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LUMINOSITY AT VERY LOW BETAS

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1. Luminosity at very low betas

When the bunch lengths are not short compared to the beta values, then the approximation of cylindrical beams over the whole interaction region which is assumed in Ref. 1) is no more valid and the integrals must extend over the longitudinal coordinate as well :

$$\mathcal{L} = \frac{v}{2\pi R\ V}\ N_1 N_2 \ , \tag{1}$$

where

$$\frac{1}{V} = 2\ \frac{\int v dt \iiint i_1(x,y,\ s+vt)\ \ i_2(x,y,\ s-vt)\ dx\ dy\ ds}{\iiint\ i_1(x,y,s)\ dx\ dy\ ds \cdot \iiint\ i_2(x,y,s)\ dx\ dy\ ds} . \tag{2}$$

V is the interaction volume in 3-dimensional space, defined by the vertex distribution obtained through integration over the time variable t .

Assuming no coupling between the motions in x,y and s and using the effective width and height defined in Ref. 1) one can write the luminosity as :

$$\mathcal{L} = \frac{c\ N_1 N_2}{2\pi R} \int \frac{\sqrt{\beta_{xo}}}{w\ \sqrt{\beta_x(s)}}\ \frac{\sqrt{\beta_{yo}}}{h\ \sqrt{\beta_y(s)}}\ g(s)\ ds \ , \tag{3}$$

where the $\sqrt{\beta}$ take care of the beam size variations around the focus, and $g(s)\ ds$ is the longitudinal crossings distribution shown in Fig. 1,a and worked out in Appendix A,

$$g(s)\ ds = \int f_1(s-t)\ \ f_2(s+t)\ dt\ ds \ , \tag{4}$$

where f_1 and f_2 are the longitudinal bunch profiles.

The variance of g is related to the variances of f_1 and f_2 in a simple way

$$V_g = \frac{1}{4} (V_{f_1} + V_{f_2}) , \tag{5}$$

(see Appendix A), which can be demonstrated in an example.

Let us assume that in the p-$\bar{p}$ collider, protons and antiprotons have bunches of 2 ns full length, with a parabolic distribution. Then

$$\sigma_{bunch} = \sqrt{V_{f_{1,2}}} = \frac{2 \text{ ns}}{2\sqrt{5}} = 0.134 \text{ m}$$

and

$$\sigma_{crossings} = \sigma_{bunch}/\sqrt{2} = 0.095 \text{ m} .$$

When the bunch length is not small compared to the beta value at the crossing point, then one should distinguish the interaction rate from the crossings rate and express the vertex distribution by :

$$h(s)\, ds = \sqrt{\frac{\beta_{xo}\, \beta_{yo}}{\beta_x(s)\, \beta_y(s)}}\; g(s)\, ds . \tag{6}$$

This new distribution shows, away from the centre, a reduction of the number of interactions.

The reduction of luminosity due to the divergences of the beams (shown by the beta functions) may be expressed as follows :

$$R(k) = \int \frac{g(s)\, ds}{\sqrt{\beta_x(s)\, \beta_y(s)}} \Bigg/ \int \frac{g(s)\, ds}{\sqrt{\beta_{xo}\, \beta_{yo}}} , \tag{7}$$

where k is a parameter proportional to the length of the interaction region and inversely proportional to the beam divergeance.

2. Numerical application

In order to get a numerical estimate for R(k) we are choosing a parabolic distribution for the bunches,

$$f(s) = \begin{cases} 1 - \left(\frac{s}{\ell}\right)^2 ds & , |s| \leq \ell \\ 0 & , |s| > \ell, \end{cases} \tag{8}$$

with $\quad V_f = \ell^2/5$.

