation of the motion with the linear friction: $\ddot x+2\gamma\dot x=0$. If we choose Lagrangian (3.11), then we have the following expressions for the generalized momentum, Hamiltonian and integrals of motion:

$$p = m\dot{x}e^{2\delta t} \tag{7.1}$$

$$H_1 = \frac{m}{2}p^2 e^{-2\delta t} \tag{7.2}$$

$$x_0 = x - \frac{p}{2\delta m}\left(1 - e^{-2\delta t}\right) = x - \frac{\dot{x}}{2\delta}\left(e^{2\delta t} - 1\right), \tag{7.3}$$

$$p_0 = p; \qquad \dot{x}_0 = \dot{x}e^{2\delta t}$$

The Green function in the coordinate representation can be easily calculated with the aid of formula (6.16):

$$G_1(x_2, t_2; x_1, t_1) = \left[i\pi m\hbar\delta^{-1}\left(1 - e^{-2\delta t}\right)\right]^{-1/2} \times \exp\left\{\frac{im\delta}{\hbar}\frac{(x_2 - x_1)^2}{1 - e^{-2\delta t}}\right\}, \qquad t = t_2 - t_1 \tag{7.4}$$

A more general expression, namely, for the Green function of the damped forced harmonic oscillator was calculated for the first time by means of Feynman's path integrals method in ref. [58]. Classical solutions for Hamiltonian (7.2) were studied, e.g., in refs. [45,48]. The Schrödinger equation with this Hamiltonian was studied in papers [59-60] in which references to other papers can be found. A detailed review of papers concerning friction in quantum mechanics is given in ref. [61].

In the momentum representation the Green function has the following form (it is the Fourier transform of eq. (7.4)):

$$G_1(p_2, t_2; p_1, t_1) = \delta(p_2 - p_1)\exp\left\{\frac{ip_2^2}{4m\delta\hbar}\left(e^{-2\delta t} - 1\right)\right\} \tag{7.5}$$

Now let us consider Lagrangian (3.12) leading to the same equation of motion. Then the Hamiltonian is given by eq. (4.4). The integrals of motion are

$$p_0 = p + 2\gamma t p_* ; \quad x_0 = x - \frac{E_0}{2\gamma p_*}\left(e^{2\gamma t} - 1\right) e^{p/p_*} \qquad (7.6)$$

$$\dot{x} = \frac{E_0}{p_*} \exp\left(p/p_*\right) \qquad (7.7)$$

One can easily calculate the Green function in the momentum representation with the aid of equations similar to eqs. (6.1) (to obtain these equations one should make the following substitutions in eqs. (6.1): $x \rightleftharpoons p$; $i \to -i$ (because in the momentum representation $\hat{x} = +i\hbar\,\partial/\partial p$). The result is as follows,

$$G_2(p_2, t_2; p_1, t_1) = \delta(p_2 - p_1 + 2\gamma p_* t) \times \exp\left\{-\frac{iE_0}{2\gamma\hbar} e^{p_2/p_*}\left(1 - e^{-2\gamma t}\right)\right\}, \quad t = t_2 - t_1 \qquad (7.8)$$

In the coordinate representation one obtains

$$G_2(x_2, t_2; x_1, t_1) = (2\pi\hbar)^{-1} \int_{-\infty}^{\infty} dp \exp\left\{\frac{ip}{\hbar}(x_2 - x_1) - \frac{2\gamma i t}{\hbar} p_* x_1 - \frac{iE_0}{2\gamma\hbar} e^{p_2/p_*}\left(e^{2\gamma t} - 1\right)\right\} \qquad (7.9)$$

