Convex hull alignment through translation

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Abstract

Given k finite point sets $A_1, \dots, A_k$ in $\mathbb{R}^2$, we are interested in finding one translation for each point set such that the union of the translated point sets is in convex position. We show that if k is part of the input, then it is NP-hard to determine if such translations exist, even when each point set has at most three points.

The original motivation of this problem comes from the question of whether a given triangulation T of a point set is the *empty shape triangulation* with respect to some (strictly convex) shape S. In other words, we want to find a shape S such that the triangles of T are precisely those triangles about which we can circumscribe a homothetic copy of S that does not contain any other vertices of T. This is the Delaunay criterion with respect to S; for the usual Delaunay triangulation, S is the circle.

1 Introduction

We study the following problem: given k finite point sets $A_1, \dots, A_k$ in $\mathbb{R}^2$, are there translations $t_1, \dots, t_k$ such that the union of all $t_i(A_i)$ is in convex position? For $k = 1$ the problem is simply convexity testing. For $k = 2$, the problem can be solved using linear programming, under the additional assumption that the order of points along the convex hull is fixed. Even without this assumption, the case $k = 2$ is solvable in polynomial time: if there is a solution for a given instance, then there is also a solution where a point p from the first set is collinear with two points q and r from the second set. For each such triple, it remains only to determine where p should be placed along the line qr, which can easily be done in polynomial time.

For general k (being part of the input), the straightforward formulation does not yield a linear program, because triples of points may come from different polygons and, thus, involve several translations, which makes the constraints quadratic. Similarly, the problem is easy if the size of the point sets is at most two: sort the line segments $s_1, \dots, s_n$ by increasing slope and translate such that the right endpoint of each s_i is identified with the left endpoint of s_{i+1}. In contrast we prove that the general problem is NP-hard by reduction from 3-SAT:

Theorem 1 *Given k finite point sets $A_1, \dots, A_k$ in $\mathbb{R}^2$, it is NP-hard to decide if there are translations $t_1, \dots, t_k$ such that the union of all $t_i(A_i)$ is in convex position.*

The reduction uses point sets of size three, as well as a regular polygon with size linear in the size of the 3-SAT formula. We also show that this regular polygon can be replaced by a set of triangles:

Theorem 2 *Given k finite point sets $A_1, \dots, A_k$ in $\mathbb{R}^2$, it is NP-hard to decide if there are translations $t_1, \dots, t_k$ such that the union of all $t_i(A_i)$ is in convex position, even if each A_i has size at most three.*

Motivation. The original motivation of this problem comes from the question of whether a given triangulation T of a point set is the *empty shape triangulation* [10, 11] with respect to some (strictly convex) shape S (Problem A). In other words, we want to find a shape S such that the triangles of T are precisely those triangles about which we can circumscribe a homothetic copy of S that does not contain any other vertices of T. This is the Delaunay criterion with respect to S. For the usual Delaunay triangulation, S is the circle.

An abstraction of this question is the following more general problem (Problem B). We are given families $(P_i^+, P_i^=, P_i^-)_{i=1,2,\dots}$ of point sets. We look for a (strictly) convex shape S with the following property: for each i, we can scale and translate S so that the three sets $P_i^+, P_i^=, P_i^-$ lie inside S, on the boundary, and outside S, respectively.

In Problem A, each $P_i^=$ is a triplet corresponding to a triangle, and P_i^- is the complementary set of points. P_i^+

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is always empty. Equivalently, we may form quadruples with three triangle points $P_i^=$ and each remaining point as a singleton set P_i^-.

We obtain a variation of this problem (Problem B′) by allowing only translation of S, but no scaling. Let Problem C be the special case of Problem B where $P_i^+ = P_i^- = \emptyset$ and let Problem C′ be the same special case of Problem B′. The problem that we consider in this paper is problem C′. Thus, our answer to problem C′ does not imply an answer to our original question, since it is *more specialized* in one respect (allowing translations only) and *more general* in another respect (considering arbitrary point sets $P_i^=$).

Related work. We are not aware of previous work on the related problems mentioned above. Regarding our main motivation, Problem A, quite some work has been devoted to studying properties of Voronoi diagrams and their duals—often triangulations—based on a particular convex distance function (see e.g. [6, 12]), but not, to the best of our knowledge, to tackling the inverse question in which we are interested here.

