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A GENERAL ANALYTICAL EXPRESSION FOR THE CHROMATICITY
OF ACCELERATOR RINGS

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ABSTRACT

We generalize the analytical approach, described in a previous note, to include magnets with inclined and/or curved boundaries. Evaluating the fringe field contribution in hard edge approximation, we arrive at relatively simple expressions for the chromaticity, general enough to cover all magnet types commonly used in proton synchrotrons and storage rings. Results are given for a number of small focusing rings for which the exact treatment of the fringe fields is important.

Distribution (of abstract)

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1. INTRODUCTION

In a previous note¹⁾ we discussed methods used in the design of LEAR to calculate the chromaticity, i.e. the linear dependence of betatron wave number Q on momentum. Our approach, derived in ref.¹⁾, starts from the equation of motion in a general time independent magnetic field²⁾. Expanding these equations up to the second order in the deviation of a particle from the reference orbit and applying appropriate transformations, we obtained a differential equation from which the tune shift $\Delta Q(\Delta p)$ can be calculated using standard accelerator theory. The resulting expressions have been evaluated in ref.¹⁾ for the case of magnets with orthogonal beam entry (sector magnets) as is appropriate for LEAR.

These analytical results agree well with those obtained by using the computer programme TRANSPORT³⁾ to calculate the chromaticity from the second order matrix elements, a formalism which was derived by S. Peggs⁴⁾. But for small machines where the bending radius ρ is no longer large compared to the dispersion D most other programmes and methods disagree with our results. We believe that, with the exception of TRANSPORT, these programmes use approximations for the edge field which are not appropriate for the small rings.

In the present report we generalize the analytical approach to include magnets with inclined and/or curved boundaries. We need to calculate the dipole, quadrupole and sextupole components of the fringe field that enters into the chromaticity formula. To obtain these components, we assume that the field passes smoothly from its value well inside the magnet to zero field in the straight section. The transition takes place in a short edge region which follows the border curve of the magnet. By evaluating this transition in hard edge approximation, we arrive at relatively simple analytical expressions for the chromaticity, general enough to cover all common magnet types used in synchrotrons and storage rings.

2. THE GENERAL FORMULA

We repeat here for convenience, the general derivation given already in ref.¹⁾ (Appendix 1).

The trajectory equations of a particle in a static magnetic field²⁾ are (note that we use for the horizontal coordinate a sign convention opposite to the one of Steffen²⁾):

$$\begin{aligned} z'' + \frac{\dot{w}}{w^2} z' &= \frac{e}{p} \frac{v}{w} [-x'B_s + (1 + hx)B_x] \\ x'' + \frac{\dot{w}}{w^2} x' - h(1 + hx) &= \frac{e}{p} \frac{v}{w} [z'B_s - (1 + hx)B_z], \end{aligned} \quad (2.1)$$

with $h(s) = \frac{1}{\rho}$ the curvature of the reference orbit, x the horizontal co-ordinate (counted positive radially outwards), z the vertical component and s the distance along the reference orbit. A prime denotes the derivative with respect to s , v the total particle velocity and $w = \dot{s}$ the velocity of the particle projection along the reference orbit. Introducing ²⁾

$$\begin{aligned} \frac{v}{w} &= \sqrt{x'^2 + z'^2 + (1 + hx)^2} \\ \frac{\dot{w}}{w^2} &= -\frac{1}{2} \frac{d}{ds} \frac{\left(\frac{v^2}{w^2}\right)}{\frac{v^2}{w^2}} \\ \frac{e}{p} &= \frac{e}{p_0 \left(1 + \frac{\Delta p}{p_0}\right)} \end{aligned} \quad (2.2)$$

and using for the magnetic field components the expressions up to second order in z and x

$$\begin{aligned} \frac{e}{p_0} B_x(x, z, s) &= -kz - rzx \\ \frac{e}{p_0} B_z(x, z, s) &= h - kx - \frac{1}{2} rx^2 - \frac{1}{2} (h'' - hk - r)z^2 \\ \frac{e}{p_0} B_s(x, z, s) &= h'z - (hh' + k')zx \end{aligned} \quad (2.3)$$

$$\text{with } k(s) = -\frac{1}{B_0 \rho} \frac{\partial B_z(0,0,s)}{\partial x} \quad \text{and } r(s) = \frac{\partial k}{\partial x} = -\frac{1}{B_0 \rho} \frac{\partial^2 B_z(0,0,s)}{\partial x^2}, \quad (2.4)$$

one can express the differential equations for the vertical and horizontal motion to second order in z, x and their derivatives :

$$\begin{aligned} z'' - (hx' + h'x)z' &= - [k - k\delta + 2khx + rx + h'x']z \\ x'' - \frac{1}{2}hx'^2 - h'xx' &= h\delta - [h^2 - k + k\delta - 2h^2\delta]x - [h^3 - 2kh - \frac{1}{2}r]x^2. \end{aligned} \quad (2.5)$$

Herein we use $\delta \equiv \frac{\Delta p}{p_0}$ for brevity.

For the chromaticity calculation, we expand these equations around the reference orbit by inserting $x = D\delta$ into the equation for the vertical and $x = \bar{x} + D\delta$ into the equation for the horizontal motion where $D(s)$ (also called α_p or η in accelerator theory) is the dispersion function defined by the periodic solution $[D(s + 2\pi R) = D(s)]$ of

$$D'' + (h^2 - k)D = h \quad (2.6)$$

Neglecting all terms of higher than first order in \bar{x}, \bar{x}' and δ , we arrive at the differential equations

$$\begin{aligned} z'' - (hD)'z'\delta + [k + (2khD - k + rD + D'h')\delta]z &= 0 \\ \bar{x}'' - (hD)'x'\delta + [h^2 - k - (2h^2 + h'D' - 2Dh^3 - k + 4khD + rD)\delta]\bar{x} &= 0. \end{aligned} \quad (2.7)$$

From these equations the familiar equations of motion can be extracted by setting $\delta \equiv 0$:

$$\begin{aligned} z'' + kz &= 0 \\ \bar{x}'' + (h^2 - k)\bar{x} &= 0 \end{aligned} \quad (2.8)$$

For the following consideration it will be useful to write eq.(2.7) in a more compact form :

$$y'' + 2gy' + (k + \Delta k)y = 0 \quad (2.9)$$

where y represents either the vertical (z) or the horizontal (\bar{x}) component and where $g \equiv -\frac{1}{2}(hD)'\delta$ and Δk are the perturbations linear in δ .

