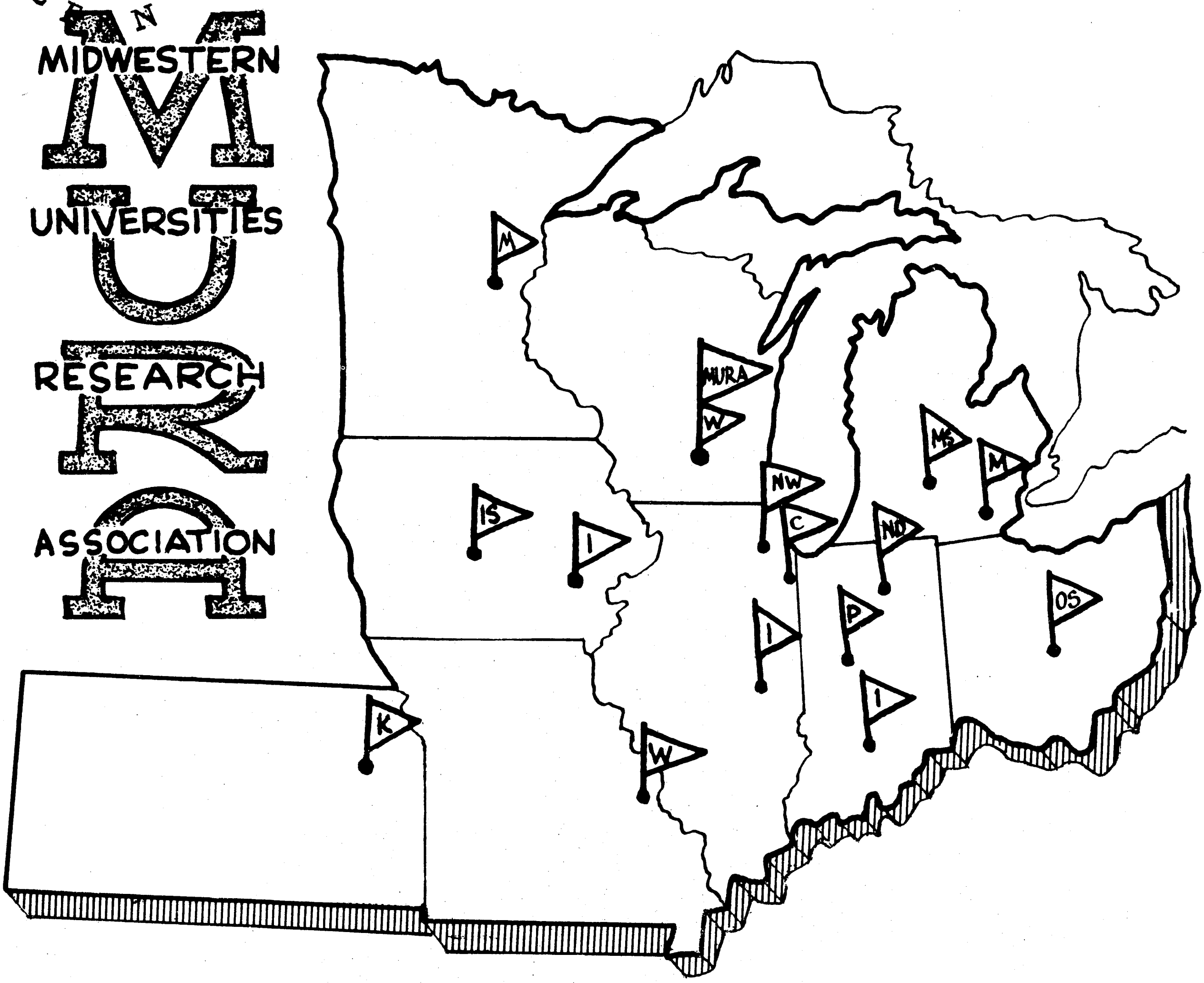

THEORY OF ACCELERATORS WITH A GENERAL MAGNETIC FIELD. II

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REPORT NUMBER 451

Printed in USA. Price $0.75. Available from the

Office of Technical Services
U. S. Department of Commerce
Washington 25, D. C.

MURA-451
Distribution List UC 28

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

2203 University Avenue, Madison, Wisconsin

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George Parzen+

April 6, 1959

ABSTRACT

This report extends the results found in the report, MURA-397, on the theory of an accelerator having a general magnetic field. The results have been extended so that they are now valid for machines with small N, N being the number of sectors in the machine, and for tunes, ν_r and ν_z, which are close to $\frac{1}{2}$ N.

*AEC Research and Development Report. Research supported by the Atomic Energy Commission, Contract No. AEC AT(11-1) 384.

+On leave from the University of Notre Dame.

I. INTRODUCTION

The results found in the report, MURA-397, for the tune ν_r, ν_z, of an accelerator having a general magnetic field begin to break down for machines with small N and for tunes ν_r or ν_z which are close to 1/2 N. This breakdown is due to the neglect of several small terms in the linear equations of motion. Also the method used in MURA-397 for finding the tune from the linear equation of motion is not valid when the tune ν_r or ν_z gets close to N/2. In this report the results for the tune will be expressed so that they are valid when the tune is near N/2, and the neglected small terms are kept to make the results valid for machines with small N.