We see that the Green functions (7.4) and (7.9) are quite different. Strictly speaking, Hamiltonians (7.2) and (4.4) describe different systems even in the classical case, because the second Hamiltonian allows not arbitrary trajectories, but only motions

with $\dot{x} > 0$ (see eq. (7.7); we suppose $E_0 > 0$, $\rho_* > 0$). However this fact is not significant, since if we imposed the additional condition $p > 0$, $\dot{x} > 0$ in the case of Hamiltonian (7.2), nonetheless the quantum systems described by the Green functions (7.4) and (7.9) would be different all the same. We cannot calculate integral (7.9) exactly, but we can obtain the approximate quasiclassical expression for this integral with the aid of the saddle point method assuming $\hbar \to 0$. If $x_2 - x_1 > 0$ then the saddle point is $\rho_s = \rho_* \ln\left[2\gamma\rho_*(x_2 - x_1)/E_0(e^{2\gamma t} - 1)\right]$, and the direction of the steepest descent is $\arg(\rho - \rho_s) = -\frac{\pi}{4}$. Therefore

$$G_2(x_2, t_2; x_1, t_1) \approx \left(\frac{\rho_*}{2\pi i\hbar\,\Delta x}\right)^{1/2} \times$$

$$\exp\left\{\frac{i\rho_*\,\Delta x}{\hbar}\ln\left[\frac{2\gamma\rho_*\,\Delta x}{eE_0(e^{2\gamma t} - 1)}\right]\right\}; \quad \Delta x = x_2 - x_1 > 0 \tag{7.10}$$

If $\Delta x < 0$, then the saddle points and the directions of the steepest descent are given by the following expressions:

$$\rho_s = \rho_*\left[\ln\left|\frac{2\gamma\rho_*\,\Delta x}{E_0(e^{2\gamma t} - 1)}\right| \pm i\pi\right];$$

$$\arg(\rho - \rho_s) = \frac{\pi}{4}$$

We choose the point with the negative imaginary part and obtain

$$G_2(x_2, t_2; x_1, t_1) \approx \left(\frac{i\rho_*}{2\pi\hbar\,|\Delta x|}\right)^{1/2} \times$$

$$\exp\left\{\frac{i\rho_*\,\Delta x}{\hbar}\ln\left[\frac{2\gamma\rho_*\,|\Delta x|}{eE_0(e^{2\gamma t} - 1)}\right] - \frac{\pi\rho_*\,|\Delta x|}{\hbar}\right\}; \quad \Delta x < 0 \tag{7.11}$$

The choice of the saddle point is determined by the physical requirement that in the classicaly forbidden region $\Delta x < 0$ the Green function must quickly decrease. Note that formula (7.11)

is nothing but the analytical continuation of formula (7.10). Moreover, one can calculate the classical action by means of eqs. (2.5) or (4.1) and verify that formula (7.10) coincides with Van-Vleck's formula (1.3).

In the case of Lagrangian (3.13) we have no other mean to calculate the Green function than Van-Vleck's formula. The result of calculations is as follows,

$$G_3(x_2,t_2;x_1,t_1)\approx\left\{\frac{m}{2\pi i\hbar}\left[\gamma\coth(\gamma t)+\gamma\frac{x_2+x_1}{x_2-x_1}\right]\right\}^{1/2}\times$$

$$\exp\left\{-\frac{2i}{\hbar}m\gamma^2x_2^2t+\frac{im\gamma}{2\hbar}(x_2-x_1)^2\coth(\gamma t)+\right. \qquad (7.12)$$

$$\left.\frac{im\gamma}{\hbar}(x_2^2-x_1^2)\ln\left[\frac{2\gamma(x_2-x_1)}{\dot{x}_*(1-e^{-2\gamma t})}\right]\right\};\quad t=t_2-t_1$$

It is interesting that in the limit $\gamma\to 0$ this approximate formula transforms to the exact Green function of the free motion.

Lagrangians (3.12) and (3.13) in the classical case are completely equivalent (both of them describe the motion with $\dot{x}>0$ if $\dot{x}_*>0$). However the quantum systems described by the Green functions (7.10) and (7.12) are undoubtedly quite different.