Using the transformation

$$y = ue^{-\int g ds} \quad (2.10)$$

we eliminate the y' term in eq. (2.9) and get

$$u'' + (k + \Delta k - g' - g^2)u = 0. \quad (2.11)$$

Dropping the g^2 - term as it is of second order in δ , we obtain the tune shift due to the Δk and g' -term applying the Courant and Snyder theory (eq. 4.31 in ref. 5) to eq. (2.11) above :

$$\Delta Q = \frac{1}{4\pi} \int (\Delta k - g') \beta ds. \quad (2.12)$$

Here $\beta(s)$ and Q are the betatron function and betatron wave number of the unperturbed motion. From eq.(2.12) we obtain the chromaticity defined as

$$\xi = \frac{\Delta Q/Q}{\delta}. \quad (2.13)$$

Thus the contribution to the chromaticity of any magnetic element can be expressed by

$$\xi_V = -\frac{1}{4\pi Q_V} \left\{ \int k \beta ds - \int r D \beta ds - 2 \int k h D \beta ds - \int [h' D' + \frac{1}{2} (h D)''] \beta ds \right\} \quad (2.14)$$

$$\xi_H = -\frac{1}{4\pi Q_H} \left\{ \int (h^2 - k) \beta ds + \int r D \beta ds + \int (h^2 - 2 D h^3 + 4 k h D) \beta ds + \int [h' D' - \frac{1}{2} (h D)''] \beta ds \right\}$$

in the vertical (index V) and horizontal (index H) plane respectively.

By adding up all contributions, one obtains the chromaticity of the ring. What remains to be done is to find for the different types of magnets, the quadrupole k and sextupole r components and to evaluate the integrals in eq. (2.14).

3. A SIMPLE METHOD TO DERIVE THE FIELD COMPONENTS IN THE FRINGE REGION

From eq. (2.3), the vertical field in the median plane can be written as

$$\frac{e}{P_0} B_z(x, 0, s) = h - kx - \frac{1}{2} r x^2 \dots \equiv h(s, x). \quad (3.1)$$

Here $h(s,x)$ may be interpreted as the curvature for a particle trajectory at $x, s, z = 0$ and $h(s,0) = h(s)$ is the curvature of the reference orbit as used above.

In the central part of the magnet h, k and r do not depend on s for "well designed" magnets or are slowly varying functions of s for more complicated magnets. Where necessary, we use the suffix "m" to refer to the values inside the central part.

From eq. (3.1), we find $k(s)$ and $r(s)$ as

$$\begin{aligned} k(s) &= - \left. \frac{\partial h(s,x)}{\partial x} \right|_{x=0} \\ r(s) &= - \left. \frac{\partial^2 h(s,x)}{\partial x^2} \right|_{x=0} \end{aligned} \tag{3.2}$$

We use these expressions for the fringe region. This almost solves our problem. We shall just have to express the partial derivatives with respect to x by those with respect to s in the edge region denoting the latter ones by

$$h' = \frac{\partial h(s,0)}{\partial s} = \frac{\partial h}{\partial s} \quad \text{and} \quad h'' = \frac{\partial^2 h(s,0)}{\partial s^2}$$

To express the transition from zero field to the value inside the magnet, we suggest the product

$$h(s,x) = H(\sigma,\xi) h_m(s,x) = H(A) h_m(s,x) \tag{3.3}$$

where $h_m(s,x)$ is the field well inside the magnet and is usually independent of s ; $H(A)$ goes from 0 to 1 at the entrance (near $A(\sigma,\xi) = 0$). The cartesian system σ, ξ has its origin in the "hard edge point", that is the point where the reference orbit enters the idealized magnetic field

border. It may be defined by

$$-\int_{-s_0}^{s_m} H(\bar{s}, x=0) d\bar{s} - s_m = 0 \quad (3.4)$$

with s_0 large enough such that $H(-s_0) \approx 0$,

s_m large enough such that $H(s_m) \approx 1$.

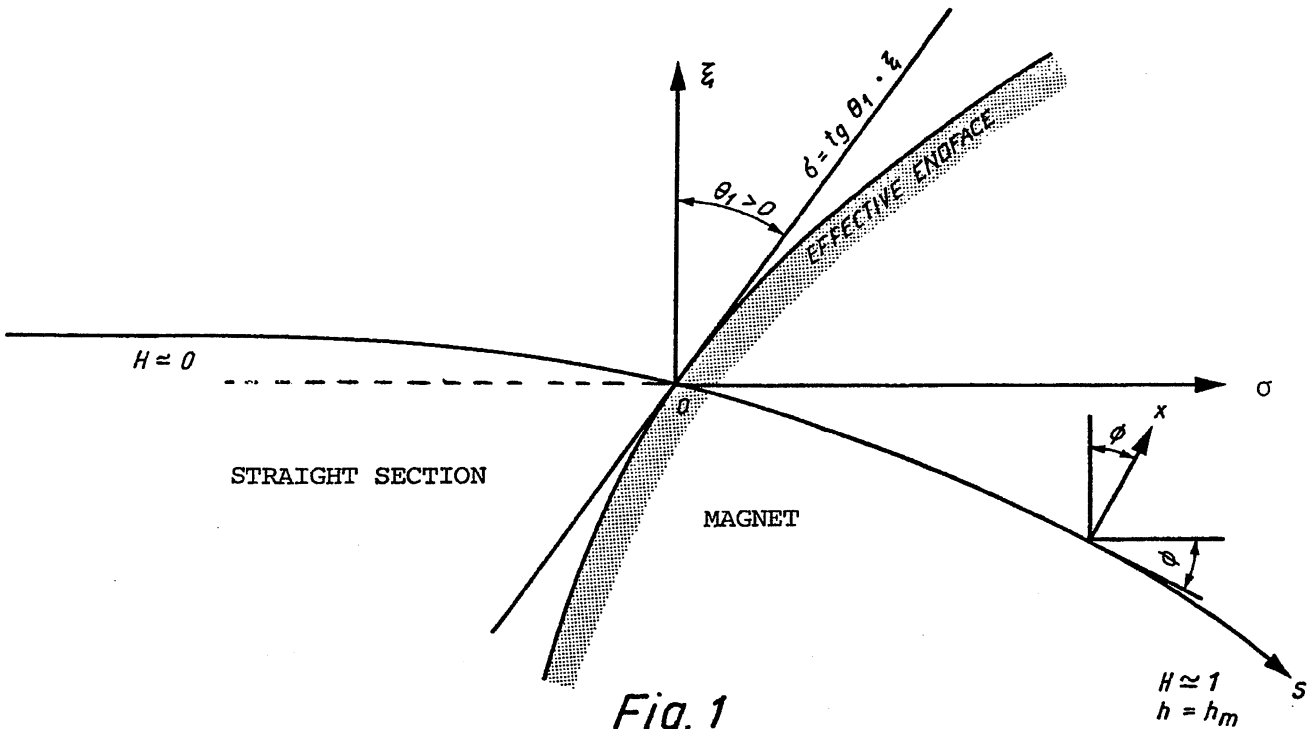


Fig. 1

From Fig. 1, one deduces :

