AADD-81

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ACCELERATOR DEPARTMENT
Internal Report

NOTE ON THE NORMAL MODE TREATMENT OF BEAM CAVITY INTERACTIONS AND THE MODE WIDTH

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May 5, 1965

I. In the analysis of the interaction between a bunched beam and a long resonant cavity,[1] the response of the cavity was obtained in terms of its resonant modes. This procedure is clearly not valid if the separation of adjacent modes is comparable with or smaller than the width of the modes. In the other extreme (mode width smaller than spacing), however, the procedure is valid.

It may also be instructive to calculate the response of the cavity to the periodic beam pulses as a driven oscillator. Rather than go through the complex algebra to show that the results are the same we will treat a simple model by both procedures to show the validity of either.

The model we shall consider is a damped vibrating string which is struck repeatedly (and periodically) at its center. This problem is similar to the cavity problem in that there are a spectrum of modes. The equation of motion for the string is

$$\frac{\partial^2 y}{\partial t^2} + 2\gamma \frac{\partial y}{\partial t} - v^2 \frac{\partial^2 y}{\partial x^2} = a\, \delta\left(x - \frac{L}{2}\right) \sum_{n=0}^{\infty} \delta(t - n\Delta)$$

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1 R.L. Gluckstern, Brookhaven National Laboratory Report AADD-38; Los Alamos Scientific Laboratory Internal Report R11/RLG-1; Particle Accelerator Conference, Washington D.C., March 10-12, 1965 (with Harold Butler).

Here 2γ is the damping constant, L is the length of the string, Δ is the interval between pulses and a is proportional to the strength of the pulses.

A. Normal Mode Solution

The boundary conditions

$$y(0) = y(L) = 0$$

determine directly the normal modes of the string, i.e.

$$y_m(x) = \sin \frac{m\pi x}{L} \qquad m = 1, 2, \cdots\cdots$$

and the natural resonant frequencies are given by (assuming $e^{i\omega t}$)

$$-\omega^2 + 2i\omega\gamma + v^2 \frac{m^2\pi^2}{L} = 0$$

$$\omega = i\gamma \pm \sqrt{v^2 \frac{m^2\pi^2}{L^2} - \gamma^2} \equiv i\gamma \pm \omega_m$$

In terms of the normal modes, one can write the general solution as

$$y(x,t) = \sum_{m=1}^{\infty} \sin \frac{m\pi x}{L} e^{-\gamma t} [A_m \sin \omega_n t + B_m \cos \omega_m t] .$$

The present picture of normal modes properly speaking applies only between pulses. What we shall calculate is the way A_m and B_m change from pulse to pulse. In order to do this we use the relation

$$\delta(x - \frac{L}{2}) = \frac{2}{L} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{L} \sin \frac{m\pi}{2} .$$

From the original equation one finds the following discontinuities

$$\delta^{(n)} y = 0 , \quad \delta^{(n)} \frac{\partial y}{\partial t} = a\, \delta(x - \frac{L}{2}) = \frac{2a}{L} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{L} \sin \frac{m\pi}{2}$$

which lead to the equations

$$(\delta^{(n)}A_m) \sin \omega_m t_n + (\delta^{(n)}B_m) \cos \omega_m t_n = 0$$

$$(\delta^{(n)}A_m) \cos \omega_m t_n - (\delta^{(n)}B_m) \sin \omega_m t_n = \frac{2a}{L} \frac{\sin \frac{m\pi}{2}}{\omega_m} e^{\gamma t_n} ,$$

from which one finds

$$\delta^{(n)} (A_m + i B_m) = \frac{2a}{L\omega_m} \sin \frac{m\pi}{2} e^{(\gamma - i\omega_m) t_n}$$

For periodic pulses, one then obtains the solution

$$A_m^{(n)} + i B_m^{(n)} = \frac{2a}{L\omega_m} \sin \frac{m\pi}{2} \frac{e^{(\gamma - i\omega_m)n\Delta} - 1}{e^{(\gamma - i\omega_m)\Delta} - 1}$$

As $n \to \infty$, one finds therefore

$$y(x, n\Delta)_{n\to\infty} = \sum_{m=1}^{\infty} \sin \frac{m\pi x}{L} e^{-\gamma n\Delta} \operatorname{Im} \left\{ \left(A_m^{(n)} + i B_m^{(n)} \right) e^{i\omega_m n\Delta} \right\}_{n\to\infty}$$

$$= \sum_{m=1}^{\infty} \sin \frac{m\pi x}{L} \frac{2a}{L\omega_m} \sin \frac{m\pi}{2} \operatorname{Im} \left\{ \frac{1 - e^{-\gamma n\Delta} e^{i\omega_m n\Delta}}{e^{(\gamma - i\omega_m)\Delta} - 1} \right\}_{n\to\infty}$$

Our final result for the normal mode treatment is therefore

$$y(x, n\Delta)_{n\to\infty} = \sum_{m=1}^{\infty} \sin \frac{m\pi x}{L} \left(\frac{2a}{L\omega_m} \right) \sin \frac{m\pi}{2} \left\{ \frac{1}{e^{(\gamma - i\omega_m)\Delta} - 1} \right\}$$

B. Driven Oscillator Treatment

In this case one expands the periodic pulses (in steady state) into a Fourier series, i.e.

$$\sum_{n=-\infty}^{\infty} \delta(t - n\Delta) = \frac{1}{\Delta} \sum_{\ell=-\infty}^{\infty} e^{\frac{2\pi i \ell}{\Delta} t}$$