II. SUMMARY OF RESULTS

Some of the more important results obtained will be listed here. Derivations and somewhat more general results are given in Sections III and IV

The equilibrium orbit corresponding to a given particle momentum is given by

$$r(\theta) = R\left(1 + x(\theta)\right) \qquad (2.1)$$

The parameter R is very nearly the average radius of the equilibrium orbit and depends on the momentum according to the relation

$$p = \frac{eR}{2c}\left\{H_0 + \left[H_0^2 + 8\sum_{n\geq 1}\frac{1}{n^2N^2}\left(RH_n'H_n + \frac{3}{2}H_n^2\right)\right]\right\}, \qquad (2.2)$$

where the H_n are evaluated at $r = R$

$x(\theta)$ is given by

$$x(\theta) = \sum_{n=-\infty}^{\infty} x_n e^{inN\theta} \qquad (2.3a$$

where

$$\chi_0 = -\frac{2}{N^4}\left(\frac{eR}{pc}\right)^2\left(RH_1'H_1 + 2H_1^2\right), \qquad (2.3b)$$

$$\chi_n = \frac{1}{n^2N^2 - E_s}\ \frac{eR}{pc}\ G_n\left[1 + \frac{3}{2}\ \frac{H_1^2}{N^2}\right] \qquad (2.3c)$$

where

$$E_s = \left(\frac{eR}{pc}\right)\left(RH_0' + 2H_0\right) - 1 \qquad (2.3d)$$

The radial tune as a function of the average radius, R, is given by

$$\nu_r^2 = \frac{N^2}{4} - \sqrt{\left(E_f - E_0\right)^2 - |g_1|^2}\ , \qquad (2.4)$$

$$\begin{aligned} E_0 = \left(1 + 3\frac{H_1^2}{N^2}\right)\Bigg[& \frac{eR}{pc}\left(RH_0' + 2H_0\right) - 1 \\ & + \left(\frac{eR}{pc}\right)\chi_0\left(R^2H_0'' + 4RH_0' + 2H_0\right) \\ & + 2\left(\frac{eR}{pc}\right)\sum_{m\geq 1}|\chi_m|\left(R^2H_m'' + 4RH_m' \right. \\ & \left. + \frac{7}{4}H_m - R^2\beta_m'^2 H_m\right)\Bigg] \\ & - \frac{9}{2}\left(\frac{eR}{pc}\right)^4\frac{H_1^4}{N^2} \end{aligned} \qquad (2.4a)$$

$$|g_n|^2 = \left(\frac{eR}{pc}\right)^2 \left[\left(R H_n' + \frac{3}{2} H_n \right)^2 + R^2 \beta_n'^2 H_n^2 \right], \tag{2.4b}$$

$$E_f = \frac{N^2}{4} - 2 \sum_{m \geq 2}^{\infty} \frac{|g_m|^2}{m^2 N^2 - 4E_0}$$

$$- \frac{9}{2} \left(\frac{eR}{pc}\right)^4 \frac{H_1^4}{N^2} \tag{2.4c}$$

$$- 9 \left(\frac{eR}{pc}\right)^4 \frac{H_1^2}{N^4} \left[\left(R H_1' + \frac{3}{2} H_1 \right)^2 - R^2 H_1^2 \beta_1'^2 \right].$$

The vertical tune is given by

$$\nu_z^2 = \frac{N^2}{4} - \sqrt{(E_f' - E_0')^2 - |f_1|^2} \tag{2.5}$$

$$E_0' = \left(1+\frac{H_1^2}{N^2}\right)\left[2\left(\frac{eR}{pc}\right)\sum_{m\geq 1} m^2 N^2 |x_m| H_m - \frac{eR}{pc} R H_0' \right.$$

$$-\left(\frac{eR}{pc}\right) x_0 \left(R^2 H_0'' + 2 R H_0'\right)$$

$$\left. - 2\left(\frac{eR}{pc}\right)\sum_{m\geq 1} |x_m| \left(R^2 H_m'' + 2 R H_m' + \frac{1}{4} H_m\right)\right] \tag{2.5a}$$

$$-\frac{1}{2}\left(\frac{eR}{pc}\right)^4 \frac{H_1^4}{N^2},$$

$$|f_n|^2 = \left(\frac{eR}{pc}\right)^2 \left[\left(R H_n' + \frac{1}{2} H_n\right)^2 + R^2 H_n^2 \beta_n'^2\right], \tag{2.5b}$$

$$E_f' = \frac{N^2}{4} - 2\sum_{m\geq 2} \frac{|f_m|^2}{m^2 N^2 - E_0'}$$

$$-\frac{1}{2}\left(\frac{eR}{pc}\right)^4 \frac{H_1^4}{N^2} \tag{2.5c}$$

$$-3\left(\frac{eR}{pc}\right)^4 \frac{H_1^2}{N^4}\left[\left(R H_1' + \frac{1}{2} H_1\right)^2 - R^2 \beta_1'^2 H_1^2\right].$$

III. DERIVATION OF TUNE RESULTS

We will start by treating the z- tune since this turns out to be the easiest case. The z- tune is also the quantity which is most likely to be affected by the neglected terms.