Nevertheless one can suppose that the difference between the Green functions (7.4), (7.10), and (7.12) is only apparent, since may be there exist some canonical transformations reducing any of these functions to the others. In a certain sense this supposition is correct. Indeed, any classical Hamiltonian can be transformed by means of a time-dependent canonical transformation to an arbitrary other Hamiltonian (provided both Hamiltonians

have no singularities or have singularities of the same nature). This statement is an evident consequence of the fact that the canonical (consequently nondegenerate) transformation $x, p \to x_0(x,p,t),\ p_0(x,p,t)$ transforms the given Hamiltonian $H(x,p,t)$ to the Hamiltonian $\tilde{H}=0$ (x_0 and p_0 are integrals of motion having the sense of the initial points in the phase space); the generating function of this canonical transformation is the classical action function satisfying the Hamilton-Jacobi equation. The analogous result is valid in quantum mechanics: if the (unitary) evolution operator $\hat{U}_1(t)$ corresponds to the system with the Hamiltonian $\hat{H}_1$, and the operator $\hat{U}_2(t)$ corresponds to the system with the Hamiltonian $\hat{H}_2$, then the time-dependent unitary operator $\hat{U}_1\hat{U}_2^{-1}$ transforms the solutions of the Schrödinger equation with the Hamiltonian $\hat{H}_2$ to the solutions of the equation with the Hamiltonian $\hat{H}_1$.

For example, let us consider classical Hamiltonians (7.2) and (4.4). To find the canonical transformation relating these Hamiltonians one should solve the equations

$$x_0(x_1, p_1, t) = x_0(x_2, p_2, t)$$

$$p_0(x_1, p_1, t) = p_0(x_2, p_2, t)$$

The left-and right-hand sides of these equations are given by eqs. (7.3), and (7.4) respectively. Solving these equations one obtains

$$p_2 = p_1 - 2\gamma t p_*;$$

$$x_2 = x_1 + \frac{1}{2\gamma}\left(1-e^{-2\gamma t}\right)\left(\frac{E_0}{p_*}e^{p_1/p_*} - \frac{p_1}{m}\right) \qquad (7.13)$$

The generating function of this transformation is

$$\Phi(p_1, x_2, t) = x_2(2\delta t p_x - p_1) + \frac{1}{2\delta}(1 - e^{-2\delta t})\left(E_0 e^{p_1/p_x} - \frac{p_1^2}{2m}\right); \qquad (7.14)$$

$$p_2 = -\frac{\partial \Phi}{\partial x_2}; \quad x_1 = -\frac{\partial \Phi}{\partial p_1}, \quad H_2 = H_1 + \frac{\partial \Phi}{\partial t}$$

We see that transformations (7.13) mix coordinates with generalized momenta. However if we believe that the variables x_1 and x_2 are not simply some generalized coordinates, but they are real space coordinates which can be measured experimentally (this assumption is the basis of our definition of the concept of quantization given in Introduction), then the transformations $x_1 \to x_2$ mixing coordinates with (unobservable?) generalized momenta must be forbidden, so that the only allowed transformations are as follows,

$$x_2 = f(x_1, t), \quad p_2 = \varphi(x_1, p_1, t) \qquad (7.15)$$

Since no canonical transformation satisfying eq. (7.15) and relating Hamiltonians H_1 (7.2) and H_2 (4.4) exists, we are to conclude that Hamiltonians H_1 and H_2 describe different physical systems, although they lead to the same equation of motion. However this difference becomes experimentally observable, in our opinion, only in quantum mechanics, because only in quantum mechanics the action becomes an observable quantity and can be measured independently (since the action is the main part of the phase of the Green function, it can be measured in principle (with evident limitations) in some interference expe-

riments). Indeed, in classical mechanics the action, Hamiltonian, Lagrangian and so on are only the auxiliary notions, because one can solve Newton's equations of motion without introducing these concepts. Roughly speaking, we have no "action-meter" in classical mechanics. Another situation takes place in quantum mechanics, where the "action-meter" can be in principle constructed, and therefore different (although classically equivalent) Lagrangians and Hamiltonians lead to physically different and distinguishable in experiments quantum systems.