The problem of finding a set of translations to place a set of points in convex position is related to certain matching and polygon placement problems. In some matching problems the goal is to find a rigid motion of one shape to make it as *similar* as possible to another shape. Here, similarity can be measured in a variety of ways, such as using Hausdorff distance [2], Fréchet distance [3] or maximizing their area of overlap [7], among others. Polygon placement problems are usually concerned with finding a transformation of a polygon to place it inside another polygon or to contain/avoid certain objects (such as points or other polygons). Multiple variants have been studied in the past, depending on the type of polygon (e.g. convex [14] or simple [4]), the type of transformation (e.g. translation versus translation and rotation [4]), and the final goal (e.g. to fit the largest possible copy of a polygon inside another one [1], or to cover as many points as possible [8]). We remark that, besides having clearly different goals, these problems are usually concerned with two single shapes or objects, whereas we are interested in translating $k > 2$ points sets altogether.

2 Reduction from 3-SAT

This section is devoted to the proof of Theorem 1. We prove the theorem by reduction from 3-SAT. Given a 3-SAT formula F with variables $x_0, \ldots, x_{n-1}$ and clauses $C_0, \ldots, C_{m-1}$, construct point sets as follows.

Let $R = r_0, r'_0, \ldots, r_{t-1}, r'_{t-1}$ be a regular[1] convex polygon on $2t = 8n + 16m$ points centered at the origin. Given an edge $r_i r'_i$ of R, we define the *pocket* P_i of $r_i r'_i$ to be the compact set bounded by the line segment $r_i r'_i$ and the lines ℓ^- through $r'_{i-1} r_i$ and ℓ^+ through $r'_i r_{i+1}$. The intersection of ℓ^- and ℓ^+ is the *apex* of pocket P_i. Our reduction relies on the following fact: if a point p is placed on an open line segment $r_i r'_i$, then placing a point q in $P_i \setminus r_i r'_i$ destroys convexity of $R \cup \{p, q\}$. We say that p *blocks* the pocket P_i. Unblocked pockets are called *free*. Note that placing a point in P_i does not prevent us from placing a point in any other pocket. We say that two convex sets are *compatible* if their union is convex.

Variable gadget. In order to encode the variables of our formula, we will first define triangles $Q_i^d = (q_i, q_{i+d}, q_{i+t/2})$ for $0 \le i < t$ and $0 \le d < t$ as follows (note that in this section all indices are taken modulo t, and that t is even). Consider the diametrically opposed segments $r_i r'_i$ and $r_{i+t/2} r'_{i+t/2}$. Place $q_{i+t/2}$ on $r_{i+t/2}$ and place q_{i+d} on the apex of P_{i+d} (Figure 1). Translate $q_{i+t/2}$ and q_{i+d} rigidly, parallel along $r_{i+t/2} r'_{i+t/2}$ until q_{i+d} hits the line segment $r_{i+d} r'_{i+d}$. Now place q_i on r_i. Note that Q_i^d can slide freely along $r_i r'_i$ until:

(1) q_i hits r_i and q_{i+d} hits the line segment $r_{i+d} r'_{i+d}$: moving further would move q_{i+d} inside the convex hull of R; or

(2) $q_{i+t/2}$ hits $r_{i+t/2}$ and q_{i+d} hits the apex of P_{i+d}: moving further would push q_{i+d} outside the pocket, removing r_{i+d} and r'_{i+d} from the joint convex hull.

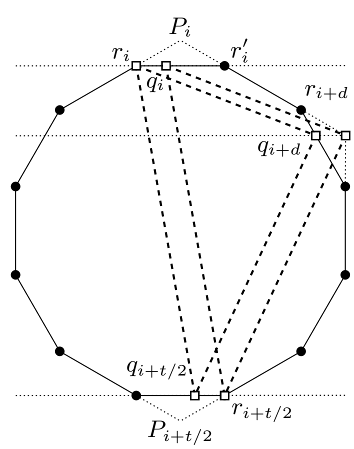

Figure 1: The construction of variable gadget Q_i^d. The figure shows the two extreme positions of the gadget.

Since $r_i r'_i$ and $r_{i+t/2} r'_{i+t/2}$ are diametrically opposed segments, moving Q_i^d in any other direction would place either q_i or $q_{i+t/2}$ inside the convex hull of R. For most of the compatible translations, the pockets P_i and

[1] A sufficiently fine rational approximation [5] of R suffices. The main property of R that is important for our reduction is that opposite sides are parallel.

$P_{i+t/2}$ are both blocked. However, for the two extreme positions (shown in Figure 1), where q_{i+d} touches the apex or the line segment, exactly one of the pockets is free.

