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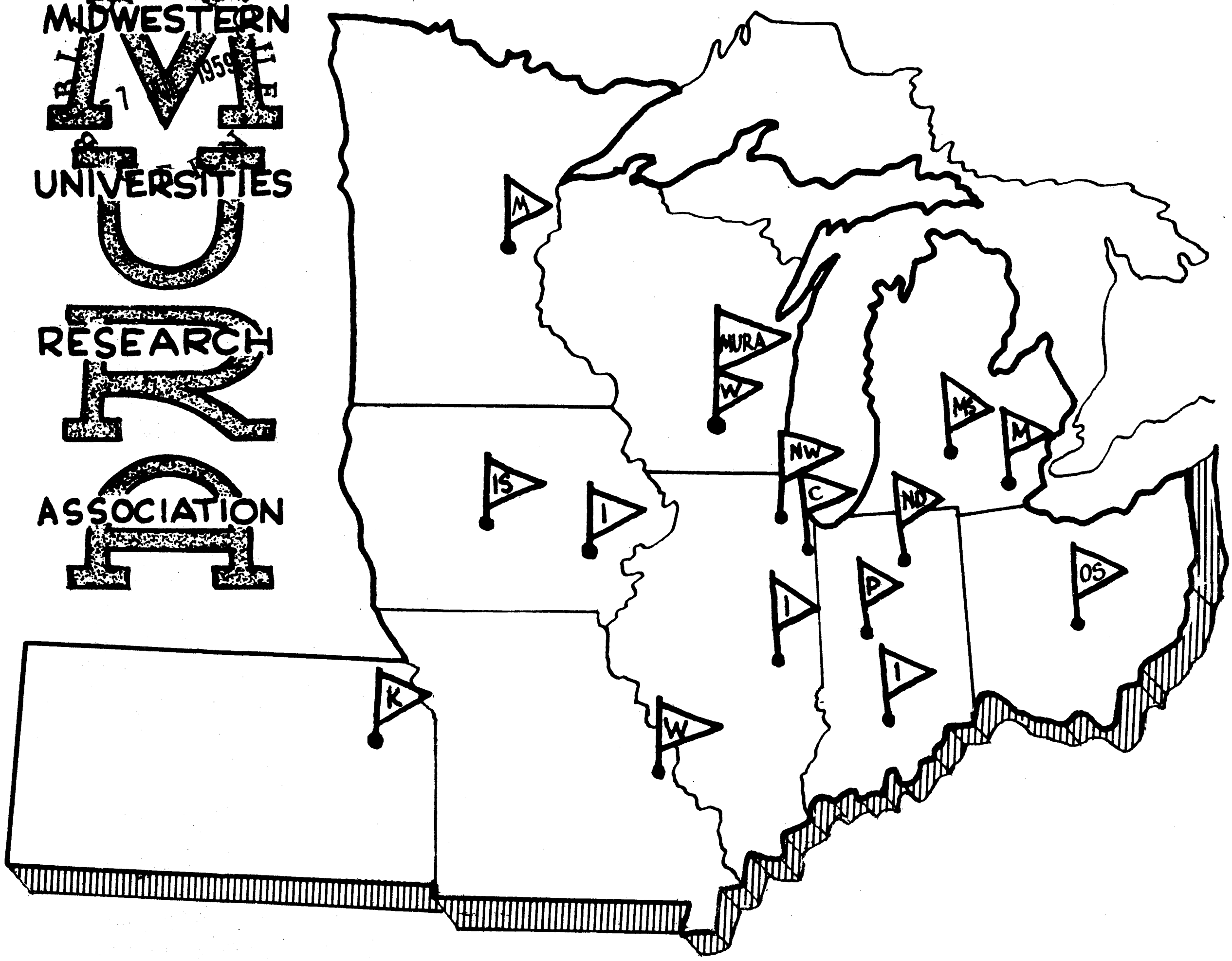

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A SUMMARY OF THE LINEAR ORBIT PROPERTIES OF AN ACCELERATOR HAVING A GENERAL MAGNETIC FIELD

George Parzen**

April 21, 1959

ABSTRACT

This report lists results for the linear orbit properties of an accelerator with a general magnetic field. The results found in MURA-397 and MURA-451 are collected here together with comments on the relative importance of the different terms. No derivations are given in this report.

