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THE RECIPROCITY THEOREM IN NEUTRON SCATTERING

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PHYSICS

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ABSTRACT

A proof of the reciprocity theorem for the neutron scattering by spherical shells is given. The theorem itself is stated in the Introduction and at the end of Section 1. The theorem does not state that the transmission of the sphere is independent of the distance between source and detector, and the correction for finite distance is calculated. In Section 4, it is verified from the elementary scattering laws that the total transmission is unity for a sphere which scatters only elastically, and a certain paradox arising from the reciprocity theorem is resolved.

The Reciprocity Theorem in Neutron Scattering

H. A. Bethe

Introduction

A very useful method for determining neutron cross sections, especially for inelastic scattering and capture, is the spherical shell method. The substance whose cross section is to be measured, is arranged as a spherical shell around an isotropic detector at its center. A source is placed at some distance R_d from the center of the sphere; ideally, R_d is infinite. The neutrons are counted in the detector both with and without the shell around it; from the ratio (transmission), the desired cross section is calculated. If the capture cross section is to be measured, a neutron counter with a flat energy response is desired; for inelastic scattering measurements, threshold detectors such as a U-238 fission counter are used. The evaluation of such experiments will be discussed in a separate report (LA-1429).

A great aid for discussion and evaluation of spherical shell experiments is the Reciprocity Theorem which states that the transmission is unchanged if source and detector are interchanged. A more precise statement of the theorem will be given below. The reciprocity theorem is well known but there seems to be no proof readily available which is directly applicable to neutron problems. This report is to supply such a proof.

Let R_1 and R_2 be the inner and outer radius of the scattering shell, r the radius of an arbitrary point in the shell, σ the total cross section for all processes, elastic, inelastic and capture, measured in cm^{-1}, and $\sigma(\theta)\, d\omega$ the elastic[1] scattering cross section into solid angle $d\omega$. Consider first a unit source, emitting one neutron[2], at the center of the shell, and consider in particular those neutrons which escape from the sphere after two scatterings, of which the first occurs in volume element $d V_1$, and the second in $d V_2$. (See Fig. 1). Generalization to more or fewer scatterings will be obvious. After escaping from the sphere, the neutrons hit an isotropic detector of area A_d, thickness t_d, and mean free path λ_d, at distance R_d from the center of the sphere.

Assuming that the first scattering volume has an area $d A_1$, and a thickness $d t_1 = d V_1 / d A_1$, the number of source neutrons hitting it is

$$I_1 = \frac{d A_1}{4\pi r_1^2} e^{-\sigma x_1} \tag{1}$$

where r_1 is the distance of $d V_1$ from the center of the sphere and $x_1 = r_1 - R_1$ is the distance the neutrons have to go through material. The fraction of I_1 which is elastically scattered in $d V_1$ is

$$\sigma_{el}\, d t_1$$

and the fraction which is scattered through an angle θ into solid angle $d\omega_1$ is[3]

$$\sigma(\theta_1)\, d t_1\, d\omega_1$$

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1. This report will be written as if a threshold detector were used so that inelastic scattering acts as absorption, and only elastically scattered neutrons can be counted by the detector. The modifications applicable for a flat-response detector are obvious.