The equation for the z- motion is (see MURA-397)

$$\frac{d}{d\theta} \frac{y'}{\sqrt{(1+x)^2 + x'^2 + y'^2}} = F_z , \tag{3.1a}$$

$$F_z = \frac{e}{p_c} \left(r' H_\theta - r H_r \right). \tag{3.1b}$$

In MURA-397, we assumed that $x' \ll 1$, $x \ll 1$, and we wrote Eq. (3.1a) as

$$y'' = (1+x) F_z \tag{3.2}$$

We will now keep the terms which were thrown away and see how much they contribute. We write

$$\frac{d}{d\theta}(a y') = \frac{y''}{1+x} + \left[\frac{d}{d\theta}(a y') - \frac{y''}{1+x} \right] , \tag{3.3a}$$

where

$$a = \left\{ (1+x)^2 + x'^2 + y'^2 \right\}^{-\frac{1}{2}} , \tag{3.3b}$$

We rewrite Eq. (3.1) as

$$y'' + \left[(1+x) \frac{d}{d\theta}(a y') - y'' \right] = (1+x) F_z , \tag{3.4}$$

which, when compared with Eq. (3.2), has the correction term

$$C_y = (1+x)\frac{d}{d\theta}(a y') - y'' , \qquad (3.5)$$

$$C_y = y''\left[(1+x)a - 1\right] + a'y'(1+x).$$

To further evaluate C_y, we expand a in powers of x, x', and y'.

$$a = \left\{(1+x)^2 + x'^2 + y'^2\right\}^{\frac{1}{2}}, \qquad (3.6a)$$

$$a = (1+x)^{-1}\left\{1 - \frac{1}{2}\frac{1}{1+x}(x'^2 + y'^2)\ldots\right\}, \qquad (3.6b)$$

$$a = (1-x)\left\{1 - \frac{1}{2}x'^2 - \frac{1}{2}y'^2 + \frac{xy'^2}{2} + \frac{xx'^2}{2} + \ldots\right\}, \qquad (3.6c)$$

$$a = 1 - x - \frac{1}{2}x'^2 - \frac{1}{2}y'^2 + xy'^2 + xx'^2 \ldots\ldots \qquad (3.6d)$$

It is not easy to see how many terms one should keep in the expansion of a until we come to evaluating the tune ν_z. One may keep in mind however that in the expansion, $x << x'$ since $x \sim (1/N)\,x'$. Also that we have $x'' \simeq 1$ as $x'' \simeq N^2 x$.

From Eq. (3.6) we find for a'

$$a' = -x' - x'x'' - y'y'' \ldots \qquad (3.7)$$

where we have kept only up to terms of $O(x')$. Thus we find

$$(1+x)a - 1 = 1 - x - \frac{1}{2}x'^2 - \frac{1}{2}y'^2 + xx'^2 + xy'^2$$
$$+ x - x^2 - \frac{1}{2}x'^2 x - \frac{1}{2}y'^2 x \qquad (3.8a)$$

$$(1+x)q-1 = -\tfrac{1}{2}x'^2 - \tfrac{1}{2}y'^2 + \tfrac{1}{2}xx'^2 + \tfrac{1}{2}xy'^2, \tag{3.8b}$$

We get for the correction term

$$\begin{aligned} C_y &= \left(-\tfrac{1}{2}x'^2 - \tfrac{1}{2}y'^2 + \tfrac{1}{2}xx'^2 + \tfrac{1}{2}xy'^2\right)y'' \\ &\quad + \left(-x' - x'x'' - y'y''\right)(1+x). \end{aligned} \tag{3.9}$$

Finally, the linear equation for the z-motion is then

$$\left(1-\tfrac{1}{2}x'^2\right)y'' + \left(E_0' - f(\theta)\right)y - x'(1+x'')y' = 0, \tag{3.10}$$

where we have kept only linear terms in y, and have used the result of MURA-397 that $(1+x)F_z$ gives $(E_0' - f(\theta))y$.

In Appendix A, the problem of finding the tune of an equation having the form of Eq. (3.10) is treated. It is shown in Appendix A, that the equation

$$y'' + Ay' + By = 0 \tag{3.11}$$

can be written in the form

$$\bar{y}'' + \left(B - \frac{A'}{2} - \frac{1}{4}A^2\right)\bar{y} = 0, \tag{3.12a}$$

where

$$y = e^{-\frac{1}{2}\int A\,d\theta}\,\bar{y}. \tag{3.12b}$$

By examining Eq. (3.12), one can see what terms must be kept in the linear equation because they contribute significantly to the tune. One sees that $A(\theta)$ can be smaller than $B(\theta)$ by the factor $1/N$, and still contribute as much to the tune as does $B(\theta)$. For Eq. (3.10), we actually have that $B \sim KN\chi'$, and $A \sim \chi'$, where $K \sim RG_m'/G_m$ and B and A can make comparable contributions provided $K \sim 1$.

For Eq. (3.10),

$$A = -\frac{\chi'(1+\chi'')}{1-\frac{1}{2}\chi'^2}, \tag{3.13a}$$

$$B = \frac{1}{1-\frac{1}{2}\chi'^2}\left(E_0' - f(\theta)\right). \tag{3.13b}$$

We expand A and B in powers of χ'^2 keeping up to terms of order χ'^2.

$$A = -\chi'(1+\chi''), \tag{3.14a}$$

$$A' = -\frac{d}{d\theta}\left[\chi'(1+\chi'')\right], \tag{3.14b}$$

$$B = \left[E_0' - f(\theta)\right]\left(1+\frac{1}{2}\chi'^2\right). \tag{3.14c}$$

Thus we find for the tune function of Eq. (3.12)

$$\begin{aligned} B - \frac{A'}{2} - \frac{1}{4}A^2 = {} & E_0' - f(\theta) \\ & + \left(E_0' - f(\theta)\right)\frac{1}{2}\chi'^2 \\ & + \frac{1}{2}\frac{d}{d\theta}\left[\chi'(1+\chi'')\right] \\ & - \frac{1}{4}\chi'^2(1+\chi'')^2. \end{aligned} \tag{3.15}$$