Writing

$$y(x,t) = \sum C_m(t) \sin \frac{m\pi x}{L}$$

one obtains the differential equation for C_m

$$\ddot{C}_m + 2\gamma \dot{C}_m + \frac{v^2 m^2 \pi^2}{L^2} C_m = \frac{2a}{\Delta L} \sin \frac{m\pi}{2} \sum_{\ell=-\infty}^{\infty} e^{\frac{2\pi i \ell}{\Delta} t}$$

with the steady state solution

$$C_m(t) = \sum_{\ell=-\infty}^{\infty} \frac{\frac{2a}{\Delta L} \sin \frac{m\pi}{2} e^{\frac{2\pi i \ell}{\Delta} t}}{-\left(\frac{2\pi\ell}{\Delta}\right)^2 + 2\gamma \left(\frac{2\pi i \ell}{\Delta}\right) + \frac{v^2 m^2 \pi^2}{L^2}} .$$

For $t = n\Delta$

$$C_m(n\Delta) = \sum_{\ell=-\infty}^{\infty} \frac{\frac{2a}{\Delta L} \sin \frac{m\pi}{2}}{\omega_m^2 + \left(\gamma + \frac{2\pi i \ell}{\Delta}\right)^2}$$

$$= - \frac{2\Delta}{2\pi^2 L} \sum_{\ell=-\infty}^{\infty} \sin \frac{m\pi}{2} \frac{1}{\left(\ell - \frac{\Delta\gamma}{2\pi} i\right)^2 - \left(\frac{\Delta\omega_m}{2\pi}\right)^2}$$

$$= - \sum_{\ell=-\infty}^{\infty} \frac{a}{2\pi\omega_m L} \sin \frac{m\pi}{2} \left[\frac{1}{\ell - \frac{\Delta}{2\pi}(i\gamma + \omega_m)} - \frac{1}{\ell - \frac{\Delta}{2\pi}(i\gamma - \omega_m)} \right]$$

Changing ℓ to $-\ell$ in the first term

$$C_m(n\Delta) = \sum_{\ell=-\infty}^{\infty} \frac{a}{2\pi\omega_m L} \sin\frac{m\pi}{2} \left[\frac{1}{\ell + \frac{\Delta}{2\pi}(\omega_m + i\gamma)} + \frac{1}{\ell + \frac{\Delta}{2\pi}(\omega_m - i\gamma)} \right]$$

$$= \frac{a}{\pi\omega_m L} \sin\frac{m\pi}{2} \operatorname{Re}\left\{ \sum_{\ell=-\infty}^{\infty} \frac{1}{\ell + \frac{\Delta}{2\pi}(\omega_m + i\gamma)} \right\} .$$

It can be shown that

$$\sum_{-\infty}^{\infty} \frac{1}{\ell+\beta} = \frac{\pi e^{i\pi\beta}}{\sin \pi\beta} = \frac{2\pi i}{1 - e^{-2i\pi\beta}} ,$$

and so

$$\operatorname{Re} \sum_{-\infty}^{\infty} \frac{1}{\ell+\beta} = 2\pi \operatorname{Im}\left\{ \frac{1}{e^{-2\pi i\beta} - 1} \right\} .$$

Therefore

$$C_m(n\Delta) = \frac{2a}{\omega_m L} \sin\frac{m\pi}{2} \operatorname{Im}\left\{ \frac{1}{e^{(\gamma - i\omega_m)\Delta} - 1} \right\}$$

which leads to $y(x,n\Delta)$ in exact agreement with the normal mode solution.

One can also show that the differential equation for $C_m(t)$ can be solved using Laplace transform techniques in order to obtain the solution for a finite number of pulses as well as an infinite number.

II. A further item of interest is the relation between the standing wave solution for the normal modes and the solution obtained by reflections of traveling waves. For a propagation constant

$$\Gamma = \alpha + i\beta$$

one has, for the forward waves, to evaluate the sum

$$S = \sum_{n=0}^{\infty} e^{-(\alpha+i\beta)2Ln} = \frac{1}{1 - e^{-2L(\alpha+i\beta)}}$$

For small values of α, the resonant behavior of this expression is exhibited by its large value near $\beta L = m\pi$. In this vicinity one can write

$$S = \frac{e^{2\alpha L}}{e^{2\alpha L} - e^{-2i(\beta L - m\pi)}}$$

For $\alpha L \ll 1$, the modes are separated. Near the resonance one then has

$$S \cong \frac{1}{2\alpha L + 2i(\beta L - m\pi)}$$

The frequency dependence of S depends on that of the propagation constant β. Specifically the resonance ω_m occurs when $\beta_m L = m\pi$. Near this value one has

$$\beta - \beta_m \simeq \frac{\partial\beta}{\partial\omega}(\omega - \omega_m) = \frac{\omega - \omega_m}{v_g}$$

where v_g is the group velocity. Hence

$$S(\omega) \simeq \frac{1}{2iL} \frac{1}{\left[\frac{\omega-\omega_m}{v_g} - i\alpha\right]} = \frac{v_g}{2iL\omega_m} \frac{1}{\left[\frac{\omega-\omega_m}{\omega_m} - \frac{i\alpha v_g}{\omega_m}\right]}$$

The "width" of the resonance is therefore

$$\frac{\delta\omega}{\omega} = \frac{\alpha v_g}{\omega} = \frac{1}{Q}$$

which turns out to be exactly the standing wave definition of Q^{-1}.

It is clear that the meaning of mode width breaks down when the mode spacing is smaller than the width. This corresponds to $2\alpha L$ being comparable with or greater than 1. In this case, the reflection treatment may be more useful.

Acknowledgement

The author would like to thank Dr. James Leiss of the National Bureau of Standards for several conversations.

Distr.: AADD External