2. The time required for emission is irrelevant.

3. Note that for isotropic scattering, $\sigma(\theta)$ is defined to be $\sigma_{el}/4\pi$

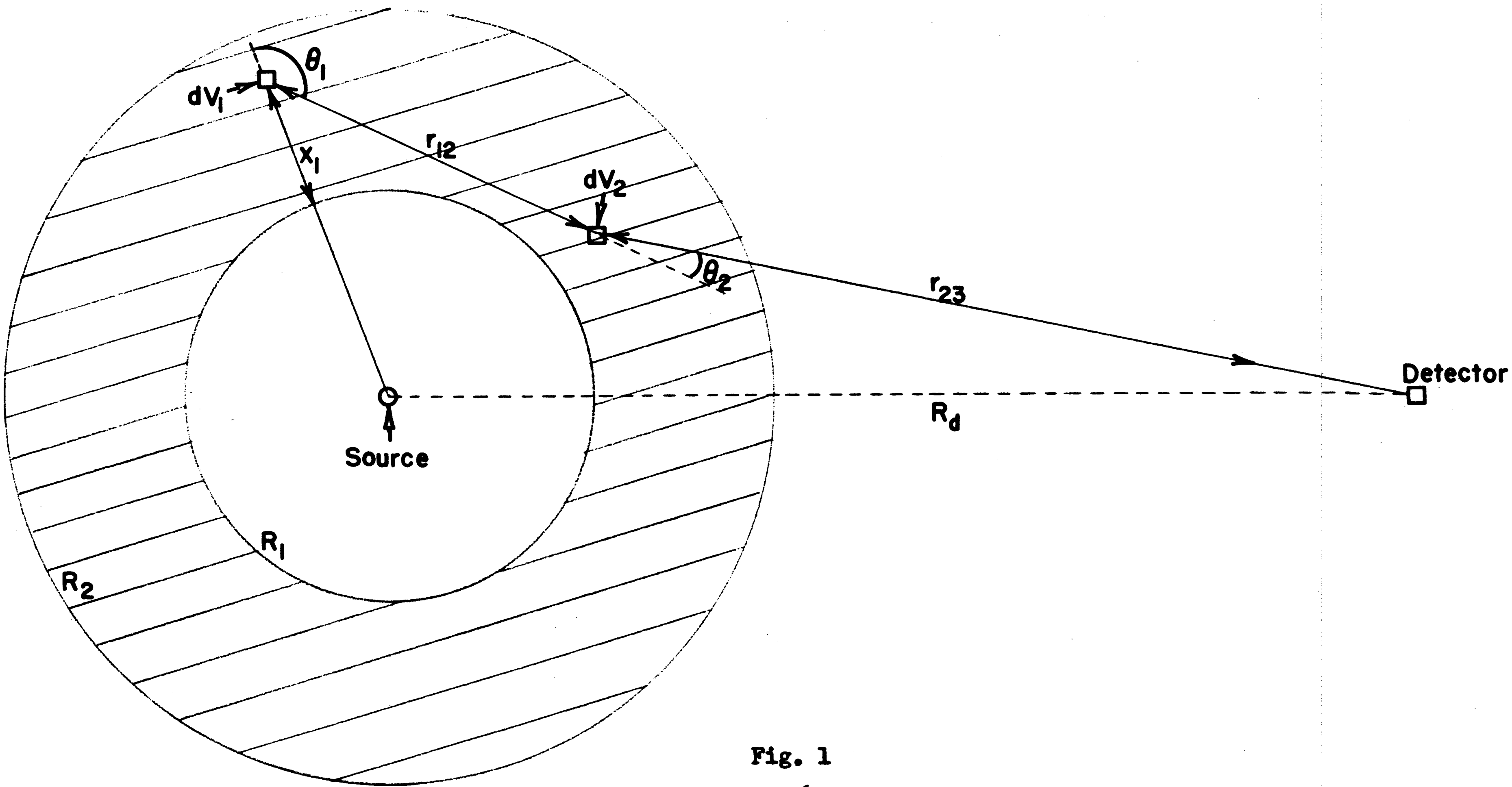

Fig. 1

Thus the number of neutrons scattered into this solid angle is

$$S_1 = \frac{d\ V_1}{4\pi r_1^{\ 2}}\ e^{-\sigma x_1}\ \sigma(\theta_1)\ d\,\omega_1 \tag{2}$$

where we have used $d\ A_1\ d\ t_1 = d\ V_1$.