Thus if we write Eq. (3.12) in the form

$$\bar{y}'' + \left(E_0' + \Delta E_0' - f(\theta) - \Delta f(\theta) \right) \bar{y} = 0 , \tag{3.16a}$$

then we find that

$$\Delta E_0' = E_0' \left(\tfrac{1}{2} \chi'^2\right)_0 - \tfrac{1}{4} \left[(\chi'^2)_0 + (\chi'^2 \chi''^2)_0 \right] , \tag{3.16b}$$

$$\Delta f_n = -\tfrac{1}{2} (\chi'')_n - \tfrac{1}{2} \left(\tfrac{d}{d\theta} (\chi' \chi'') \right)_n . \tag{3.16c}$$

We combine the corrections given by Eqs. (3.16) with the results for E_0' and f_n found in MURA-397 to find the following corrected values of E_0' and f_n,

$$\begin{aligned} E_0' = \left[1 + \tfrac{1}{2} (\chi'^2)_0 \right] \Big[& 2 \frac{eR}{pc} \sum_{m \geq 1} \omega_m^2 |\chi_m| H_m - \frac{eR}{pc} R H_0' \\ & - \frac{eR}{pc} \chi_0 \left(R^2 H_0'' + 2 R H_0' \right) \\ & - 2 \left(\frac{eR}{pc} \right) \sum_{m \geq 1} |\chi_m| \left(R^2 H_m'' + 2 R H_m' - R^2 H_m \beta_m'^2 \right) \Big] \\ & - \tfrac{1}{4} \left[(\chi'^2)_0 + (\chi'^2 \chi''^2)_0 \right] , \end{aligned} \tag{3.18a}$$

$$\begin{aligned} f_n = & \frac{eR}{pc} \left(R G_n' \right) \\ & - \tfrac{1}{2} \left[(\chi'')_n + i \omega_n (\chi' \chi'')_n \right] . \end{aligned} \tag{3.18b}$$

In Eqs. (3.18), the x_n are the Fourier coefficients of the equilibrium orbit $x(\theta)$ and the corrected values of x_n will be given in Section IV on the equilibrium orbit. One may note that

$$\frac{1}{2}(x'^2)_0 = \left(\frac{eR}{p_0}\right)^2 \sum_{n\geq 1} \frac{H_n^2}{\omega_n^2} , \tag{3.18c}$$

$$-(x'')_m = \frac{eR}{p_0} H_m , \tag{3.18d}$$

$$\frac{1}{4}(x'^2 x''^2)_0 = \frac{1}{2}\left(\frac{eR}{p_0}\right)^4 \frac{H_1^4}{N^2} \tag{3.18e}$$

$$-\frac{1}{2} i \omega_n (x'x'')_n = \begin{cases} -\left(\frac{eR}{p_0}\right)^2 G_1^2 , & n = 2 , \\ 0 , & n \neq 2 \end{cases} \tag{3.18f}$$

where in the last two results, we assume we can neglect all the harmonics but the first. It may be noted that Eq. (3.18f) says that even if the magnetic field has just a first harmonic, there may still be an appreciable second harmonic in $f(\theta)$ given by

$$f_2 = -\left(\frac{eR}{p_0}\right)^2 G_1^2 \tag{3.18g}$$

This second harmonic, f_2, can be comparable to the first harmonic f_1 if the flutter $(eR/p_0) H_1$ is comparable to the field gradient $(eR/p_0) K H_1'$, which will happen in large flutter machines with small N and field gradients, like a low energy two-way FFAG machine.

The tune ν_2 can be found using the above results for E_n' and f_n. In MURA-397 the smooth approximation formula

$$\nu_z^2 = E_0' + 2 \sum_{m \geq 1} \frac{|f_m|^2}{\omega_m^2} \tag{3.19}$$

was used. This formula is not valid when ν_z is close to 1/2 N. We will use here another result which will be derived in a future report that is valid in the entire range $0 \leq \nu_z \leq \frac{1}{2} N$. This result is

$$\nu_z^2 = \frac{N^2}{4} - \sqrt{(E_f' - E_0')^2 - |H_1|^2} , \tag{3.20a}$$

$$E_f' = \frac{N^2}{4} - \sum_{n \geq 2} \frac{2\,|f_n|^2}{\omega_n^2 - 4E_0'} + \sum_{n_1+n_2+n_3=0} \frac{f_{n_1} f_{n_2} f_{n_3}}{\omega_{n_1}^2 (\omega_{n_1} + \omega_{n_2})^2} . \tag{3.20b}$$

Eq. (3.20) for the tune may be considered an expansion in powers of f and it is correct up to and including terms of order f^3 for $\nu^2 << (N/2)^2$ and is correct up to terms of order f when $\nu^2 \cong (N/2)^2$. Usually the f^3 term may be safely neglected, except for machines with small N and large flutter. If we assume that the higher harmonics past the first in the magnetic field may be neglected, then the f^3 term contributes the amount

$$\sum_{n_1+n_2+n_3=0} \frac{f_{n_1} f_{n_2} f_{n_3}}{\omega_{n_1}^2 (\omega_{n_2} + \omega_{n_3})^2} = 3 \left(\frac{f_1^2 f_{-2}}{N^4} + c.c. \right) \tag{3.21a}$$

$$= \frac{-3}{N^4} \left(\frac{eR}{p_0} \right)^4 H_1^2 \left[(R H_1' + \tfrac{1}{2} H_1)^2 - R^2 H_1^2 B_1'^2 \right] \tag{3.21b}$$

The second harmonic f_2 generated by H_1 will also contribute to E_f through

$$- 2 \frac{|f_2|^2}{\omega_2^2 - 4E_0'} = -\frac{1}{2} \left(\frac{eR}{pc}\right)^4 \frac{H_1^{\,4}}{N^2} \, . \tag{3.21c}$$

It may be noted that this contribution will exactly cancel the $-\frac{1}{4}\left(x'^2 x''^2\right)$ term in E_0'.