$$\sigma(s, x) = \int_0^s \cos\phi(\bar{s}) d\bar{s} + x \sin\phi(s) \quad (3.5)$$

$$\xi(s, x) = -\int_0^s \sin\phi(\bar{s}) d\bar{s} + x \cos\phi(s)$$

where

$$\phi(s) = \int_{-s_0}^s h(\bar{s}) d\bar{s} \quad (3.6)$$

is the angle between the (instantaneous) s -direction and the σ -axis (which is almost the extension of the reference orbit in the straight section).

A "hard edge line" can be defined by taking eq. (3.4) for $x \neq 0$. When using the function $H(A)$ introduced above to describe the "field rise" in the edge region, it is general enough to take

$$A = \sigma - \text{tg}\theta_1 \xi - \frac{1}{2} \tau \xi^2 \quad (3.7)$$

so that $A = 0$ describes a tilted and curved end face whose radius of curvature is $1/(\tau \cos^3 \theta_1)$. Higher powers of ξ do not contribute to the chromaticity.

The azimuthal extent of the fringe field scales with the gap height g . For a gradient magnet g is a function of the radial position and thus contributes to the derivative $\partial A / \partial x$. But this contribution goes to zero in hard edge approximation ($g \rightarrow 0$) where $H(A)$ becomes the unit step function and dH/dA the δ function. In this approximation eqs. (3.3) and (3.7) can be considered as appropriate.

4. THE QUADRUPOLE AND THE SEXTUPOLE COMPONENT FOR A GRADIENT MAGNET WITH TILTED AND CURVED END FACES

The fact that the edge function $H(A)$ depends on A only allows one to replace the partial derivatives for k and r (eq. (3.2)) by derivatives with respect to s . Using

$$\frac{\partial H}{\partial x} = \frac{dH}{dA} \frac{\partial A}{\partial x}$$

$$\frac{\partial^2 H}{\partial x^2} = \frac{d^2 H}{dA^2} \left(\frac{\partial A}{\partial x} \right)^2 + \frac{dH}{dA} \frac{\partial^2 A}{\partial x^2}$$

one gets, with A given by eq. (3.7), along with eqs. (3.1) to (3.6), the quadrupole component k

$$\begin{aligned} k &= Hk_m - h_m \frac{dH}{dA} \{ \sin \phi - (\text{tg}\theta_1 + \tau \xi) \cos \phi \} \\ &= Hk_m + \frac{dH}{dA} \frac{\sin(\theta - \phi)}{\cos \theta} h_m \end{aligned}$$

where we have put

$$\text{tg} \theta = \text{tg} \theta_1 + \tau \xi.$$

We eliminate $\frac{dH}{dA}$ using

$$\begin{aligned} h' &= h_m \frac{dH}{dA} \left\{ \cos \phi + (\operatorname{tg} \theta_1 + \tau \xi) \sin \phi \right\} (1 + xh) \Big|_{x=0} \\ &= \frac{dH}{dA} \frac{\cos(\theta - \phi)}{\cos \theta} h_m \end{aligned} \quad (4.1)$$

and get

$$k = Hk_m + h' \operatorname{tg}(\theta - \phi). \quad (4.2)$$

For the sextupole component r we obtain

$$r = Hr_m - 2 \frac{dH}{dA} \frac{\sin(\theta - \phi)}{\cos \theta} k_m + \frac{dH}{dA} \cos^2 \phi \tau h_m - \frac{d^2 H}{dA^2} \frac{\sin^2(\theta - \phi)}{\cos^2 \theta} h_m$$

In order to eliminate $\frac{d^2 H}{dA^2}$ we need

$$h'' = \frac{d^2 H}{dA^2} \frac{\cos^2(\theta - \phi)}{\cos^2 \theta} h_m + \frac{dH}{dA} \left[\frac{\sin(\theta - \phi)}{\cos \theta} h - \tau \sin^2 \phi \right] h_m$$

which yields, together with eq. (4.1)

$$\begin{aligned} r &= Hr_m - 2H'k_m \operatorname{tg}(\theta - \phi) - h'' \operatorname{tg}^2(\theta - \phi) + hh' \operatorname{tg}^3(\theta - \phi) \\ &\quad + \tau h' \frac{\cos \theta}{\cos(\theta - \phi)} \left[1 - \frac{\sin^2 \phi}{\cos^2(\theta - \phi)} \right] \end{aligned} \quad (4.3)$$

For most cases it is sufficient to consider k and r in hard edge approximation. Then, eqs. (4.2) and (4.3) simplify to

$$k = Hk_m + h_m \operatorname{tg} \theta_1 \delta(s) \quad (4.4)$$

$$r = Hr_m + \left\{ \tau h_m - \operatorname{tg} \theta_1 \left[2k_m + h_m^2 \left(1 + \frac{\operatorname{tg}^2 \theta_1}{2} \right) \right] \right\} \delta(s) - h_m \operatorname{tg}^2 \theta_1 \delta'(s) \quad (4.5)$$

Signs are correct for an entrance. Signs for the exit depend on sign convention for θ and τ . Adopting that of Fig. 3, eq.(4.4) remains unchanged whilst r for the exit becomes

$$r = Hr_m + \{ \tau h_m - \text{tg } \theta_2 [2k_m + h_m^2 (1 + \frac{\text{tg}^2 \theta_2}{2})] \} \delta(s) + h_m \text{tg}^2 \theta_2 \delta'(s) \quad (4.6)$$

A magnet with upstream-downstream symmetry has the property of doubling - never of cancelling - the contributions from the entrance at the exit. This is also true for the last term of eq. (4.6) since, when folded with another function, $-\delta'(s)$ yields the derivative at the hard edge point and that has opposite signs at entrance and exit.