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This report lists results for the linear orbit properties of an accelerator with a general magnetic field. The results found in a previous report, MURA-39' have been extended in the report, MURA-451, so that they now hold for machines with small N, and for tunes ν_r or ν_z which are close to $\frac{1}{2}$ N. No derivations will be given here. Derivations and somewhat more general results are given in the reports, MURA-397 and MURA-451.

We write the median plane field as

$$H_z = -\sum_{n=-\infty}^{\infty} G_n(r)\, e^{inN\theta}, \tag{1a}$$

$$G_n(r) = H_n(r)\, e^{-i\beta_n(r)}, \tag{1b}$$

where H_n and β_n are real, and $2\pi/N$ is the period of the machine.

Equilibrium Orbit

The equilibrium orbit corresponding to a given particle momentum p is given by r(θ)

$$r(\theta) = R\,(1 + x(\theta)). \tag{2}$$

The parameter, R, is very nearly the average radius of the equilibrium orbit and depends on the momentum according to the relationship

$$p = \frac{eR}{2c}\left\{ H_0 + \sqrt{H_0^2 + 8\sum_{n\geq 1}\frac{1}{n^2N^2}\left(RH_n' H_n + \frac{3}{2}H_n^2\right)}\right\} \tag{3}$$

where H_0 and H_n are evaluated at r = R. It is usually sufficient to keep just the n = 1 term in the sum. In what follows, the magnetic field is always to be evaluated at r = R.

$\chi(\theta)$, the oscillation of the equilibrium orbit, is given by

$$\chi(\theta) = \sum_{n=-\infty}^{\infty} \chi_n e^{inN\theta}, \tag{4a}$$

where

$$\chi_0 = -\frac{2}{N^4}\left(\frac{eR}{pc}\right)^2 (RH_1'H_1 + 2H_1^2), \tag{4b}$$

$$\chi_n = \frac{1}{n^2N^2 - E_s}\left(\frac{eR}{pc}\right) G_n \left[1 + \frac{3}{2}\left(\frac{eR}{pc}\right)^2 \frac{H_1^2}{N^2}\right], \tag{4c}$$

where

$$E_s = \left(\frac{eR}{pc}\right)(RH_0' + 2H_0) - 1. \tag{4d}$$

One may usually put $\chi_0 = 0$, as it is small, and one may usually neglect E_s in Eq. (4c) for χ_n.

Because χ_0 is not zero, the average radius of the equilibrium orbit is more accurately given by

$$R_{av} = R(1 + \chi_0) \tag{5}$$

However, χ_0 is quite small and may usually be neglected. We will refer to R as the average radius in the following.

<u>Radial Tune</u>

The radial tune ν_r depends on the particle momentum, p, or on the average radius, R. If ν_r is not very close to $\frac{1}{2}$ N, then ν_r depends on R according to

$$\nu_r^2 = \varepsilon_0 + 2 \sum_{m \geq 1} \frac{|g_m|^2}{m^2 N^2 - 4\varepsilon_0} \tag{6a}$$

$$+ 9\left(\frac{eR}{pc}\right)^4 \frac{H_1^2}{N^4}\left[\left(RH_1' + \frac{3}{2}H_1\right)^2 - R^2 \beta_1'^2 H_1^2\right],$$

where

$$\varepsilon_0 = \left[1 + 3\left(\frac{eR}{pc}\right)^2 \frac{H_1^2}{N^2}\right] \times \left[\left(\frac{eR}{pc}\right)\left(RH_0' + 2H_0\right) - 1\right. \tag{6b}$$

$$+ \frac{eR}{pc}\, \chi_0 \left(R^2 H_0'' + 4RH_0' + 2H_0\right)$$

$$+ 2\,\frac{eR}{pc} \sum_{m \geq 1} |\chi_m| \left(R^2 H_m'' + 4RH_m'\right.$$

$$\left.\left. + \frac{7}{4} H_m - R^2 \beta_m'^2 H_m^2\right)\right],$$

$$|g_m|^2 = \left(\frac{eR}{pc}\right)^2 \left[\left(RH_m' + \frac{3}{2}H_m\right)^2 + R^2 \beta_m'^2 H_m^2\right]. \tag{6c}$$