A result for E_f' which should have a wide range of application is

$$E_f' = \frac{N^2}{4} - 2\left(\frac{eR}{pc}\right)^2 \sum_{m \geq 2} \frac{1}{\omega_m^2 - 4E_0'} \Big| \left(RH_m' + \frac{1}{2}H_m\right)^2 + R^2 \beta_m'^2 H_,$$

$$- \frac{1}{2}\left(\frac{eR}{pc}\right)^4 \frac{H_1^{\,4}}{N^2} \tag{3.22}$$

$$- 3\left(\frac{eR}{pc}\right)^4 \frac{H_1^{\,2}}{N^4} \left[\left(RH_1' + \frac{1}{2}H_1\right)^2 - R^2 \beta_1'^2 H_1^{\,2} \right] .$$

The r - tune

The equation for the r - motion is

$$\frac{d}{d\theta} \frac{x'}{\sqrt{(1+x)^2 + x'^2}} - \frac{(1+x)}{\sqrt{(1+x)^2 + x'^2}} = F_r \, , \tag{3.23a}$$

$$F_r = \frac{e}{pc} \, r H_z \tag{3.23b}$$

In MURA-397, we assumed that $x' \ll 1$, $x \ll 1$ and we wrote Eq. (3.23) as

$$x'' - (1+x) = (1+x) F_r \, . \tag{3.24}$$

We keep now the thrown-away terms and we write

$$\frac{d}{d\theta}(ax') = \frac{x''}{1+x} + \left[\frac{d}{d\theta}(ax') - \frac{1}{1+x}\, x''\right], \tag{3.25a}$$

$$a(1+x) = 1 + [a(1+x) - 1]. \tag{3.25b}$$

We rewrite Eq. (3.23)as

$$x'' - (1+x) + \Big\{ (1+x)\frac{d}{d\theta}(ax') - x'' - (1+x)[a(1+x) - 1] \Big\} = (1+x) F_r, \tag{3.26}$$

which when compared with Eq. (3.24) has the correction term C_x

$$x'' - (1+x) + C_x = (1+x) F_r, \tag{3.27a}$$

$$C_x = (1+x)\frac{d}{d\theta}(ax') - x'' - (1+x)[a(1+x) - 1], \tag{3.27b}$$

$$C_x = x''[(1+x)a - 1] + (1+x)a'x' - (1+x)[a(1+x) - 1]. \tag{3.27c}$$

We now expand C_x in powers of x and x'. Using the expansion for $(1+x)a - 1$ and for a' found previously, we find that

$$
\begin{aligned}
C_x = \; & x'' \left[-\tfrac{1}{2} x'^2 + \tfrac{1}{2} x x'^2 \right] \\
& + x' \left(-x' - x' x'' \right) \\
& - (1+x) \left[-\tfrac{1}{2} x'^2 + \tfrac{1}{2} x x'^2 \right] ,
\end{aligned} \tag{3.28a}
$$

$$
\begin{aligned}
C_x = \; & x'' \left[-\tfrac{1}{2} x'^2 + \tfrac{1}{2} x x'^2 \right] \\
& + x'^2 \left(-1 - x'' \right) \\
& + \tfrac{1}{2} x'^2 ,
\end{aligned} \tag{3.28b}
$$

where we have dropped the y-dependent terms.

To find the linear equation for the r- motion, we have to expand C_x about the equilibrium orbit, $x = x_s(\theta)$. We write

$$x = x_s + u \tag{3.29}$$

and expand in powers of u.

$$
\begin{aligned}
C_x = \; & (x_s'' + u'') \left[-\tfrac{1}{2} (x_s' + u')^2 + \tfrac{1}{2} (x_s + u)(x_s' + u')^2 \right] \\
& + (x_s' + u')^2 \left(-1 - x_s'' - u'' \right) \\
& + \tfrac{1}{2} (x_s' + u')^2 ,
\end{aligned} \tag{3.30a}
$$

$$
\begin{aligned}
C_x = {} & u''\left(-\tfrac{1}{2}\, x_s'^2 - x_s'^2\right) \\
& + u'\left\{-x_s''\, x_s' + 2\, x_s'(-1 - x_s'') + x_s'\right\} \\
& + u\left\{x_s''\, \tfrac{1}{2}\, x_s'^2\right\},
\end{aligned}
\tag{3.30b}
$$

$$
\begin{aligned}
C_x = {} & u''\left(-\tfrac{3}{2}\, x_s'^2\right) \\
& + u'\left(-x_s' - 3\, x_s'\, x_s''\right) \\
& + u\left(\tfrac{1}{2}\, x_s'^2\, x_s''\right),
\end{aligned}
\tag{3.30c}
$$

where we have kept only the leading terms in the coefficients of u'', u', u.

We find then for the linear $r-$ equation of motion, using Eq. (3.27a) and (3.30c),

$$
\begin{aligned}
& \left(1 - \tfrac{3}{2}\, x_s'^2\right) u + \left(E_0 - g(0)\right) u \\
& \qquad + \tfrac{1}{2}\, x_s''\, x_s'^2\, u + \left(-x_s' - 3\, x_s'\, x_s''\right) u' = 0
\end{aligned}
\tag{3.31}
$$

where we have used the result of MURA-397 that $(1 + x)\, F_r + u$ gives $\left(E_0 - g(0)\right) u$.