One could think the different quantum systems were obtained because we considered an "exotic" system - a system with friction.

B) Therefore let us consider another - quite usual - system, namely, a charged particle moving in a uniform magnetic field (the two-dimensional motion). In this case both equivalent Lagrangians L_1 (3.15) and L_2 (3.16) are nonsingular. In the case of the Lagrangian L_1 the Green function is well-known [57,3]:

$$G_1(x_2,y_2,t;x_1,y_1,0) = \frac{m\omega}{4\pi i\hbar \sin(\frac{1}{2}\omega t)} \times$$
$$\exp\left\{\frac{im\omega}{4\hbar}\cot(\frac{1}{2}\omega t)\left[(x_2-x_1)^2+(y_2-y_1)^2\right] + \frac{im\omega}{2\hbar}(x_1y_2 - x_2y_1)\right\} \qquad (7.16)$$

However the same equations of motion can be obtained also from the Hamiltonian

$$H_2 = \frac{1}{2m}\cos\omega t\,(p_x^2 - p_y^2) - \frac{1}{m}\sin\omega t\,p_x p_y \qquad (7.17)$$

Applying to this special example of general quadratic Hamiltonians the scheme of the previous section, one can obtain the following Green function:

$$G_2(x_2,y_2,t;\,x_1,y_1,0) = \frac{m\omega}{4\pi i \sin(\frac{1}{2}\omega t)} \times$$

$$\exp\left\{\frac{im\omega}{4\hbar}\cot(\tfrac{1}{2}\omega t)\left[(x_2-x_1)^2+(y_2-y_1)^2\right] - \frac{im\omega}{2\hbar}(x_2-x_1)(y_2-y_1)\right\} \qquad (7.18)$$

which differs from the Green function (7.16). The physical difference between the Lagrangians L_1 and L_2 consists in different relations between the velocities and generalized momenta: in the first case

$$\dot{x} = \frac{1}{m}p_x + \frac{\omega}{2}y;$$
$$\dot{y} = \frac{1}{m}p_y - \frac{\omega}{2}x \qquad (7.19)$$

and in the second case

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \frac{1}{m}\begin{pmatrix}\cos\omega t & -\sin\omega t\\ -\sin\omega t & -\cos\omega t\end{pmatrix}\begin{pmatrix}p_x\\ p_y\end{pmatrix} \qquad (7.20)$$

Note that the Hamiltonian

$$H_1 = \frac{1}{2m}\left(p_x + \frac{m\omega}{2}y\right)^2 + \frac{1}{2m}\left(p_y - \frac{m\omega}{2}x\right)^2 \qquad (7.21)$$

can be transformed to the Hamiltonian (7,17) by means of the canonical transformation

$$p_x^{(2)} = \frac{1}{2}(1+\cos\omega t)p_x^{(1)} - \frac{1}{2}\sin\omega t\, p_y^{(1)} + \frac{m\omega}{4}\sin\omega t\, x^{(1)} - \frac{m\omega}{4}(1-\cos\omega t)y^{(1)};$$

$$p_y^{(2)} = \frac{1}{2}\sin\omega t\, p_x^{(1)} + \frac{1}{2}(1+\cos\omega t)\, p_y^{(1)} + \frac{m\omega}{4}(1-\cos\omega t)\, x^{(1)} + \frac{m\omega}{4}\sin\omega t\, y^{(1)};$$

$$x^{(2)} = \frac{\sin\omega t}{m\omega}(\cos\omega t - 1)\, p_x^{(1)} + \frac{\cos\omega t}{m\omega}(\cos\omega t - 1)\, p_y^{(1)} + (\cos\omega t + \frac{1}{2}\sin^2\omega t)\, x^{(1)} - \frac{1}{2}\sin\omega t(2-\cos\omega t)\, y^{(1)}; \tag{7.22}$$

$$y^{(2)} = \frac{1}{m\omega}(\cos\omega t - 1 - \sin^2\omega t)\, p_x^{(1)} - \frac{\sin\omega t}{m\omega}(1+\cos\omega t)\, p_y^{(1)} + \frac{1}{2}\sin\omega t\cos\omega t\, x^{(1)} + \frac{1}{2}(1+\cos^2\omega t)\, y^{(1)}$$