5. EVALUATION OF THE INTEGRALS

Integrating eq.(2.14) by parts, using ⁵⁾

$$\beta' = -2\alpha, \quad \alpha' = k\beta - \gamma$$

and eq.(2.6) to express D'' in terms of D one gets

$$\begin{aligned} \xi_V = & -\frac{1}{4\pi Q_V} \{ \int k\beta ds - \int rD\beta ds - \int h(kD\beta + D\gamma) ds - \int h'D'\beta ds \\ & - \frac{1}{2} (h'D\beta + hD'\beta + 2hD\alpha) \Big|_{\text{entr.}}^{\text{exit}} \} \end{aligned} \quad (5.1)$$

$$\begin{aligned} \xi_H = & -\frac{1}{4\pi Q_H} \{ \int (h^2 - k)\beta ds + \int rD\beta ds + \int h(2kD\beta + 2D'\alpha - D\gamma) ds \\ & - \frac{1}{2} (h'D\beta - hD'\beta + 2hD\alpha) \Big|_{\text{entr.}}^{\text{exit}} \} \end{aligned}$$

In order to simplify the integration we use the hard edge model as discussed above to describe the longitudinal field distribution $h(s)$ over a magnet:

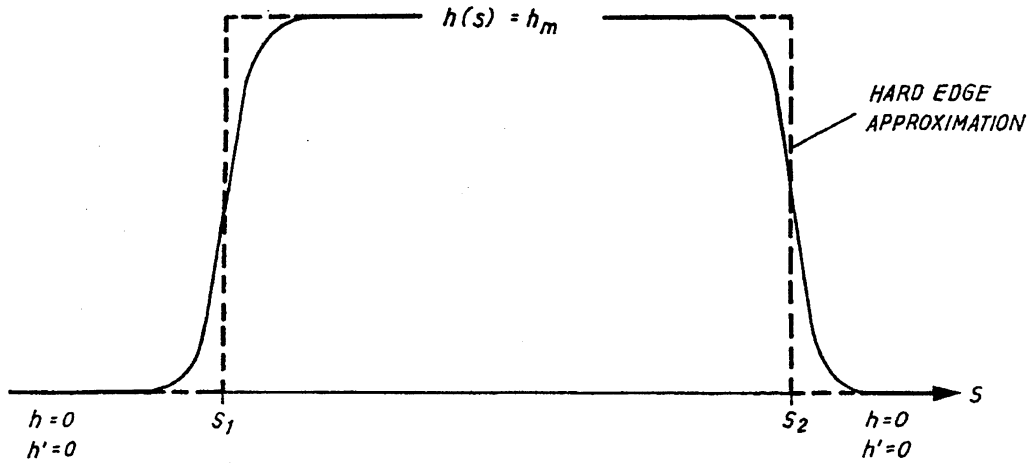


Fig. 2

Thus, the function $h(s)$ is constant everywhere except at the points s_1 and s_2 where $h'(s)$ becomes $\frac{h_m}{m}$ times the delta function $\delta(s)$. The following integration rule⁶⁾ holds the n -th derivative with respect to s of the delta function:

$$\int_{s-\epsilon}^{s+\epsilon} \delta^{(n)}(s-t) f(t) dt = (-1)^n f^{(n)}(s) \quad (5.2)$$

Since in hard edge approximation we shall need only the r.h.s. of eqs. (4.4) and (4.5) we omit the suffix "m" from hereon. Thus all h , k and r refer to the central part.

Assuming that the central part of the magnet goes from s_1 to s_2 we evaluate the contributions to the chromaticity in three steps:

- i) entrance (suffix "1")
- ii) central part
- iii) exit (suffix "2")

Central part

The contribution can be found from eq.(5.1). Using the hard edge approximation one gets

$$\Delta \xi_V = - \frac{1}{4\pi Q_V} \left\{ \int_{s_1}^{s_2} k\beta ds - \int_{s_1}^{s_2} rD\beta ds - \int_{s_1}^{s_2} h(kD\beta + D\gamma) ds - \frac{1}{2}(hD'\beta + 2hD\alpha) \Big|_{s_1}^{s_2} \right\}$$

$$\Delta \xi_H = - \frac{1}{4\pi Q_V} \left\{ \int_{s_1}^{s_2} (h^2-k)\beta ds + \int_{s_1}^{s_2} rD\beta ds + \int_{s_1}^{s_2} h(2kD\beta + 2D'\alpha - D\gamma) ds + \frac{1}{2}(hD'\beta - 2hD\alpha) \Big|_{s_1}^{s_2} \right\} \quad (5.3)$$

The various integrals can be expressed analytically by the values at s_1 . With ℓ defined by $\ell = s_2 - s_1$ and introducing the following abbreviations

$$C = \begin{cases} \cos(\sqrt{K} \ell) \\ \cosh(\sqrt{|K|} \ell) \end{cases} \quad S = \begin{cases} \frac{\sin(\sqrt{K} \ell)}{\sqrt{K}} \\ \frac{\sinh(\sqrt{|K|} \ell)}{\sqrt{|K|}} \end{cases} \quad \text{for } K = h^2 - k \begin{cases} > 0 \\ < 0 \end{cases}$$

$$C_2 = \begin{cases} \cos(2\sqrt{K_\beta} \ell) \\ \cosh(2\sqrt{|K_\beta|} \ell) \end{cases} \quad S_2 = \begin{cases} \frac{\sin(2\sqrt{K_\beta} \ell)}{\sqrt{K_\beta}} \\ \frac{\sinh(2\sqrt{|K_\beta|} \ell)}{\sqrt{|K_\beta|}} \end{cases} \quad \text{for } K_\beta \begin{cases} > 0 \\ < 0 \end{cases}$$

where $K_\beta = \begin{cases} k & \text{vertically} \\ K & \text{horizontally} \end{cases}$

one obtains

$$\int_{s_1}^{s_2} K_\beta \beta ds = \frac{1}{2} \{ (K_\beta \beta + \gamma) \ell + \alpha(C_2 - 1) + \frac{1}{2}(K_\beta \beta - \gamma) S_2 \} \quad (5.4)$$

$$\int_{s_1}^{s_2} D\beta ds = \frac{1}{2K_\beta} (K_\beta \beta + \gamma) \left[\left(D - \frac{h}{K}\right) S - D' \frac{1}{K} (C - 1) + \frac{h}{K} \ell \right]$$

$$+ \frac{h}{2KK_\beta} \left[\frac{1}{2}(K_\beta \beta - \gamma) S_2 + \alpha(C_2 - 1) \right] \quad (5.5)$$