As was done in treating the z- tune, we eliminate the μ' term in Eq. (3.31) by using the transformation

$$\mu = e^{-\frac{1}{2}\int A\, d\theta}\, \bar{\mu} \quad , \tag{3.32a}$$

and get

$$\frac{d^2}{d\theta^2}\bar{\mu} + \left(B - \frac{A'}{2} - \frac{1}{4}A^2\right)\bar{\mu} = 0 \, , \tag{3.32b}$$

where

$$A = - \frac{x_s'\,(1 + 3\, x_s'')}{1 - \frac{3}{2}\, x_s'^{\,2}} \, , \tag{3.32c}$$

$$B = \frac{1}{1 - \frac{3}{2}\, x_s'^{\,2}}\,\left(E_0 - g(\theta)\right) . \tag{3.32d}$$

We expand A and B in powers of $x_s'^{\,2}$, keeping just the leading term,

$$A = -\, x_s'\,(1 + 3\, x_s'') \, , \tag{3.33a}$$

$$A' = -\frac{d}{d\theta}\left[x_s'\,(1 + 3\, x_s'')\right] , \tag{3.33b}$$

$$B = \left[E_0 - g(\theta)\right]\left[1 + \frac{3}{2}\, x_s'^{\,2}\right] . \tag{3.33c}$$

Thus we find for the tune function

$$B - \frac{A'}{2} - \frac{1}{4}A^2 = (E_0 - g(\theta))$$

$$+ (E_0 - g(\theta))\frac{3}{2}x_s'^2$$

$$+\frac{1}{2}\frac{d}{d\theta}\left[x_s'(1+3x_s'')\right] \tag{3.34}$$

$$-\frac{1}{4}x_s'^2(1+3x_s'')^2 .$$

If we write Eq. (3.32) in the form

$$\frac{d^2}{d\theta^2}\bar{u} + (E_0 + \Delta E_0 - g(\theta) - \Delta g(\theta))\bar{u} = 0, \tag{3.35}$$

then we find that

$$\Delta E_0 = E_0 \frac{3}{2}(x_s'^2)_0 - \frac{1}{4}\left[(x_s'^2)_0 + 9(x_s' x_s'')_0\right], \tag{3.36a}$$

$$\Delta g_n = -\frac{1}{2}(x_s'')_n - \frac{3}{2}\left(\frac{d}{d\theta}(x_s' x_s'')\right)_n . \tag{3.36b}$$

$$E_0 = \left[1 + \frac{3}{2}(\chi'^2)_0\right]\Big[\frac{eR}{pc}\left(RH_0' + 2H_0\right) - 1$$

$$+ \frac{eR}{pc}\, x_0\left(R^2 H_0'' + 4RH_0' + 2H_0\right)$$

$$+ 2\,\frac{eR}{pc}\sum_{m\geq 1} |x_m|\,\big(R^2 H_m'' + 4RH_m' + 2H_m$$

$$- R^2 H_m \beta_m'^2\big)\Big] \qquad (3.37a)$$

$$- \frac{1}{4}\left[(\chi'^2)_0 + 9\,(\chi'^2\chi''^2)_0\right],$$

$$g_n = -\frac{eR}{pc}\left(RG_n' + 2G_n\right)$$

$$- \frac{1}{2}\left[(\chi_s'')_n + i\,3\,w_n\,(\chi_s'\chi_s'')_n\right],$$

(3.37b)

where in most cases we can take

$$\frac{1}{2}(\chi_s'^2)_0 = \left(\frac{eR}{pc}\right)^2 \sum_{m\geq 1} \frac{H_m^2}{w_m^2}, \qquad (3.37c)$$

$$-(\chi_s'')_m = \frac{eR}{pc}\, H_m, \qquad (3.37d)$$

$$\frac{9}{4}(\chi_s'^2\chi_s''^2)_0 = \frac{9}{2}\left(\frac{eR}{pc}\right)^2 \frac{H_1^2}{N^2} \qquad (3.37e)$$

$$-\frac{3}{2}\, i\, w_n \left(x_s' x_s'' \right)_n = \begin{cases} -3\left(\frac{eR}{pc}\right)^2 G_1^{\,2} & ; n = 2 \\ 0 & ; n \neq \pm 2 \,. \end{cases} \tag{3.37f}$$

Eq. (3.37f) shows the existence of a large second harmonic in for machines with small N and large flutter,

$$g_2 = -3\left(\frac{eR}{pc}\right)^2 G_1^{\,2} \tag{3.38}$$

The tune ν_r is given by

$$\nu_r^{\,2} = \frac{N^2}{4} - \sqrt{(E_f - E_0)^2 - |g_1|^2}\,, \tag{3.39a}$$

$$E_f = \frac{N^2}{4} - \sum_{m \geq 2} \frac{2\,|g_m|^2}{w_m^2 - 4E_0} + \sum_{n_1+n_2+n_3=0} \frac{g_{n_1} g_{n_2} g_{n_3}}{w_{n_1}^2 (w_{n_1} + w_{n_2})^2}\,. \tag{3.39b}$$

For the g^3 term in E_f, we may take

$$\sum \frac{g_{n_1} g_{n_2} g_{n_3}}{w_{n_1}^2 (w_{n_2} + w_{n_3})^2} = 3\left(\frac{g_1^{\,2} g_{-2}}{N^4} + c.c. \right) \tag{3.40a}$$

$$= -9\left(\frac{eR}{pc}\right)^4 \frac{H_1^2}{N^4} \left[\left(RH_1' + \frac{3}{2} H_1\right)^2 - R^2 B_1'^{\,2} H_1^{\,2} \right] \tag{3.40b}$$

The second harmonic g_2 generated by G_1 will also contribute to E_f through

$$-2\,\frac{|g_2|^2}{\omega_2^2-4E_0} = -\frac{9}{2}\left(\frac{eR}{pc}\right)^4 \frac{H_1^{\,4}}{N^2} \tag{3.41}$$

which will just cancel the $-\frac{9}{4}\left(\chi_s'^2\,\chi_s''^2\right)_0$ term in E_0.