The crossings distribution is obtained by introducing the above distribution into Equ. 4. Since the bunches are symmetric the crossings distribution will also be symmetric and it is enough to solve Equ. 4 for $s \geq 0$ only; hence,

$$g(s) = \int_{\frac{s}{\ell}-1}^{1-\frac{s}{\ell}} \left[1 - \left(\frac{s}{\ell} + t\right)^2\right] \left[1 - \left(\frac{s}{\ell} - t\right)^2\right] dt \ , \ s \geq 0 \tag{9}$$

which can be integrated to give

$$g(s) = \begin{cases} 1 - 5\left(\frac{s}{\ell}\right)^2 + 5\left(\frac{s}{\ell}\right)^3 - \left(\frac{s}{\ell}\right)^5 & , \ 0 \leq s \leq \ell \\ 0 & , \ s > \ell \end{cases} \tag{10}$$

with $V_g = \ell^2/10$.

The functions f(s) and g(s) are shown on Fig. 1, b.

In order for Equ. 7 to be easely integrated, we further assume that $\beta_x(s) \approx \beta_y(s)$ so that we can call $\beta(s) \approx \sqrt{\beta_x(s)\ \beta_y(s)}$ and describe the s-dependence in the interaction region by $\beta(s) = \beta_o + s^2/\beta_o$.

The luminosity reduction then reads

$$R(k) = \int_0^{\ell} \frac{g(s)\ ds}{1 + s^2/\beta_0^2} \Bigg/ \int_0^{\ell} g(s)\ ds \ . \tag{11}$$

This last integral gives

$$\int_0^{\ell} \left[1 - 5\left(\frac{s}{\ell}\right)^2 + 5\left(\frac{s}{\ell}\right)^3 - \left(\frac{s}{\ell}\right)^5\right] ds = \frac{5\ell}{12} , \tag{12}$$

and Equ. 11 can be rewritten

$$R(k) = \frac{12}{5k} \int_0^{k} \left[1 - \frac{5}{k^2} x^2 + \frac{5}{k^3} x^3 - \frac{x^5}{k^5}\right] \frac{dx}{1 + x^2} , \tag{13}$$

where $x = s/\ell$ and $k = \ell/\beta_o$.

Equ. 13 can be integrated by the use of standard integrals and yields :

$$R(k) = \frac{12}{5} \left\{ \left(1 + \frac{5}{k^2}\right) \frac{\text{atan } k}{k} - \frac{1}{2k^4} \left(5 + \frac{1}{k^2}\right) \left[\ell n \left(1 + k^2\right) - k\right] - \frac{21}{4k^2} \right\} . \tag{14}$$

Fig. 2 shows the importance of the effect in terms of the parameter $k = \ell/\beta_o$. Note that $\ell = \sqrt{\ell_1^2/2 + \ell_2^2/2}$ is the quadratic mean of the half bunch length and putting $\beta_o \approx \sqrt{\beta_{xo}\,\beta_{yo}}$ is an approximation which is, of course, the better the closser β_{xo} and β_{yo} are.

3. Acknowledgements

I should like to thank G. von Holtey and C. Iselin for useful discussions.

Reference

C. Rubbia, "Measurement of the Luminosity of $p\bar{p}$ collider ..." CERN $p\bar{p}$ Note 38, November 14, 1977.

APPENDIX A : Crossings distribution g(s) ds

Let us assume that the longitudinal distribution of the two bunches, shown on Fig. 1,a are given by :

$$f_1(s)\ ds \quad \text{and} \quad f_2(s)\ ds\ ,$$

with the following moments :

$$\int_{-\infty}^{+\infty} f_1(s)\ ds = \int_{-\infty}^{+\infty} f_2(s)\ ds = 1$$

$$\bar{s}_1 = \int_{-\infty}^{+\infty} f_1(s)\ s\ ds\ , \quad \bar{s}_2 = \int_{-\infty}^{+\infty} f_2(s)\ s\ ds \tag{A1}$$

$$\overline{s_1^2} = \int_{-\infty}^{+\infty} f_1(s)\ s^2\ ds\ , \quad \overline{s_2^2} = \int_{-\infty}^{+\infty} f_2(s)\ s^2\ ds\ ,$$

and the variances :

$$V_{f_1} = \overline{s_1^2} - \bar{s}_1^2\ , \quad V_{f_2} = \overline{s_2^2} - \bar{s}_2^2$$