$$+ \frac{1}{4K_\beta - K} \left\{ \left(D - \frac{h}{K}\right) \alpha (K \cdot S \cdot S_2 + 2C \cdot C_2 - 2) + D' \alpha (2S \cdot C_2 - C \cdot S_2) \right.$$

$$\left. + \frac{1}{2K_\beta} (K_\beta \beta - \gamma) \left[\left(D - \frac{h}{K}\right) (2K_\beta \cdot C \cdot S_2 - K \cdot S \cdot C_2) + D' (2K_\beta \cdot S \cdot S_2 + C \cdot C_2 - 1) \right] \right\}$$

$$\text{For } K_{\beta} = 0 : \int_{s_1}^{s_2} D\beta ds = D\beta\ell + (D'\beta - 2D\alpha)\frac{\ell^2}{2} + (D\gamma - 2D'\alpha + \frac{1}{2}h\beta)\frac{\ell^3}{3} + (D'\gamma - h\alpha)\frac{\ell^4}{4} + h\gamma\frac{\ell^5}{10}$$

$$\int_{s_1}^{s_2} hD(k\beta + \gamma) ds = h(k\beta + \gamma) \left\{ (D - \frac{h}{K})S + \frac{1}{K}[D'(1-C) + h\ell] \right\} \quad (5.6)$$

$$\text{For } K = 0 : \int_{s_1}^{s_2} hD(k\beta + \gamma) ds = h(k\beta + \gamma) \left\{ D\ell + \frac{1}{2}D'\ell^2 + \frac{1}{6}h\ell^3 \right\}$$

Since horizontally one has $C^2 = C^2 - 1$, $S^2 = 2SC$ and $K = K_{\beta}$ the corresponding integral can be written in terms of S , C and K only:

$$\begin{aligned} \int_{s_1}^{s_2} h(2kD\beta + 2D'\alpha - D\gamma) ds &= h(K\beta + \gamma) \left(\frac{k}{K} - \frac{1}{2} \right) \left\{ (D - \frac{h}{K})S + \frac{1}{K}[D'(1-C) + h\ell] \right\} \\ &+ h^2 \frac{1}{K} \left(\frac{k}{K} + \frac{1}{2} \right) \{ (K\beta - \gamma)SC - 2\alpha KS^2 \} \\ &+ \frac{1}{3}h \{ 2(C^3 - 1) [2\alpha(D - \frac{h}{K}) (\frac{k}{K} + \frac{1}{2}) - \frac{1}{K}D'(K\beta - \gamma)] \} \\ &- 2KS^3 [2\alpha D' (\frac{k}{K} + \frac{1}{2}) + (D - \frac{h}{K})(K\beta - \gamma)] \\ &+ S(1 + 2C^2) [2\alpha D' + (D - \frac{h}{K})(K\beta - \gamma) (\frac{k}{K} + \frac{1}{2})] \\ &- (3C - 1 - 2C^3) [2\alpha(D - \frac{h}{K}) - \frac{1}{K}D'(K\beta - \gamma) (\frac{k}{K} + \frac{1}{2})] \} \end{aligned} \quad (5.7)$$

$$\begin{aligned} \text{For } K = 0 : \int_{s_1}^{s_2} h(2kD\beta + 2D'\alpha - D\gamma) ds &= h \{ (2kD\beta + 2D'\alpha - D\gamma)\ell + (kD'\beta - 2kD\alpha + h\alpha \\ &- \frac{3}{2}D'\gamma)\ell^2 + (hk\beta - 4kD'\alpha + 2kD\gamma - \frac{5}{2}h\gamma)\frac{\ell^3}{3} + k(D'\gamma - h\alpha)\frac{\ell^4}{2} + hk\gamma\frac{\ell^5}{5} \} \end{aligned}$$

Entrance

In order to carry out the integration of eq. (2.14) over the fringe region, we insert k and r as given in chapter 4, eq. (4.4) and eq. (4.5) respectively. We must be careful with the functions α and D' because of their discontinuity at the edges:

$$\begin{aligned} \text{vertical: } \alpha &= \alpha_1 + \beta h \operatorname{tg}\theta_1 H(s) ; \text{ horizontal: } \alpha = \alpha_1 - \beta h \operatorname{tg}\theta_1 H(s) \\ D' &= D'_1 + Dh \operatorname{tg}\theta_1 H(s) \end{aligned} \quad (5.8)$$

But their average values $\langle \alpha \rangle$ and $\langle D' \rangle$, needed for the integration, are simply obtained by letting $\langle H(s) \rangle = \frac{1}{2}$ in eq. (5.8).

Inserting the eqs. (4.4) and (4.5) into the eq. (2.14), using eq. (2.6) and the hard edge approximation (where H becomes the step function and H' the delta function), one finds for the contribution of the entrance edge applying the relation (5.2)

$$\begin{aligned} \Delta \xi_{V_1} = & -\frac{1}{4\pi Q_V} \{ h\beta_1 \text{tg}\theta_1 - \tau h D_1 \beta_1 + \text{tg}\theta_1 [2k + h^2(1 + \text{tg}\theta_1)] D_1 \beta_1 \\ & - \frac{1}{2} h^2 D_1 \beta_1 \text{tg}^3 \theta_1 - h \text{tg}^2 \theta_1 \langle \frac{d}{ds}(D\beta) \rangle - h^2 D_1 \beta_1 \text{tg}\theta_1 - 2h\beta_1 \langle D' \rangle \\ & + \frac{1}{2} h \langle \frac{d}{ds}(D\beta) \rangle - \frac{1}{4} h^2 D_1 \beta_1 \text{tg}\theta_1 \} \end{aligned}$$

$$\begin{aligned} \Delta \xi_{H_1} = & -\frac{1}{4\pi Q_H} \{ -h\beta_1 \text{tg}\theta_1 + \tau h D_1 \beta_1 - \text{tg}\theta_1 [2k + h^2(1 + \text{tg}^2 \theta_1)] D_1 \beta_1 \\ & + \frac{1}{2} h^2 D_1 \beta_1 \text{tg}^3 \theta_1 + h \text{tg}^2 \theta_1 \langle \frac{d}{ds}(D\beta) \rangle + 2h^2 D\beta \text{tg}\theta_1 + \frac{1}{2} h \langle \frac{d}{ds}(D\beta) \rangle \\ & - \frac{1}{4} h^2 D_1 \beta_1 \text{tg}\theta_1 \} \end{aligned}$$