In the above result for the tune ν_r, it is usually sufficiently accurate to keep just the first term in the sum over m. The term involving χ_0 may usually be neglected. The third term in the equation for ν_r^2 may be usually neglected except for machines with large flutter and small N where it may be a 10% correction. All quantities are to be evaluated at r = R.

A more general result for ν_r which is also valid when ν_r is near $\frac{N}{2}$ is given by

$$\nu_r^2 = \frac{N^2}{4} - \sqrt{(\mathcal{E}_f - \mathcal{E}_0)^2 - |g_1|^2}\,, \tag{7a}$$

where

$$\mathcal{E}_f = \frac{N^2}{4} - 2\sum_{m\geq 2} \frac{|g_m|^2}{m^2N^2 - 4\mathcal{E}_0} \tag{7b}$$

$$- 9\left(\frac{eR}{pc}\right)^4 \frac{H_1^2}{N^4}\left[(RH_1' + \tfrac{3}{2}H_1)^2 - R^2\beta_1'^2 H_1^2\right]$$

This last result should be used if $|\mathcal{E}_f - \mathcal{E}_0| \simeq |g_1|$.

If $|g_1| \ll |\mathcal{E}_f - \mathcal{E}_0|$, then the radical may be expanded giving the previous result. In the above expression for $\mathcal{E}_f$, it is usually sufficiently accurate to just put $\mathcal{E}_f = \frac{1}{4}N^2$.

Vertical Tune

The vertical tune ν_z depends on the particle momentum, p, or on the average radius R. ν_z depends on R according to

$$\nu_z^2 = \mathcal{E}_0' + 2\sum_{m\geq 1} \frac{|f_m|^2}{m^2N^2 - 4\mathcal{E}_0'} \tag{8a}$$

$$+ 3\left(\frac{eR}{pc}\right)^4 \frac{H_1^2}{N^4}\left[(RH_1' + \tfrac{1}{2}H_1)^2 - R^2\beta_1'^2 H_1^2\right]$$

where

$$\begin{aligned}\mathcal{E}_0' = \left[1 + \left(\frac{eR}{pc}\right)^2 \frac{H_1^2}{N^2}\right] \times \Bigg[& 2\frac{eR}{pc}\sum_{m\geq 1} m^2N^2|x_m|H_m - \frac{eR}{pc}RH_0' \\ & - \frac{eR}{pc}x_0(R^2H_0'' + 2RH_0') \\ & - 2\frac{eR}{pc}\sum_{m\geq 1}|x_m|\left(R^2H_m'' + 2RH_m' + \tfrac{1}{4}H_m - R^2\beta_m'^2 H_m\right)\Bigg]\end{aligned} \tag{8b}$$

$$|f_m|^2 = \left(\frac{eR}{pc}\right)^2 \left[\left(R H_m' + \frac{1}{2} H_m\right)^2 + R^2 \beta_m'^2 H_m^2 \right] . \tag{8c}$$

A more general result for ν_z which is also valid when ν_z is near $\frac{N}{2}$ is given by

$$\nu_z^2 = \frac{N^2}{4} - \sqrt{(\varepsilon_f' - \varepsilon_0')^2 - |f_1|^2} , \tag{9a}$$

where

$$\varepsilon_f' = \frac{N^2}{4} - 2 \sum_{m \geq 2} \frac{|f_m|^2}{m^2 N^2 - 4 \varepsilon_0'} \tag{9b}$$

$$- 3 \left(\frac{eR}{pc}\right)^4 \frac{H_1^2}{N^4} \left[\left(R H_1' + \frac{1}{2} H_1\right)^2 - R^2 H_1^2 \beta_1'^2 \right] .$$

The remarks made about the relative importance of the terms in the case of ν_r hold equally well for ν_z . The radical in ν_z^2 may be usually expanded giving the previous result for ν_z^2 . In the above equation the H_n and β_n are to be evaluated at r = R.