Then the longitudinal distribution of the points where particles will cross each other is given by

$$g(s)\ ds = \left[v \int_{-\infty}^{+\infty} f_1(s - vt)\ f_2(s + vt)\ dt \right] ds\ , \tag{A2}$$

which is a convolution product but with unusual variables and normalization. Let us first compute the norm n , adopting a time like variable t instead of vt for the following integration :

$$n = \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} f_1(s - t)\ \ f_2(s + t)\ dt\ . \tag{A3}$$

With a first substitution : $u = s + t$, $du = dt$, one has

$$n = \int_{-\infty}^{+\infty} f_2(u)\, du \int_{-\infty}^{+\infty} f_1(2s - u)\, ds \, , \tag{A4}$$

and by further use of $w = 2s - u$, $dw = 2\, ds$, one gets

$$\boxed{n = \frac{1}{2}} \tag{A5}$$

For the first moment of $g(s)\, ds$ we have

$$\bar{s} = \frac{1}{n} \int_{-\infty}^{+\infty} g(s)\, s\, ds \, , \text{ or}$$

$$\bar{s} = 2 \int_{-\infty}^{+\infty} s\, ds \int_{\infty}^{\infty} f_1(s - t)\, f_2(s + t)\, dt \, , \tag{A6}$$

which by substituting : $u = s + t$, $du = dt$, becomes

$$\bar{s} = 2 \int_{-\infty}^{+\infty} f_2(u)\, du \int_{-\infty}^{+\infty} f_1(2s - u)\, s\, ds \, ; \tag{A7}$$

and by : $w = 2s - u$, $dw = 2\, ds$, we get

$$\bar{s} = \frac{1}{2} \int_{-\infty}^{+\infty} f_2(u)\, du \int_{-\infty}^{+\infty} f_1(w)\, (u + w)\, dw \, , \tag{A8}$$

and finally, using Equ. A1, we have

$$\boxed{\bar{s} = \frac{1}{2} (\overline{s_1} + \overline{s_2})} \, . \tag{A9}$$

For the second moment,

$$\overline{s^2} = \frac{1}{n} \int_{-\infty}^{+\infty} g(s)\, s^2 ds \, ,$$

a similar algebraic sequence leads to

$$\overline{s^2} = \frac{1}{4} (\overline{s_1^2} + \overline{s_2^2} + 2\, \overline{s_1}\, \overline{s_2}) \, . \tag{A10}$$