Carrying out the derivatives with respect to s and substituting the proper averages one obtains

$$\begin{aligned} \Delta \xi_{V_1} = & -\frac{1}{4\pi Q_V} \{ \tau_1 h \beta_1 D_1 + \beta_1 \text{tg}\theta_1 (h + 2k D_1) - h^2 \beta_1 D_1 \text{tg}\theta_1 (\frac{3}{2} - \text{tg}^2 \theta_1) \\ & - h \alpha_1 D_1 (1 - 2 \text{tg}^2 \theta_1) - h \beta_1 D_1' (\frac{3}{2} + \text{tg}^2 \theta_1) \} \quad (5.9) \end{aligned}$$

$$\begin{aligned} \Delta \xi_{H_1} = & -\frac{1}{4\pi Q_H} \{ \tau_1 h \beta_1 D_1 - \beta_1 \text{tg}\theta_1 (h + 2k D_1) + h^2 \beta_1 D_1 \text{tg}\theta_1 (\frac{3}{2} + \text{tg}^2 \theta_1) \\ & - h \alpha_1 D_1 (1 + 2 \text{tg}^2 \theta_1) + h \beta_1 D_1' (\frac{1}{2} + \text{tg}^2 \theta_1) \} \end{aligned}$$

Exit

The contribution of the exit edge reads, with the sign convention as defined in Fig. 3,

$$\Delta \epsilon_{V_2} = -\frac{1}{4\pi Q_V} \left\{ -\tau_2 h \beta_2 D_2 + \beta_2 \operatorname{tg} \theta_2 (h + 2kD_2) - h^2 \beta_2 D_2 \operatorname{tg} \theta_2 \left(\frac{3}{2} - \operatorname{tg}^2 \theta_2 \right) \right. \\ \left. + h \alpha_2 D_2 (1 - 2 \operatorname{tg}^2 \theta_2) + h \beta_2 D_2' \left(\frac{3}{2} + \operatorname{tg}^2 \theta_2 \right) \right\} \quad (5.10)$$

$$\Delta \epsilon_{H_2} = -\frac{1}{4\pi Q_H} \left\{ \tau_2 h \beta_2 D_2 - \beta_2 \operatorname{tg} \theta_2 (h + 2kD_2) + h^2 \beta_2 D_2 \operatorname{tg} \theta_2 \left(\frac{3}{2} + \operatorname{tg}^2 \theta_2 \right) \right. \\ \left. + h \alpha_2 D_2 (1 + 2 \operatorname{tg}^2 \theta_2) - h \beta_2 D_2' \left(\frac{1}{2} + \operatorname{tg}^2 \theta_2 \right) \right\}$$

6. A CHROMATICITY FORMULA FOR PRACTICAL CALCULATION

Combining the three formulae for the contributions of the entrance, eq. (5.8), the central part, eq. (5.3), and the exit, eq. (5.9), using eq. (5.7) and the equivalent relations for the end field region one arrives at a practical formula to calculate the chromaticity contribution of a magnet element in terms of the lattice functions at the reference orbit:

$$\Delta \epsilon_V = -\frac{1}{4\pi Q_V} \left\{ \int_{s_1}^{s_2} k \beta ds - \int_{s_1}^{s_2} r D \beta ds - \int_{s_1}^{s_2} h D (k + \gamma) ds \right. \\ \left. + \left[\operatorname{tg} \theta (h \beta + 2Dk\beta) - h \operatorname{tg}^2 \theta (\beta D' - 2\alpha D - h D \beta \operatorname{tg} \theta) - \beta h D' - \tau h \beta D \right] \Big|_{s_1}^{s_2} \right. \\ \left. + \left[\operatorname{tg} \theta (h \beta + 2Dk\beta) + h \operatorname{tg}^2 \theta (\beta D' - 2\alpha D + h D \beta \operatorname{tg} \theta) + \beta h D' - \tau h \beta D \right] \Big|_{s_1}^{s_2} \right\} \\ \Delta \epsilon_H = -\frac{1}{4\pi Q_H} \left\{ \int_{s_1}^{s_2} (h^2 - k) \beta ds + \int_{s_1}^{s_2} r D \beta ds + \int_{s_1}^{s_2} h (2kD\beta + 2D'\alpha - D\gamma) ds \right. \\ \left. + \left[-\operatorname{tg} \theta (h \beta + 2Dk\beta) + h \operatorname{tg}^2 \theta (\beta D' - 2\alpha D + h D \beta \operatorname{tg} \theta) + \tau h \beta D \right] \Big|_{s_1}^{s_2} \right. \\ \left. + \left[-\operatorname{tg} \theta (h \beta + 2Dk\beta) - h \operatorname{tg}^2 \theta (\beta D' - 2\alpha D - h D \beta \operatorname{tg} \theta) + \tau h \beta D \right] \Big|_{s_1}^{s_2} \right\} \quad (6.1)$$

where s_1 = beginning of the central part ($H = 1$)
 s_2 = end " " " " "
 $"1"$ = entrance of the fringe region ($H = 0$)
 $"2"$ = exit " " " "
 θ = entrance or exit angle of the trajectory
 $\frac{1}{\tau \cos^3 \theta}$ = radius of curvature of the end faces
 $h = \frac{1}{\rho}$ = curvature of the reference orbit

$$k = -\frac{1}{B\rho} \frac{\partial B_z(0,0,S)}{\partial x} \quad \text{is the quadrupole component}$$

$$r = -\frac{1}{B\rho} \frac{\partial^2 B_z(0,0,S)}{\partial x^2} \quad \text{is the sextupole component}$$

} of the magnet profile

D, D' are the dispersion function and its derivative
 α, β, γ are the Twiss functions.

The various integrals were already evaluated, see eqs. (5.4) to (5.7).