The second scattering volume, of area $d\ A_2$ and thickness $d\ t_2$, subtends at $d\ V_1$ the solid angle

$$d\,\omega_1 = d\ A_2/r_{12}^{\ 2} \tag{3}$$

where r_{12} is the distance from $d\ V_1$ to $d\ V_2$. The number of neutrons hitting $d\ A_2$ is given by (2) with (3), multiplied by an attenuation factor

$$e^{-\sigma x_{12}} \tag{4}$$

where x_{12} is the distance traveled through material from $d\ V_1$ to $d\ V_2$. This is equal to r_{12} if the path is entirely through material but less than r_{12} if the path crosses the cavity inside the shell. The scattering probability in $d\ V_2$ is $\sigma_{el}\ d\ t_2$, and the absolute number of neutrons scattered through θ_2 into $d\,\omega_2$ is

$$S_2 = \frac{d\ V_1}{4\pi\ r_1^{\ 2}}\ e^{-\sigma x_1}\ \sigma(\theta_1)\ \frac{d\ V_2}{r_{12}^{\ 2}}\ e^{-\sigma x_{12}}\ \sigma(\theta_2)\ d\,\omega_2 \tag{5}$$

Now the detector subtends at $d\ V_2$ the solid angle

$$d\,\omega_2 = A_d/r_{23}^{\ 2} \tag{6}$$

where r_{23} is the distance from $d\ V_2$ to the detector. The detector will count the fraction t_d/λ_d of the neutrons incident upon it. Therefore the number of neutrons detected is

$$D_{12} = \frac{d\,V_1}{4\pi r_1^2} e^{-\sigma x_1} \sigma(\theta_1) \frac{d\,V_2}{r_{12}^2} e^{-\sigma x_{12}} \sigma(\theta_2) \frac{d\,V_d}{r_{23}^2 \lambda_d} e^{-\sigma x_{23}} \tag{7}$$

where x_{23} is the part of the path r_{23} which goes through material. If the shell were absent, the number of counts in the detector would be

$$D_o = \frac{d\,V_d}{4\pi R_d^2 \lambda_d} \tag{8}$$

The contribution of twice-scattered neutrons to the <u>transmission</u> of the shell is therefore

$$T_2 = \frac{\int D_{12}}{D_o} = \iint \frac{d\,V_1}{r_1^2} \sigma(\theta_1) \frac{d\,V_2}{r_{12}^2} \sigma(\theta_2) \frac{R_d^2}{r_{23}^2} e^{-\sigma(x_1 + x_{12} + x_{23})} \tag{9}$$,

the integrals over $d\,V_1$ and $d\,V_2$ going over the entire volume of the shell. This expression may easily be generalized to any other number of scatterings. The total transmission is the sum of the contributions from neutrons which have suffered 0, 1, 2, ... collisions.

Now consider the <u>inverse problem</u>. The source, of unit strength, is now outside the shell, the detector, of the same volume and λ_d as before, in the center of the shell. The neutrons follow the same path as before but in the reverse direction; that is, they go from the source to $d\,V_2$, are there scattered to $d\,V_1$, and then go to the center detector. The number of source neutrons hitting $d\,V_2$ is now

$$\frac{d\,A_2}{4\pi r_{23}^2} e^{-\sigma x_{23}}$$

The number of those scattered at $d\,V_2$ which hit $d\,V_1$ is

$$\frac{d\,V_2}{4\pi r_{23}^2}\, e^{-\sigma x_{23}}\, \sigma(\theta_2)\, \frac{d\,A_1}{r_{12}^2}\, e^{-\sigma x_{12}}$$

and the number detected by the center detector is

$$D_{21} = \frac{d\,V_2}{4\pi r_{23}^2}\, e^{-\sigma x_{23}}\, \sigma(\theta_2)\, \frac{d\,V_1}{r_{12}^2}\, e^{-\sigma x_{12}}\, \sigma(\theta_1)\, \frac{d\,V_d}{r_1^2 \lambda_d}\, e^{-\sigma x_1} \qquad (10).$$

This is obviously equal to (7). Similarly, the identity can be proved for any other number of scatterings. Thus the reciprocity theorem is proved:

The number of neutrons detected by an isotropic detector coming from a unit source at the center of a spherical shell of material and detected by an isotropic detector outside the shell is equal to the number detected if the positions of source and detector are interchanged.