And from Equ. A9 and Equ. A10 the variance of g can be computed

$$\boxed{V_g = \frac{1}{4}\left(V_{f_1} + V_{f_2}\right)} \, . \tag{A11}$$

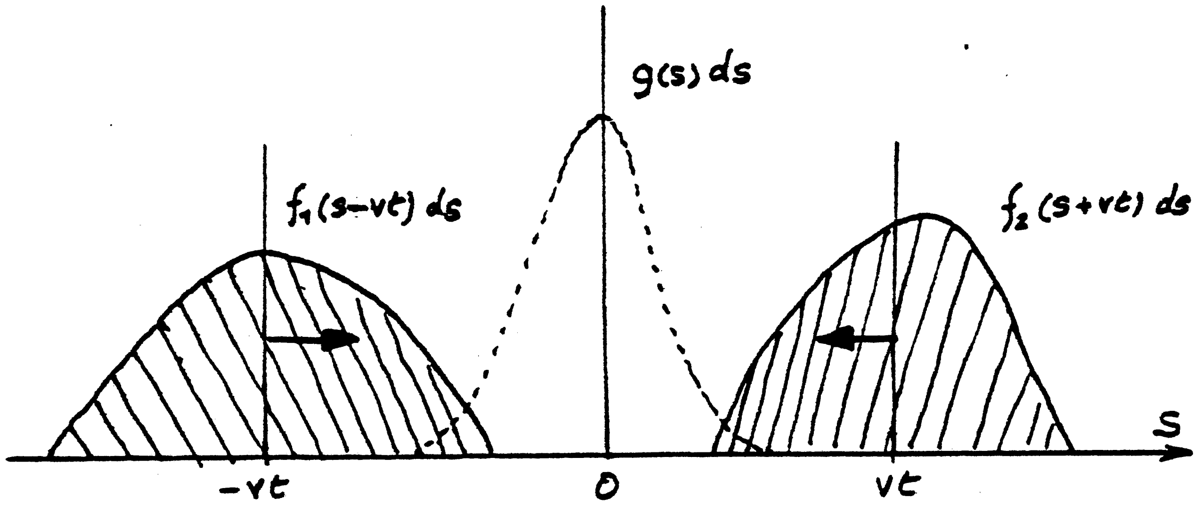

Fig. 1, a

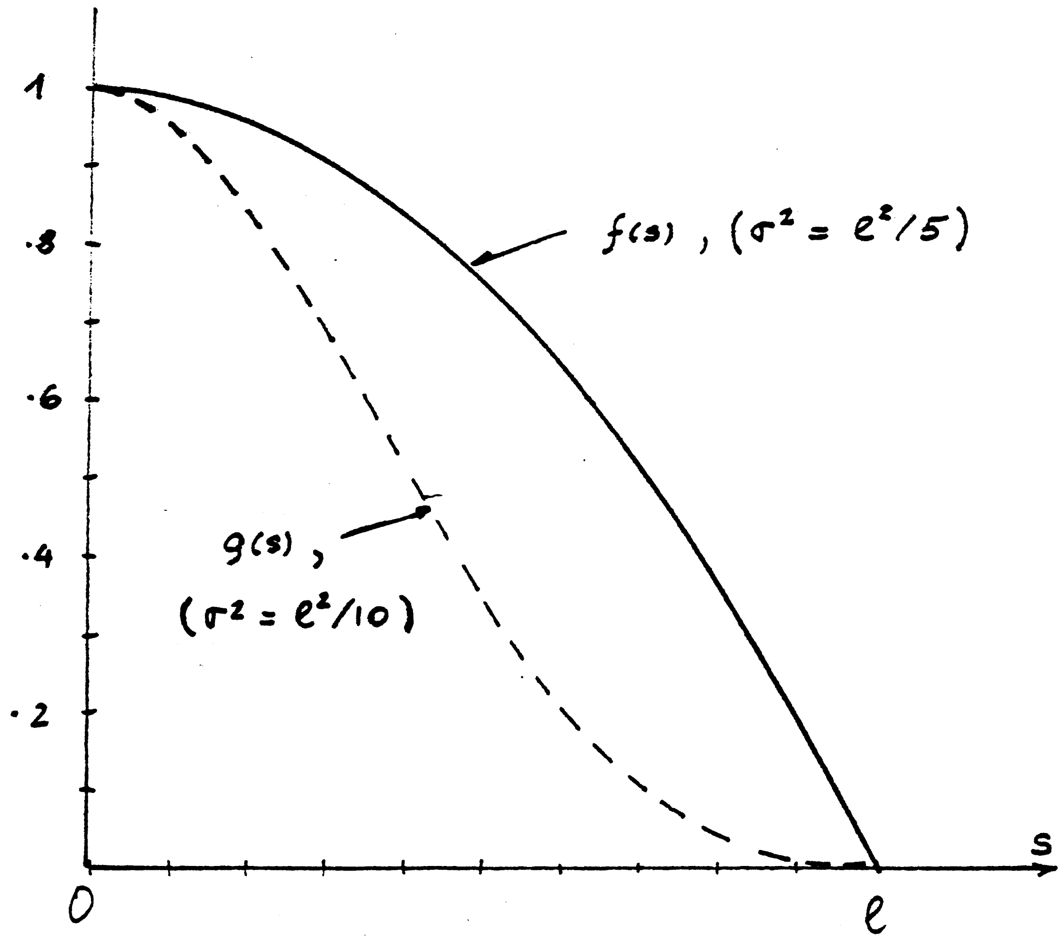

Fig. 1, b

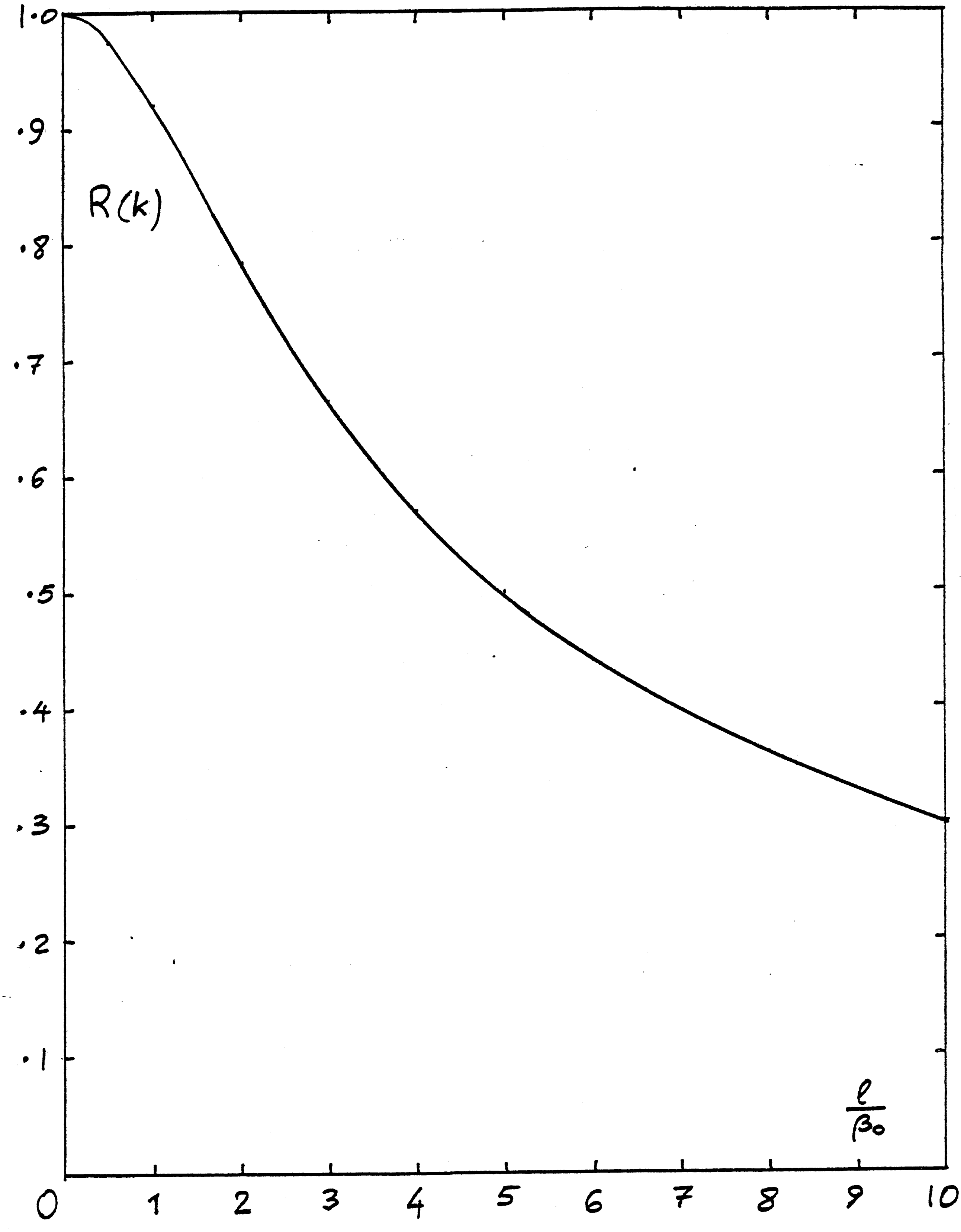

Fig. 2