We say that the *state* of Q_i^d is *true* if Q_i^d does not block P_i, *false* if it does not block $P_{i+t/2}$ and *undefined* if P_i and $P_{i+t/2}$ are both blocked. After defining the other two gadgets, we will associate some Q_j^d with each variable x_i, with the interpretation that x_i is true if and only if the state of Q_j^d is true and $\bar{x}_i$ is true if and only if the state of Q_j^d is false. If the state of Q_j^d is undefined, then both x_i and $\bar{x}_i$ are false. Note that if R and a translation of Q_i^d are compatible, then x_i and $\bar{x}_i$ are not both true.

Copy gadget. Given a variable gadget Q_i^d, we can copy the state of Q_i^d with the gadget Q_{i+k}^{d-k}, provided $k \notin \{0, t/2, d\}$. This can be seen as follows. Assume for the moment that Q_i^d is in one of its two extreme positions. The gadget Q_{i+k}^{d-k} shares the pocket P_{i+d} with Q_i^d (see Figure 2). If Q_i^d touches the apex of P_{i+d} (as in Figure 2), then so must Q_{i+k}^{d-k} in order to be compatible with R and the translation of Q_i^d. Similarly, if Q_i^d touches the open line segment $r_{i+d}r'_{i+d}$, then so must Q_{i+k}^{d-k}. Hence, the state of Q_{i+k}^{d-k} is completely determined by the state of Q_i^d. Specifically, the state of P_i is copied to P_{i+k} and the state of $P_{i+t/2}$ is copied to $P_{i+k+t/2}$. If Q_i^d is not in an extreme position, then neither is Q_{i+k}^{d-k} and hence both states are undefined.

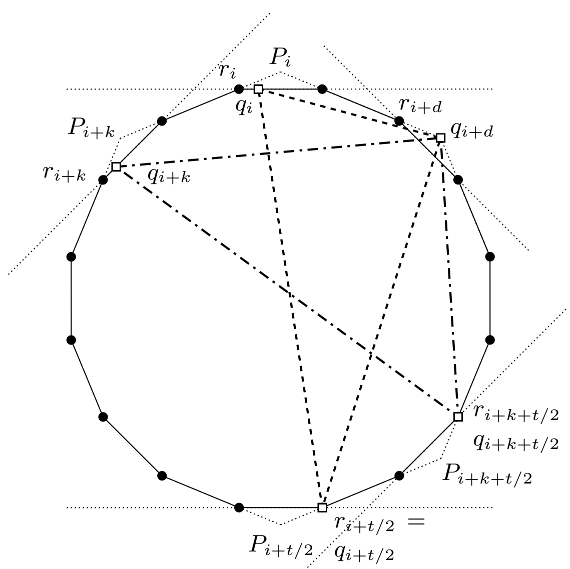

Figure 2: The construction of copy gadget Q_{i+k}^{d-k} with $k = -1$.

Clause gadget. Given i, j and k, let T_{ijk} be the triangle whose vertices t_i, t_j, t_k are the midpoints of the segments $r_ir'_i$, $r_jr'_j$ and $r_kr'_k$, respectively. For each clause C on variables x, y, z, we will use three copy gadgets to copy the states of the literals x (or $\bar{x}$), y (or $\bar{y}$) and z (or $\bar{z}$) of C to pockets P_i, P_j, P_k such that T_{ijk} contains the origin. Some care must be taken to ensure that this clause does not interfere with existing gadgets, but we will cover this issue in the subsection below. Figure 3 shows an example for a clause $C = x \vee \bar{y} \vee z$. Only the relevant corner of each copy gadget is shown.

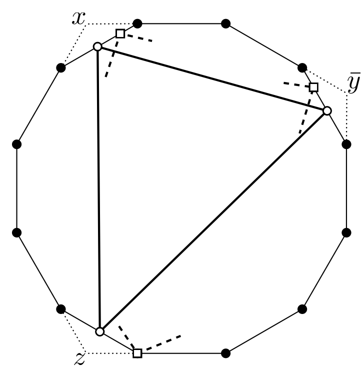

Figure 3: The construction of a clause gadget for a clause $C = x \vee \bar{y} \vee z$.

Note that x and $\bar{y}$ are both in the false (or undefined) state, whereas z is in the true state. We now scale T_{ijk} by a factor of $1 + \varepsilon$ for ε sufficiently small. Since triangle T_{ijk} contains the origin, it no longer fits inside R. If it did not contain the origin, a small translation towards the origin would potentially bring it back in convex position with R. If one of the pockets P_i, P_j or P_k is free, e.g. P_i, then we can translate T_{ijk} such that t_j is again on $r_jr'_j$ and t_k is again on $r_kr'_k$ and t_i is inside P_i. This translation of T_{ijk} is compatible with R: hence, the clause C is satisfied. If all three pockets are blocked, then there is no translation for which T_{ijk} is compatible with R and hence C is not satisfied.