The theorem holds for any number of scatterings in the shell, for any angular distribution $\sigma(\theta)$ of the scattering, and for any relation of elastic to inelastic scattering, capture, etc. It holds for any thickness of the shell and any ratio of outer to inner radius, including the case of a solid sphere.

2. Generalizations

From the derivation, and also from intuition, it is clear that the theorem still holds if the material, or its density, changes in any way as a function of the radius. The main change is that the attenuation exponentials no longer depend simply on the distance traveled through material, such as $e^{-\sigma x_{12}}$, but are replaced by such expressions as

$$\exp - \int \sigma(x)\, d\, x$$

where the integral is extended along the straight line between the two scattering volumes $d\, V_1$ and $d\, V_2$, and $\sigma(x)$ is the local value of the total cross section at the point x along that path. But still the same exponential occurs for both directions of the path. In addition, $\sigma(\Theta)$ is replaced by $\sigma(\Theta, r)$, i.e. it depends now on the radius r at which the scattering occurs.

The theorem also still holds if <u>both</u> detector and source are surrounded by spherical shells.

In fact, the reciprocity theorem <u>as such</u> holds for <u>any</u> arrangement of source and detector. However, its main usefulness is for spherical arrangements. With a spherical distribution of material surrounding the source, the number of neutrons emitted in any direction is directly given by the total emission which usually can be calculated more easily. In particular, for spherical arrangements of materials which scatter purely elastically, the number of neutrons emitted into any solid angle is the same as from the bare source itself (Sec. 4); it is this fact which makes spherical shells so suitable for the measurement of inelastic scattering or capture.

3. Finite and Infinite Distance of Detector

The transmission (9) contains the factor R_d^2/r_{23}^2 which is, in general, different from unity. However, if the detector is infinitely distant from the sphere, this factor reduces to unity. Therefore the case of infinite detector distance gives a simpler result than that of finite distance. Thus the reciprocity theorem does not mean that the "transmission" of the sphere, as defined in (9), is independent of the distance between source and detector.

This result can also be understood in a different way which will lead to a quantitative evaluation. Let T be the total transmission of the shell with the source in the center and the detector at infinite distance. Then, for unit source strength, T neutrons will cross any sphere around the source, of arbitrary radius R as long as $R > R_2$. Now let $n(\psi)\, d\omega$ be the flux at R of neutrons moving at an angle ψ with the radius, and within the solid angle $d\omega$. The flux is defined as the number crossing a unit area perpendicular to the direction of motion. Then the net flux through a unit area of the sphere is

$$T/4\pi R^2 = \int n(\psi) \cos\psi d\omega \tag{11}$$

On the other hand, an isotropic detector placed at R will detect

$$D = \int n(\psi)\, d\omega \quad V_d/\lambda_d \tag{12}$$

with the same notations as in Section 1. In other words, the efficiency of the detector is proportional to[4]

$$\langle 1/\cos\psi \rangle \tag{13}$$

the average being taken with the weight factor $n(\psi) \cos\psi$.

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4. That this is so, can be seen most directly for a thin foil detector placed parallel to the surface of the sphere. In this case, the thickness of the detector traversed is $t_d/\cos\psi$.

(13) shows that a detector at finite distance will always record a larger apparent transmission than one at infinite distance. The enhancement factor (13) can be evaluated, using Fig. 2. The trigonometric sine theorem shows that

$$\sin \psi = \sin \theta \; r/R_d \tag{14}$$

where r is the radius at which the last scattering in the shell takes place, and θ the angle between the radius and the neutron direction after this scattering. Therefore

$$\left\langle \frac{1}{\cos \psi} \right\rangle = \left\langle \left(1 - \frac{r^2}{R_d^{\,2}} \sin^2 \theta \right)^{-1/2} \right\rangle \tag{15}$$

For large enough detector distance, this may be written

$$1 + \frac{1}{2} R_d^{-2} \left\langle r^2 \sin^2 \theta \right\rangle \tag{16}$$

so that the correction goes as the inverse square of the detector distance. In calculating the average in (16), all neutrons coming directly from the source should be taken into account with r = 0.