A result for E_f which should have wide application is

$$E_f = \frac{N^2}{4} - 2\left(\frac{eR}{pc}\right)^2 \sum_{m\geq 2} \frac{1}{\omega_m^2-4E_0}\left[\left(RH_m' + \frac{3}{2}H_m\right)^2 + R^2\beta_m'^2 H_m^2\right]$$
$$-\frac{9}{2}\left(\frac{eR}{pc}\right)^4 \frac{H_1^{\,4}}{N^2} \tag{3.42}$$
$$-9\left(\frac{eR}{pc}\right)^4 \frac{H_1^2}{N^4}\left[\left(RH_1'+\frac{3}{2}H_1\right)^2 - R^2\beta_1'^2 H_1^2\right].$$

IV. EQUILIBRIUM ORBIT

In the previous section we found the correction to the tune due to certain terms in the equations of motion which are small and are of the order of χ' but which can contribute significantly in some special cases. It is clear that in treating the tune we should have first found the effect of these small terms on the equilibrium orbit since the equilibrium orbit is used in calculating the tune. We will calculate now the effect on the equilibrium orbit.

In treating the r- tune we found the corrected r- equation of motion which we write as

$$x'' - (1+x) + C_x = (1+x) F_r , \tag{4.1a}$$

$$\begin{aligned} C_x = \; & x'' \left[-\tfrac{1}{2} x'^2 + \tfrac{1}{2} x x'^2 \right] \\ & + x'^2 \left(-1 - x'' \right) \\ & + \tfrac{1}{2} x'^2 , \end{aligned} \tag{4.1b}$$

$$C_x = -\tfrac{1}{2} x'^2 - \tfrac{3}{2} x'^2 x'' \tag{4.1c}$$

Now following the same procedure as was used in MURA-397, we can write Eq. (4.1) as

$$x'' + \left(E_s - \bar{g}(\theta) \right) x = f(\theta) - C_x \tag{4.2}$$

As in MURA-397, we find the solution

$$x(\theta) = \sum_n a_n \frac{u_n(\theta)}{E_s - E_n} , \tag{4.3a}$$

$$a_n = \int d\theta \, u_n(\theta) \left[f(\theta) - C_x \right] , \tag{4.3b}$$

$$f(\theta) = 1 + \left(\frac{eR}{pc} \right) H_z . \tag{4.3c}$$

Now we choose R so that $a_0 = 0$,

$$\int d\theta\, u_0^* \left[f(\theta) - C_x \right] = 0 , \tag{4.4a}$$

$$u_0 = 1 - \sum_{n \neq 0} \frac{\bar{g}_n}{w_n^2} e^{i w_n \theta} , \tag{4.4b}$$

$$\bar{g}_n = \frac{eR}{pc} \left(R G_n' + 2 G_n \right) \tag{4.4c}$$

Thus

$$f_0 - \sum_{n \neq 0} \frac{g_{-n}}{w_n^2} f_n - \left[C_x \right]_0 = 0 \tag{4.5a}$$

or

$$1 - \frac{eR}{pc} H_0 - 2 \left(\frac{eR}{pc} \right)^2 \sum_{n \geq 1} \frac{1}{w_n^2} \left(R H_n' H_n + 2 H_n^2 \right) + \frac{1}{2} \left(\chi'^2 \right)_0 = 0 \tag{4.5b}$$

In evaluating $\left(\chi'^2 \right)_0$, we can use the uncorrected value for $\chi(\theta)$.

Thus

$$\frac{1}{2} \left(\chi'^2 \right)_0 = \sum_{n \geq 1} \left(\frac{eR}{pc} \right)^2 \frac{H_n^2}{w_n^2} . \tag{4.6}$$

Thus we find

$$1 - \frac{eR}{pc} H_0 - 2 \left(\frac{eR}{pc} \right)^2 \sum_{n \geq 1} \frac{1}{w_n^2} \left(R H_n' H_n + \frac{3}{2} H_n^2 \right) = 0 \tag{4.7a}$$

or

$$p = \frac{eR}{2c} \left\{ H_0 + \sqrt{H_0^2 + 8 \sum_{n \geq 1} \frac{1}{w_n^2} \left(R H_n' H_n + \frac{3}{2} H_n^2 \right)} \right\} \tag{4.7b}$$

We now find the other a_n,

$$a_n = \frac{1}{E_s - E_n} \int d\theta \, u_n^*(\theta) \left[f(\theta) - C_x \right] \tag{4.8}$$

Here it is accurate enough to put

$$u_n(\theta) \simeq e^{i w_n \theta}, \tag{4.9a}$$

$$E_n \simeq w_n^2 . \tag{4.9b}$$

Then

$$a_n = \frac{1}{E_s - w_n^2} \left(f_n - [C_x]_n \right) \tag{4.10}$$

For $n = 1$, which is usually the dominant term.

$$\begin{aligned} -[C_x]_1 &= 3 \left(x'^2 x'' \right)_1 \\ &= -3 \left(\frac{eR}{pc} \right)^3 \frac{H_1^2}{N^2} G_1 , \end{aligned} \tag{4.11a}$$

$$a_1 = \frac{1}{N^2 - E_s} \frac{eR}{pc} G_1 \left[1 + \frac{3}{2} \left(\frac{eR}{pc} \right)^2 \frac{H_1^2}{N^2} \right] . \tag{4.11b}$$