The sign convention for θ, τ and ρ are all positive as shown in the following figure:

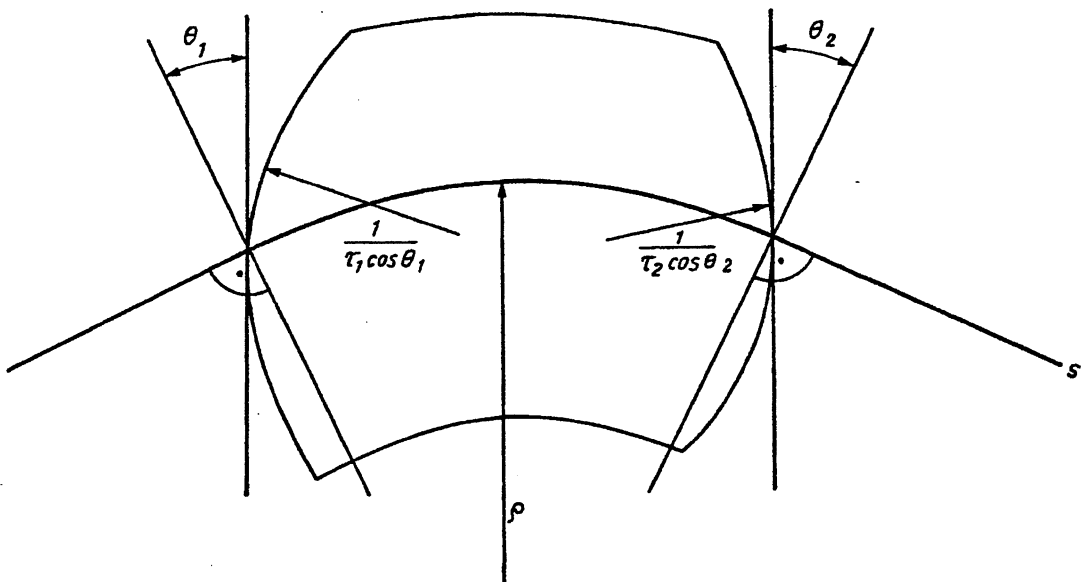


Fig. 3

Summing the contributions of all magnetic elements of the circumference one obtains the total chromaticity.

7. EXAMPLES

Eqs. (6.1) were evaluated to obtain the natural chromaticity of a number of small focusing rings. The results, summarized in Table 1, agree with the chromaticity obtained by the method of Peggs⁴⁾ which uses the program TRANSPORT³⁾. All other computer programs known to us gave significantly different results for our small rings where D/ρ is not negligible. Details about the selected machines may be seen from Table 1 and from the computer outputs attached.

8. REFERENCES

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Distribution (open)

Distribution (of abstract)

PS Scientific Staff

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TABLE 1: CHARACTERISTICS AND CHROMATICITIES FOR SOME SMALL RINGS

	Type of bending magnets	Circumference/ 2π R (m)	Maximum bending radius ρ (m)	Maximum dispersion D (m)	Nominal working point Q_H ; Q_V	Natural chromaticity ξ_H ; ξ_V
LEAR	sector	12.5	4.64	3.63	2.27 ; 2.78	-1.42; -2.69
TARN	sector	5.06	1.32	1.52	2.42 ; 2.26	-0.68; -0.63
AA	parallel ended	24.91	7.30	10.01	2.28 ; 2.28	-0.65; -1.50
PSR	parallel ended	14.36	4.06	2.41	3.25 ; 2.25	-0.82; -1.30
SIS12A	parallel ended	31.25	10.0	2.71	4.2 ; 3.4	-1.51; 1.43
BPS-BOOSTER	parallel ended	22.35	9.87	1.74	4.73 ; 4.80	-1.10; -1.47

APPENDIX

Outputs of the Twiss functions
for machines used as test cases

- LEAR, TARN
- AA, PSR, SIS12A
- BPSB

LOW ENERGY ANTI-PROTON RING (running-in)

***** CER N: L E A R (QH = 2.27, QV = 2.78)

CIRCUMFERENCE = 78.53982 4 HALF SUPERPERIODS

WEDGE MAGNETS ALL VALUES AT EXIT OF ELEMENTS

Table with columns: NO, ELEM, L(M), DIST(M), ANG(MR), K(1/H2), BETAV(M), RETAH(M), ALPHAV, ALPHAH, MUV/2PI, MUH/2PI, ALPHAP(M), ALPHAP I. Contains data for elements 1 through 12.

DP/P CUSMU(H) Q(H) QPRIME(H) RETAMAX(H) COSMU(V) Q(V) QPRIME(V) BETAMAX(V) XMAX(H) GAMMA TR.

0.0000 -0.91140 2.2700 -2.7987 12.5258 -0.33874 2.7800 -5.7822 21.3860 3.6340 -6.5412

* edge focusing to account for stray fields at end of magnet blocks, kl = -0.00257, 0.00324, 0.02444 for the elements 6, 8, 11 respectively.

***** TOKYO: T A R N (QH = 2.42147, QV = 2.25941)

CIRCUMFERENCE = 31.79488 8 SUPERPERIODS

WEDGE MAGNETS ALL VALUES AT EXIT OF ELEMENTS

Table with columns: NO, ELEM, L(M), DIST(M), ANG(MR), K(1/H2), BETAV(M), RETAH(M), ALPHAV, ALPHAH, MUV/2PI, MUH/2PI, ALPHAP(M), ALPHAP I. Contains data for elements 1 through 7.

DP/P CUSMU(H) Q(H) QPRIME(H) RETAMAX(H) COSMU(V) Q(V) QPRIME(V) BETAMAX(V) XMAX(H) GAMMA TR.

0.0000 -0.32501 2.4215 -2.5552 5.1616 -0.20233 2.2594 -2.3667 5.4240 1.5216 2.0037

ANTIPROTON ACCUMULATOR * (operating)