In the case of thin shells, the average of r^2 for the neutrons which have been scattered can be estimated very easily; it is about $R_1 R_2$. Further, if all angles θ were equally probable, $\left\langle \sin^2 \theta \right\rangle = \frac{2}{3}$. Actually, the neutrons having suffered only one collision are distributed as the differential cross section $\sigma(\theta)$, i.e. a large fraction of them is at small θ, giving a small contribution to (16). Those scattered more than once are probably nearly isotropic just after scattering, but those moving in nearly tangential directions, θ near 90°, cannot escape easily because they have to traverse a large thickness of material. This reduces the average of $\sin^2 \theta$ below 2/3. Explicit calculations of the corrections for finite distance are given in LA-1429.

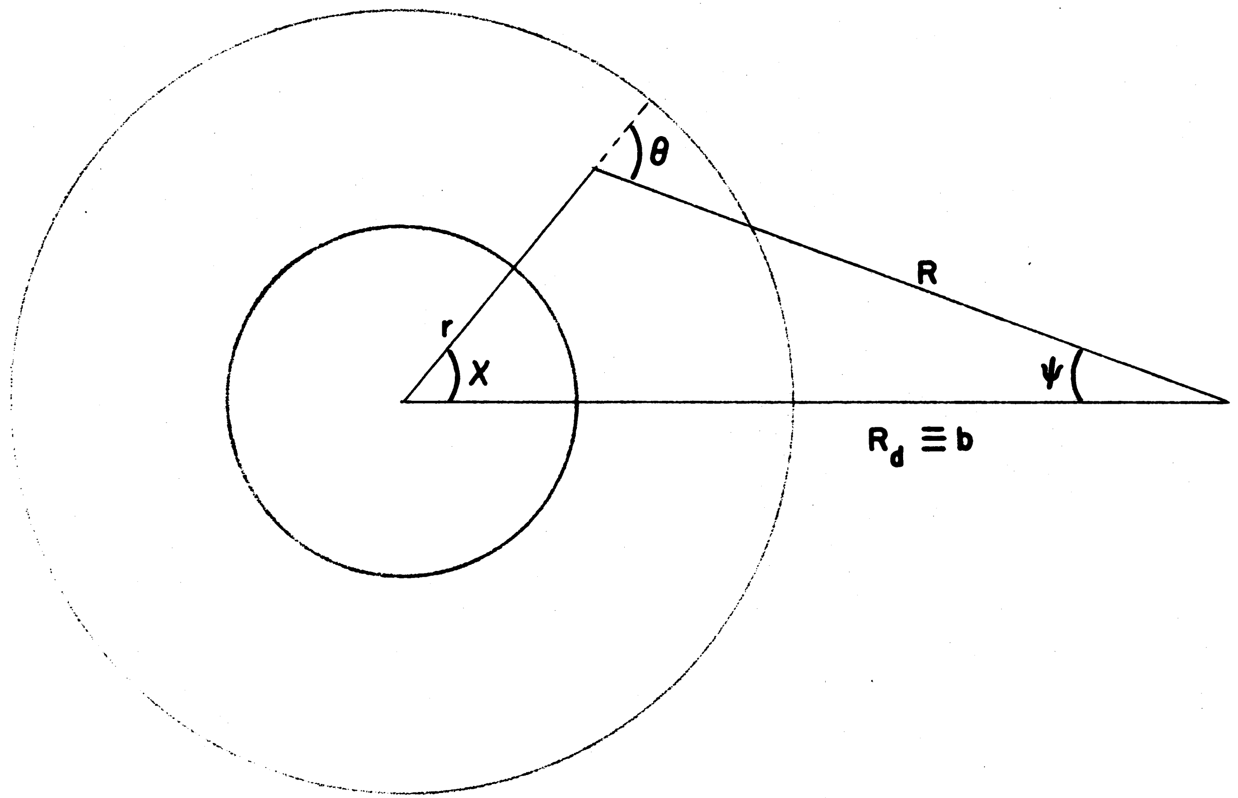

Fig. 2

It remains to obtain the result (15) directly from the transmission formula (9). For this calculation, we shall set $R_d = b$ and the distance from the last scattering to the detector (previously r_{23}) equal to R. We shall further introduce the number of neutrons $n(r,\theta) d\,V\, d\,\omega$ which (a) have their last scattering (before escaping from the spherical shell) in the volume element d V at r, and (b) whose direction of motion immediately after this last scattering is in the solid angle $d\,\omega$ at an angle θ with the radius. ($n(r,\theta)$ is meant to include neutrons having suffered <u>any</u> number of scatterings. Neutrons escaping without scattering are counted at $r = \theta = 0$). Then the total number escaping from the sphere is

$$T = \int d\,V \int 2\pi \sin\theta \, d\,\theta \; n(r,\theta) \tag{17}$$

and the number which will hit a detector of area A_d at a very large distance b is