The correction we have found for a_1 is small, and as the other a_n are usually not as important as a_1, we can write for all the

$$a_n = \frac{1}{W_n^2 - E_s} \frac{eR}{pc} G_n \left[1 + \frac{3}{2} \left(\frac{eR}{pc}\right)^2 \frac{H_1^2}{N^2} \right]. \tag{4.12}$$

If we expand $\chi(\theta)$ in a Fourier series,

$$\chi(\theta) = \sum_n \chi_n e^{i W_n \theta} \tag{4.13}$$

we may note that $\chi_0 \neq 0$ as the term $a_1 u_1(\theta)$ has a zero harmonic component since

$$u_1(\theta) = e^{iN\theta} \left(1 - \sum_{n \neq 0} \frac{\bar{g}_n}{W_n^2 + 2NW_n} e^{iW_n\theta} \right) \tag{4.14}$$

$$= e^{iN\theta} + \frac{\bar{g}_{-1}}{N^2} - \frac{\bar{g}_1}{3N^2} e^{i2N\theta},$$

$$\chi_0 = a_1 \frac{\bar{g}_{-1}}{N^2} + \text{c.c.}, \tag{4.15a}$$

or

$$\chi_0 = -2 \left(\frac{eR}{pc}\right)^2 \frac{(RH_1H_1' + 2H_1^2)}{N^4}. \tag{4.15b}$$

Finally we can write $\chi(\theta)$ as

$$\chi(\theta) = \sum \chi_n e^{iW_n\theta}, \tag{4.16a}$$

$$\chi_0 = -2 \left(\frac{eR}{pc}\right)^2 \frac{1}{N^4} (RH_1'H_1 + 2H_1^2), \tag{4.16b}$$

$$\chi_n = \left(\frac{eR}{pc}\right) \frac{1}{W_n^2 - E_s} G_n \left(1 + \frac{3}{2}\left(\frac{eR}{pc}\right)^2 \frac{H_1^2}{N^2}\right). \tag{4.16c}$$

We may note that as $x_0 \neq 0$, R is not actually the average radius of the equilibrium orbit. The average radius is given by $R(1+x_0)$. This correction is rather small and we shall refer to R as the average radius and usually ignore the correction.

Frequency of revolution

We will now calculate the frequency of revolution, $\omega/2\pi$, of the particle in the equilibrium orbit. To do this we must first calculate the length of the equilibrium orbit, L. L is given by

$$L = \int_0^{2\pi} d\theta \sqrt{r^2 + r'^2}, \tag{4.17a}$$

$$L = R \int_0^{2\pi} d\theta \sqrt{(1+x)^2 + x'^2}, \tag{4.17b}$$

$$L = R \int_0^{2\pi} d\theta \left((1+x) + \frac{1}{2} x'^2 \right), \tag{4.17c}$$

$$L = 2\pi R \left(1 + x_0 + \frac{1}{2} (x'^2)_0 \right). \tag{4.17d}$$

In MURA-397, we neglected x and x'^2 in Eq. (4.17) and found that

$$L = 2\pi R \tag{4.18}$$

Using the results found for x_0, we now have

$$L = 2\pi R \left[1 + \left(\frac{eR}{pc}\right)^2 \frac{H_1^2}{N^2} - 2\left(\frac{eR}{pc}\right)^2 \frac{1}{N^4} \left(R H_1' H_1 + 2 H_1^2 \right) \right]. \tag{4.19}$$

The frequency of rotation may be found from

$$\omega = 2\pi \frac{v}{L} , \tag{4.20a}$$

$$\omega = \frac{v}{R}\left[1 - x_0 - \frac{1}{2}(x'^2)_0\right], \tag{4.20b}$$

$$\omega = \frac{v}{R}\left[1 - \left(\frac{eR}{pc}\right)^2 \frac{H_1^2}{N^2} + 2\left(\frac{eR}{pc}\right)^2 \frac{1}{N^4}\left(RH_1'H_1 + 2H_1^2\right)\right]. \tag{4.20c}$$

APPENDIX A

We start with the equation

$$y'' + Ay' + By = 0 , \tag{A.1}$$

and to eliminate the y' term, we make the transformation

$$y = C\bar{y} , \tag{A.2a}$$

$$y' = C'\bar{y} + C\bar{y}' , \tag{A.2b}$$

$$y'' = C\bar{y}'' + 2C'\bar{y}' + C''\bar{y} . \tag{A.2c}$$

Thus Eq. (A.1) becomes

$$C\bar{y}'' + \bar{y}'(2C' + AC) + \bar{y}(C'' + AC' + BC) = 0 \tag{A.3}$$

We then determine C from

$$2C' + AC = 0 , \tag{A.4a}$$

$$C = e^{-\frac{1}{2}\int A\, d\theta} \tag{A. 4b}$$

We find for C' and C''

$$C' = -\frac{A}{2} C , \tag{A. 5a}$$

$$\begin{aligned} C'' &= -\frac{A'}{2} C - \frac{A}{2} C' \\ &= -\frac{A'}{2} C + \frac{A^2}{4} C . \end{aligned} \tag{A. 5b}$$

Thus

$$\frac{C''}{C} + A\frac{C'}{C} + B = -\frac{A'}{2} + \frac{A^2}{4} + B , \tag{A. 6}$$

and our transformed equation becomes

$$\bar{y}'' + \left(B - \frac{A'}{2} + \frac{A^2}{4}\right)\bar{y} = 0 , \tag{A. 7}$$

$$y = e^{-\frac{1}{2}\int A\, d\theta}\, \bar{y} .$$