Since this transformation does not belong to the class of allowed transformations (7.15), we conclude that the Green functions (7.16) and (7.18) describe physically different quantum systems. Note in addition that Hamiltonian H_2 cannot be obtained from H_1 by means of any gauge transformation of electromagnetic potentials.

If $\omega = 0$, we obtain two different Green functions describing the free motion:

$$G_\pm(x_2, y_2, t; x_1, y_1, 0) = \frac{m}{2\pi\hbar t}\exp\left\{\frac{im}{2\hbar t}\left[(x_2 - x_1)^2 \pm (y_2 - y_1)^2\right] - \frac{i\pi}{4}(1 \pm 1)\right\}; \tag{7.23}$$

$$H_\pm = \frac{1}{2m}(p_x^2 \pm p_y^2) \tag{7.24}$$

The Hamiltonians $H_\pm$ are related by means of the time-dependent

canonical transformation mixing coordinates and generalized momenta:

$$p_x^{(-)} = p_x^{(+)}; \quad p_y^{(-)} = p_y^{(+)}; \quad x^{(-)} = x^{(+)};$$

$$y^{(-)} = y^{(+)} - \frac{2t}{m} p_y^{(+)}; \quad H_- = H_+ + \frac{\partial \Phi}{\partial t}; \qquad (7.25)$$

$$\Phi(p_x^{(+)}, p_y^{(+)}, x^{(-)}, y^{(-)}, t) = -p_x^{(+)} x^{(-)} - p_y^{(+)} y^{(-)} - \frac{t}{m}\left[p_y^{(-)}\right]^2$$

8. Criteria of the choice of action functionals.

Since different variational formulations of the same classical equations of motion lead, as we have just seen, to different quantum systems, we are to discuss how to choose the most correct action functionals. In the case of forces satisfying eq. (2.10) the most correct Lagrangian has, of course, the known form $L_0(x, \dot{x}, t) = \frac{m}{2}\dot{x}^2 + \frac{e}{c} A(x,t)\dot{x} - \varphi(x,t)$, because it has many advantages: it belongs to the class of identity Lagrangians, it describes in a unique manner the interaction with various external fields (whereas other equivalent Lagrangians, e.g. Lagrangian (3.16), describe the motions only in the special configurations of fields), it possesses definite invariance properties with respect to Galilei's and gauge transformations, etc.

But what shall we do if we wish to quantize a general classical system (2.2), when the forces do not satisfy eq. (2.10), for example, a system with friction like (3.3)? One of possible answers is as follows: one can say that such forces are not fundamental, so that equations like (3.3) are approximate even in the

classical case, and the systems described by these equations are in fact open; consequently, to try to describe such systems in terms of pure quantum-mechanical states, i.e. in terms of wave functions, is the meaningless pastime, because such systems can be described correctly only in terms of density matrix. It is difficult to object to such a viewpoint, since in this approach all difficulties disappear indeed. Let us note only that in such an approach the nonuniqueness of quantization of the given classical system (2.2) becomes selfevident from the out-set, since in this case one should introduce auxiliary variables or parameters (thermostat) describing the influence of the external world on the system under study, and this can be done in an infinite number of different ways.

However we wish to investigate none the less another possibility - to try to quantize general system (2.2) in the framework of pure quantum states, due to the following reasons. Firstly, in a general case we have no evident method of introducing auxiliary variables or parameters to obtain the evolution equation for the density matrix. Secondly, our study may help us to understand deeper why in usual cases the Lagrangian L_0 to the best extent corresponds to reality. Finally, maybe we will succeed in future in finding such a method of quantization which can be applied to a more wide class of systems than now available.