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***** CERN: A A (QH = 2.2811, QV = 2.2834)  
CIRCUMFERENCE = 156.49240           2 HALF SUPERPERIODS  
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
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NO	ELEM	L(M)	DIST(M)	ANG(CMR)	K(1/MS2)	BETA(H)	BETA(M)	ALPHAV	ALPHA	MUV/2PI	MUH/2PI	ALPHAP(M)	ALPHAP1
INITIAL													
1	DD1H	3750	0.0000	0.00000	0.00000	9.26265	7.35956	0.00000	0.00000	0.00000	0.00000	0.0205	0.00000
2	S3	7500	3750	0.00000	1569347	9.07481	7.54242	49724	-49122	0.06649	0.08004	0.0207	0.00121
3	GF1	4000	6.1455	0.00000	0.00000	7.91270	18.09176	29586	-44050	1.2571	0.08842	0.0276	0.00121
4	SA	4000	6.1455	0.00000	0.00000	9.46592	18.09176	85748	-1.54633	1.3980	0.09509	0.0269	0.00322
5	SA	4000	6.1455	0.00000	0.00000	13.92835	17.40935	47548	1.54633	1.4604	0.09863	0.0256	0.00322
6	S2	4000	12.6661	0.00000	0.00000	33.39899	7.70763	2.04566	1.63304	1.8960	0.16357	0.0281	0.00787
7	SD2	4000	11.9186	0.00000	1569347	34.51013	7.22762	1.40513	56309	1.8773	1.7211	1.154	0.70787
8	GF2	7500	11.9186	0.00000	0.00000	33.51521	7.06496	2.60185	38104	1.9306	1.6920	2.154	0.96766
9	GF2	7500	15.0356	0.00000	0.00000	11.73092	16.89124	1.26619	-31711	1.4100	1.8749	1.1917	0.96766
10	S3	7500	15.0356	0.00000	0.00000	10.29897	16.89124	56607	-1.31745	2.2808	2.444	2.2021	3.68414
11	SD1	3750	26.4571	0.00000	1569347	21.33051	6.85871	1.22594	38323	3.1435	3.7623	6.21042	3.68414
12	RI	3750	26.4571	0.00000	0.00000	21.04532	7.16978	1.8164	51459	3.9433	4.0297	9.35162	3.58152
13	RI	3750	29.7246	445.38000	0.00000	21.42533	7.16978	1.8164	51459	3.9433	4.0297	9.35162	3.58152
14	SD2	2500	52.2270	0.00000	0.00000	15.17883	17.99221	1.82692	-1.80890	4.0606	4.5189	7.96544	8.10728
15	SD3	2500	52.2270	0.00000	0.00000	4.69162	18.63062	1.3036	-1.41651	4.3074	4.8503	10.01459	8.10728
16	S2	2500	35.4806	0.00000	0.00000	6.69346	12.66973	1.3036	1.41651	4.3074	4.8503	10.01459	8.10728
17	SI	3775	38.3708	445.38000	0.00000	10.77346	12.66973	58342	1.56530	5.6012	5.5508	7.93475	8.14436
18	SI	3775	38.3708	445.38000	0.00000	11.19229	12.66973	58342	1.56530	5.6012	5.5508	7.93475	8.14436
19	DD4H	3750	39.1231	0.00000	1569347	11.43117	7.65176	0.00000	5.0670	5.6566	5.6255	6.19397	3.61860
DP/P	CUSHU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	COSH(U)	Q(V)	QPRIME(V)	BETAMAX(V)	GAMMA TR.	XMAX(H)	GAMMA TR.		
0.0000	63466	2.2811	-2.7670	19.7566	.62908	2.2834	-1.2581	34.5101		10.0147	2.4287		

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* Note: a simplified model of the AA with straight parallel ended magnets is used in the present calculation.

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***** LUS ALAMOS SL: P S R (QH = 3.25, QV = 2.25)  
CIRCUMFERENCE = 90.2240           10 SUPERPERIODS  
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
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NO	ELEM	L(M)	DIST(M)	ANG(CMR)	K(1/MS2)	BETA(H)	BETA(M)	ALPHAV	ALPHA	MUV/2PI	MUH/2PI	ALPHAP(M)	ALPHAP1
INITIAL													
1	SL	2.2865	0.0000	0.00000	0.00000	6.62673	5.02783	-1.12290	1.27794	0.00000	0.00000	1.82525	-254888
2	SD	4500	2.2865	0.00000	4510538	13.54534	1.92182	1.90300	0.8050	0.3879	0.3154	1.24246	-254888
3	SD	4500	3.2365	0.00000	0.00000	9.19774	2.19604	1.18255	0.4040	0.4448	0.17142	1.8332	0.16108
4	SD	4500	3.2365	0.00000	0.00000	12.80034	2.91159	1.10501	0.4073	0.4983	0.19990	1.9057	0.16108
5	BEND	2.5495	5.2359	628.31850	0.00000	4.80021	11.08146	1.42041	-2.48478	0.0236	0.6886	2.00392	665948
6	SD	4500	6.2359	0.00000	0.00000	3.64914	13.44886	1.13753	-2.77611	1.1951	0.27473	2.30360	665948
7	SD	2.2865	9.0224	0.00000	0.00000	3.27544	13.40967	1.34281	-2.47539	1.4336	0.28043	2.40804	-254888
DP/P	CUSHU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	COSH(U)	Q(V)	QPRIME(V)	BETAMAX(V)	GAMMA TR.	XMAX(H)	GAMMA TR.		
0.0000	-45399	3.2500	-4.5658	13.6097	.15643	2.2500	-1.4867	13.9197		2.4080	3.1370		

XX

***** DARMSTADT: S I S 12A (QH = 4.2, QV = 3.4)
CIRCUMFERENCE = 196.34999 12 SUPERPERIODS
XX

SCHWERIONEN INJEKTOR SYNCHROTRON (planned)

STRAIGHT MAGNETS ALL VALUES AT EXIT OF ELEMENTS
XX

NO	ELEM	L(M)	DIST(M)	ANG(MR)	K(1/M ²)	BETAV(M)	BETAH(M)	ALPHAV	ALPHAH	MUV/2PI	MUH/2PI	ALPHAP(M)	ALPHAP1
	INITIAL		0.0000			18.20022	5.47556	.80180	.67284	0.00000	0.00000	1.23869	-.249882
1	BEND	2.6180	3.6180	261.79940	0.0000000	13.53241	3.76990	.91972	-.01383	.02667	.09646	.93269	.013423
3	BEND	2.6180	6.2360	0.00000	0.0000000	12.39441	3.89997	.81106	-.18628	.03467	.12357	.94142	.013423
4	L3	6.2565	12.4725	0.00000	0.0000000	18.28832	9.64146	.68333	-.47295	.07625	.20848	1.31690	.276728
5	QYF	.9900	13.4625	0.00000	0.0000000	6.66561	27.95001	-.42402	-.33287	.23548	.28442	2.71329	.276728
6	QYF	.5100	13.9725	0.00000	-.4091095	10.99608	22.16806	-4.51956	-.57092	.26129	.29035	2.71329	.276728
7	QYF	.9900	14.9625	0.00000	0.0000000	16.11285	15.12999	-5.51331	6.23923	.26977	.30943	2.71329	.276728
8	DRF	1.7300	16.6925	0.00000	0.3621130	21.24661	18.59764	-5.95790	1.35183	.28333	.35000	1.67099	-.249882
	DP/P	COSMU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	COSMU(V)	Q(V)	QPRIME(V)	BETAMAX(V)	XMAX(H)	XMAX(V)	GAMMA TR.	
0.0000	-.58779		4.2000	-6.7841	27.9500	-.20791	3.4000	-3.6080	21.2446	3.0483	3.0483	5.4555	

XX