$$D = T\, A_d/4\pi b^2 \tag{18}.$$

On the other hand, the number hitting a detector of the same area at a finite distance b is, by definition of $n(r,\theta)$:

$$D' = \int d\,V\, n(r,\theta)\; A_d/R^2 \tag{19}$$

From this we can calculate an effective transmission, using the infinite-distance formula (18):

$$T' = 4\pi b^2 \int d\,V\, n(r,\theta)/R^2 \tag{20}.$$

This formula is obviously similar to (9) in that it contains the characteristic factor b^2/R^2 (there called ${R_d}^2/{r_{23}}^2$).

Referring to Fig. 2, the volume element may be written

$$d\,V = 2\pi \sin\chi\, d\chi\, r^2 dr \tag{21}$$

We have, also from Fig. 2,

$$R^2 = r^2 + b^2 - 2\,b\,r\cos\chi$$

and therefore, at fixed r (of course, b is always fixed):

$$R\,dR = b\,r\sin\chi\,d\chi$$

Hence

$$T' = 4\pi b \cdot 2\pi \int r\,dr \int n(r,\theta)\,dR/R \tag{22}$$

where θ is given from Fig. 2 by

$$\cos\theta = \frac{b^2 - r^2 - R^2}{2\,R\,r} \tag{23}$$

In order to obtain a formula similar to (17), we wish to introduce $\sin\theta\,d\theta$ in (22) instead of d R. Differentiating (23) at constant r gives

$$2\,r\sin\theta\,d\theta = \left(1 + \frac{b^2 - r^2}{R^2}\right) d\,R$$

from which

$$\frac{d\,R}{R} = \frac{2\,r\,R}{b^2 - r^2 + R^2}\sin\theta\,d\theta = \frac{r\sin\theta\,d\theta}{r\cos\theta + R} \tag{24}$$

where (23) has been used again. Using it once more, we may write

$$R^2 + 2\,R\,r\cos\theta = b^2 - r^2$$

$$(R + r\cos\theta)^2 = b^2 - r^2\sin^2\theta$$

Inserting in (22) and setting $4\pi r^2 dr = d\,V$, we get

$$T' = \int d\,V \int 2\pi \sin\theta\,d\theta\; n(r,\theta)\,\frac{b}{\sqrt{b^2 - r^2\sin^2\theta}} \tag{25}$$

This is the same as (17), with each scattered neutron multiplied by the characteristic factor

$$\left(1 - r^2\sin^2\theta/b^2\right)^{-1/2}$$

Thus we have re-derived formula (15).