First of all it is interesting to note that not all possible action functionals discussed in Sec.2 can be used to quantize a classical system. If we suppose that in the limit $\hbar \to 0$ the asymptotic Van-Vleck's formula (1.3) should be valid (since this

formula naturally arises from Feynman's path integral [3], it is difficult to imagine such a method of quantization that would not lead to it) and take into account the important relation (6.4), then quasilocal functionals like (2.7) should be excluded from consideration. For example, in the case of quasilocal functional (2.7) the classical action is $S_{cl}(x_2,t_2;x_1,t_1)=\frac{1}{2}(x_2-x_1)^4/(t_2-t_1)^2$. Then, calculating the integral

$$I=\int G(x_2,t_2;x',t')\,G(x',t';x_1,t_1)\,dx' \approx$$

$$\int exp\left\{\frac{i}{\hbar}\left[S_{cl}(x_2,t_2;x',t')+S_{cl}(x',t';x_1,t_1)\right]\right\}dx'$$

by means of the saddle point method, one obtains instead of the correct expression following from (6.4):

$$I=G(x_2,t_2;x_1,t_1)\approx exp\left\{\frac{i}{2\hbar}\,\frac{(x_2-x_1)^2}{(t_2-t_1)^2}\right\}$$

the following wrong result:

$$I\approx exp\left\{\frac{i}{2\hbar}\,\frac{(x_2-x_1)^2}{\left[(t_2-t')^{2/3}+(t'-t_1)^{2/3}\right]^3}\right\}$$

The reason of this failure is the nonadditivity of the nonlocal functional (2.7). Consequently, we have to restrict ourselves with local additive functionals (2.5). (It should be noted nevertheless that there exist approaches to the problem of quantization in which nonlinear modofocations of the Schrödinger equation are considered. In such approaches nonadditive actions arise quite naturally [31]). Then in the quasiclassical approximation

(1.3) equation (6.4) is always fulfilled (the most simple way to prove this statement is to use Feynman's path integrals method [2-3]). However which of the infinite number of equivalent Lagrangians $L\{f\}$ should be chosen? It seems to us that it is impossible to give a convincing answer to this question without analyzing the problem of measurements in quantum mechanics (see in this connection also ref.[33]). Indeed, different equivalent Lagrangians lead to different relations between the velocity and the generalized momentum. Therefore the following problem arises: what can we measure in reality: the kinematic momentum $m\dot{x}$ or the generalized momentum p ? For example, if a charged particle moves in a uniform magnetic field, then the standard Hamiltonian (7.21) leads to the conclusion that in the quantum case the kinematic momenta $m\dot{x}$ and $m\dot{y}$ cannot be measured simultaneously, while the equivalent Hamiltonian (7.17) does not leads to such a restriction. Thus the problem of the choice of the correct Hamiltonian can be solved experimentally. However we did not see anywhere the physical explanation of the impossibility of measuring $\dot{x}$ and $\dot{y}$ simultaneously in the presence of a magnetic field. A similar situation takes place in the case of the motion with friction. If one believes that the uncertainty relation $\langle\Delta\hat{x}\rangle\langle\Delta\hat{p}\rangle \geq \hbar/2$ relates to the operator of the <u>generalized momentum</u>, then none of the quantum systems obtained from Lagrangians (3.11), (3.12), (3.13) and from other equivalent Lagrangians is better or worse than others, from the point of view of mathematics (physically all these systems are different); in such a case the question which of these Lagrangians is the most correct is meaningless. But if the uncertainty relation holds

for the operator of the kinematic momentum (note that usually, illustrating the uncertainty relation in the examples like the propagation of particles through slits etc., we implicitly have in mind just the kinetic momentum), then all these Lagrangians are incorrect, and quantization in terms of wave functions is impossible. This example shows once more that the problem of the choice of the correct action is in fact not mathematical but physical problem.