Frequency of Revolution

The going-around frequency of revolution, $\omega / 2\pi$, depends on the particle momentum or the velocity v according to

$$\omega = \frac{v}{R} \left\{ 1 - \left(\frac{eR}{pc}\right)^2 \frac{H_1^2}{N^2} + 2 \left(\frac{eR}{pc}\right)^2 \frac{1}{N^4} \left(R H_1' H_1 + 2 H_1^2\right) \right\} . \tag{10}$$

It is often good enough to take $\omega \simeq v / R$.

Floquet Solutions

The linear motion is given by a linear combination of Floquet solutions. The radial motion is given by

$$u = a\, u_{\nu_r}(\theta) + c.c. \tag{11}$$

where

$$u_{\nu_r}(\theta) = e^{i\nu_r\theta}\left\{1 - \sum_{n=-\infty}^{\infty} \frac{g_n}{(nN+\nu_r)^2 - E_0} e^{inN\theta} - \frac{\bar{g}_2}{4N^2} e^{i2N\theta}\right\}, \tag{12a}$$

where

$$E_0 = \mathcal{E}_0 - \frac{9}{2}\left(\frac{eR}{pc}\right)^4 \frac{H_1^4}{N^2} \tag{12b}$$

$$\bar{g}_2 = -3\left(\frac{eR}{pc}\right)^2 G_1^2 \tag{12c}$$

The $\bar{g}_2$ term and the difference of E_0 from $\mathcal{E}_0$ are only important for small N machines with larger flutter.

One can usually replace the denominator in the $e^{inN\theta}$ term by $N^2 + 2\nu_r N$, unless ν_r is close to 1/2 N.

The vertical linear motion is given by the linear combination of Floquet functions

$$y = b\, y_{\nu_z}(\theta) + c.c. \tag{13}$$

where

$$y_{\nu_z}(\theta) = e^{i\nu_z\theta}\left\{1 - \sum_{n=-\infty}^{\infty} \frac{f_n}{(nN+\nu_z)^2 - E_0'} e^{inN\theta} - \frac{\bar{f}_2}{4N^2} e^{i2N\theta}\right\}, \tag{14a}$$

$$E_0' = \mathcal{E}_0' - \frac{1}{2}\left(\frac{eR}{pc}\right)^4 \frac{H_1^4}{N^2}, \tag{14b}$$

$$\bar{f}_2 = -\left(\frac{eR}{pc}\right)^2 G_1^2. \tag{14c}$$

Fixed-Frequency Machines

In fixed-frequency machines, one has the added condition that the frequency of revolution, $\omega/2\pi$, is constant. The result given above for ω, Eq. (10), now becomes an added relation between the velocity v and the average radius R. We have $v = v(R)$, where

$$v(R) = \omega R\left\{1 + \left(\frac{e}{m\omega c}\right)^2 (1-R^2)\frac{H_1^2}{N^2} - 2\left(\frac{e}{m\omega c}\right)^2 \frac{(1-R^2)(RH_1H_1' + 2H_1^2)}{N^4}\right\}.$$

The average magnetic field $\bar{H}_0(R)$ necessary to give a constant revolution frequency is given by

$$H_0(R) = \frac{m\omega c}{e}\left\{\frac{v}{\omega R}\frac{1}{\left[1-\left(\frac{v}{c}\right)^2\right]^{\frac{1}{2}}} - \frac{2\omega R}{v}\left[1-\left(\frac{v}{c}\right)^2\right]^{\frac{1}{2}} \times \sum_{m \geq 1} \frac{1}{m^2N^2}\left(RH_m' H_m + \frac{3}{2}H_m^2\right)\right\}, \tag{16}$$

where $v = v(R)$ is given by Eq. (15). It is usually sufficiently accurate to keep just the first term in the sum over m.