Because of the reciprocity theorem, the formulae of this section are also applicable to the case where the source is outside and the detector inside the sphere. The calculations are clearly much simpler if they are done with the assumption of the source inside.

4. Total Transmission

Consider a substance which has only elastic scattering. Then all neutrons produced by the source will escape from the sphere, the transmission calculated for infinite detector distance is unity. We shall now prove this from the formula for transmission.

If t is the shell thickness, then $e^{-\sigma t} \frac{d\omega}{4\pi}$ is the number of neutrons escaping without any collision into solid angle $d\omega$. Those escaping after one collision are

$$\int \frac{dV_1}{4\pi r_1^2} e^{-\sigma(x_1 + x_2)} \sigma(\Theta_1) \, d\omega \tag{26}$$

where x_2 is the distance from the first scattering to the surface of the sphere. Now let us, for the moment, neglect the absorption after the first scattering, $e^{-\sigma x_2}$. Let us introduce a polar coordinate system with sphere center as origin and the final direction of the neutron as axis. Then $dV_1 = r_1^2 \, dr_1 \sin\Theta_1 \, d\Theta_1 \, d\psi$ because Θ_1 is the angle between the radius vector r_1 and the polar axis. Then (26) becomes

$$\frac{d\omega}{4\pi} \int dr_1 \, e^{-\sigma x_1} \int \sigma(\Theta_1) \sin\Theta_1 \, d\Theta_1 \, d\psi \tag{27}$$

But the last integral is exactly the total elastic cross section which by assumption is equal to the total cross section σ. Furthermore, $dr_1 = dx_1$ and $\sigma \int dr_1 \, e^{-\sigma x_1} = 1 - e^{-\sigma t}$. Thus, if the attenuation after the collision is neglected, the neutrons having suffered one collision, plus those escaping without collision, equal exactly the total number from the source, $d\,\omega/4\pi$.

Now consider the neutrons lost by attenuation after a first collision at $d\,V_1$, viz.

$$\frac{d\,V_1}{4\pi\, r_1^2}\, e^{-\sigma x_1} \left(1 - e^{-\sigma x_2}\right) \sigma(\theta_1)\, d\,\omega \tag{28}$$

and compare with it the number of neutrons suffering two collisions, given by (5). In Eq. (5) the attenuation of the outgoing neutrons, $e^{-\sigma x_{23}}$ (cf. Eq. 6) has already been neglected. Then, just as before, and using the spherical symmetry of the whole problem, we may put $d\,V_2 = r_{12}^2\, dr_{12} \sin\theta_2 \, d\,\theta_2 \, d\,\phi_2$. Integrating over angles, the differential cross section $\sigma(\theta_2)$ turns into the integral cross section σ, and integrating over $dr_{12} = dx_{12}$, we get $\sigma \int dr_{12}\, e^{-\sigma x_{12}} = 1 - e^{-\sigma x_2}$. Thus the second collisions just compensate the loss of first-collided neutrons by attenuation.

In a similar way, the proof can be extended to collisions of any order. Thus it is shown that the total number of neutrons emerging from the sphere is equal to the number emitted from the source, as it must be if there are only elastic collisions.

There is, of course, no simple way of calculating total transmission in the case when there is inelastic scattering.

A Paradox

We have just proved that all the neutrons emitted by a source will get out through a spherical shell of any material which has only elastic scattering, no matter how thick the shell. From the reciprocity theorem it follows then that a detector inside a large sphere of elastically scattering material will detect exactly as many neutrons of an incident plane-parallel beam as if the sphere were absent.

This sounds at first sight paradoxical: Surely, very many of the neutrons of the beam will be reflected by the outside of the sphere, and very few only will get to the center especially if the thickness of the sphere is many mean free paths. The solution of this paradox is that those neutrons which do get to the center will cross the internal cavity of the sphere many times which enhances the probability of being detected. The reciprocity theorem states then that the increase due to multiple traversals exactly compensates the decrease due to reflection of neutrons from the outside of the sphere.