However it may be possible that the ambiguities discussed are due to the model and, consequently, degenerate nature of the systems concerned: they correspond to very special forces $f_j(t, x, \dot{x})$ and they are one-or two-dimensional, while the real systems consist of many particles moving in the three-dimensional space.

Maybe for Lagrangians describing systems consisting of many interacting particles there exists a principle which permits to select the only physical Lagrangian from an infinite number of equivalent Lagrangians. Our conjecture is that from all equivalent Lagrangians $L\{f\}$ (with the same forces f) one can always select uniquely the physical Lagrangian L^* using the additivity principle - i.e. requiring that if the interaction between the particles is turned off then the physical Lagrangian must be equal to the sum of one-particle Lagrangians L_n dependent only on variables of n-th particle (cf. [33]):

$$\lim_{\{g\}\to 0} L^*(t, x_1, \dot{x}_1, \dots x_N, \dot{x}_N / \{g\}) = \sum_{n=1}^{N} L_n(t, x_n, \dot{x}_n) \quad (8.1)$$

The symbol $\{g\}$ means the set of coupling constants describing the interactions between particles (principle (8.1) can be appli-

ed only if Lagrangians are continuous functions of $\{q\}$, but the most of physical systems satisfy this requirement). This principle is the consequence of the fact that the world in which we live is governed by quantum laws after all. Therefore if the action of the system of noninteracting particles were not equal to the sum of individual actions of each particle, then the wave function of the system would not be factorized, and this phenomenon would be very strange.

Let us consider as an example the following system:

$$\ddot{x} + \gamma(\dot{x} - \dot{y}) = 0$$
$$\ddot{y} + \gamma(\dot{y} - \dot{x}) = 0 \qquad (8.2)$$

Let us consider for the sake of simplicity only equivalent Lagrangians dependent only on $t, \dot{x}$, and $\dot{y}$. Then one can easily verify that the general solution of eq. (3.2) is

$$L(t, \dot{x}, \dot{y}) = L_1(\dot{x}+\dot{y}) + e^{-2\gamma t} L_2\left((\dot{x}-\dot{y})e^{2\gamma t}\right) \qquad (8.3)$$

L_1 and L_2 being arbitrary functions. Let us require that in the case $\gamma = 0$ the Lagrangian would have the form $L = L'(\dot{x}) + L'(\dot{y})$, where L' is an unknown function. Then the functional equation

$$L_1(\dot{x}+\dot{y}) + L_2(\dot{x}-\dot{y}) = L'(\dot{x}) + L'(\dot{y}) \qquad (8.4)$$

uniquely determines all three functions L_1, L_2, and L', so that the Lagrangian L^* must have the following form:

$$L^*(t, \dot{x}, \dot{y}) = \frac{m}{4}(\dot{x}+\dot{y})^2 + \frac{m}{4} e^{2\gamma t}(\dot{x}-\dot{y})^2 \qquad (8.5)$$

A more detailed study of the additivity principle will be given elsewhere.

Another unsolved problem is whether it is possible to quantize (without introducing auxiliary variables) classical systems like (3.4) which have no equivalent Lagrangian. We can prove that the classical integrals of motion $x_0(x, \dot{x}, t)$ and $\dot{x}_0(x, \dot{x}, t)$ for system (3.4) cannot be transformed to the integrals $x_0(x, p, t)$ and $p_0(x, p, t)$ satisfying eq. (6.9) by means of any linear transformation of the allowed type (7.15), so that the integrals of motion method hardly can be applied to this system. Maybe one should in such cases replace the system of the second-order differential equations by the equivalent system of the first-order equations, for which equivalent Lagrangians always exist [33]. However such an approach is also ambiguous, since in the cases when the initial second-order system is derivable from a Lagrangian the actions calculated with the aid of this Lagrangian and of the Lagrangian of the equivalent first-order system are, generally speaking, different. In more detailes this problem will be discussed elsewhere.

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