Selecting the gadgets. We are now ready to explicitly define the gadgets to be used for the given 3-SAT formula F with variables $x_0, \ldots, x_{n-1}$ and clauses $C_0, \ldots, C_{m-1}$. We partition the polygon into paths $A, B'_1, B_2, Q, B'_3, X, A', B_1, B'_2, Q', B_3, X'$ as shown in Figure 4. The paths A, Q, A' and Q' all have length $2n$. The other paths have length $2m$. Hence R has size $2t = 8n + 16m$. We will not use the paths X, Q' and X'. Associate with each variable x_i the variable gadget Q_i^{2m+n}. This places the variable gadgets in paths A, Q and A'. For each clause $C_i = \{\ell_a, \ell_b, \ell_c\}$ add copy gadgets as follows. If $\ell_a = \bar{x_a}$ (i.e. the literal is negative), then add Q_{i+n}^{a+2m-i} to copy the value of x_a to the ith pocket in B'_1. This copies the value of $\bar{x_a}$ into the ith pocket in B_1. Alternatively, if $\ell_a = x_a$ (i.e. the literal is positive), then add $Q_{i+3n+4m}^{a-2n-2m-i}$ to copy the value of x_a to the ith pocket in B_1. Copy ℓ_b and ℓ_c analogously to

B_2 and B_3 and add the clause gadget corresponding to the selected pocket. Note that any triangle of vertices selected from B_1, B_2, B_3 contains the origin.

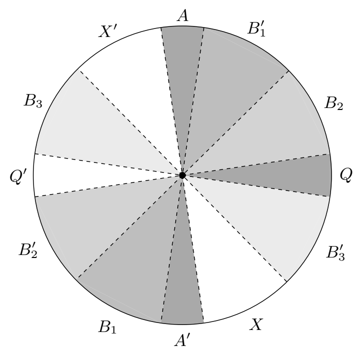

Figure 4: Placing the gadgets on the polygon.

Correctness. If our 3-SAT formula F is satisfiable, then set the state of the initial variable gadgets according to a satisfying assignment of F. The copy gadgets preserve the state of these initial variable gadgets, so each clause gadget has at least one free pocket. Hence, the union of R with all gadgets is in convex position. Conversely, if F is unsatisfiable, then suppose for the sake of obtaining a contradiction that the union of R with all gadgets is in convex position. Consider the assignment α of F corresponding to the state of the variable gadgets (this may set some x_i and $\bar{x_i}$ both to false). Since F is unsatisfiable, there must be a clause C that is not satisfied by assignment α. Since the copy gadgets preserve the state of the original variable gadgets, all pockets of C are blocked. Hence, C is not compatible with the other gadgets and the union of R with all gadgets cannot be in convex position, which yields a contradiction. Theorem 1 follows.

3 Replacing a regular polygon by triangles

In this section we show that we can replace the regular polygon from our reduction above by a set of triangles. This will prove Theorem 2.

Proposition 3 *Let $p_1, p_2, \ldots, p_n$ be the vertices of a regular n-gon in counterclockwise order. Let us consider the family of triangles*

$$\mathcal{T} = \{\triangle p_1p_2p_3, \triangle p_2p_3p_4, \ldots, \triangle p_np_1p_2, \\ \triangle p_1p_2p_{n-2}, \triangle p_2p_3p_{n-1}, \ldots, \triangle p_np_1p_{n-3}\}.$$

Let S be a point set obtained by translating each triangle in $\mathcal{T}$ independently. If S is in convex position, then S consists of the vertices of a regular n-gon.

The rest of this section is devoted to the proof of Proposition 3. For a given set S of points in the plane, let $CH(S)$ denote the *vertices* of the convex hull of S.

Let $T \subseteq \mathcal{T}$. Suppose that every triangle $T_i \in T$ is translated according to some vector λ_i, and let S_{T_λ} be the resulting set of points. We call S_{T_λ} a *geometric placement* of T. A placement of T is called *combinatorial* if the order of the vertices of T around the convex hull is known, but not the exact position of the triangles. We say that S_{T_λ} satisfies the *convex condition* if all points of S_{T_λ} belong to $CH(S_{T_\lambda})$. In this case we might also say that the placement of the triangles is *valid.* We say that $p \in S_{T_\lambda}$ *violates* the convex condition if $p \notin CH(S_{T_\lambda})$.

Every vertex of the regular n-gon belongs to several triangles of $\mathcal{T}$. To distinguish the distinct copies of the vertex, we use superscripts, so for example p_1^i will denote vertex p_1 in some particular triangle of $\mathcal{T}$. We set $T_1 = \triangle p_1^1p_2^1p_3^1$, $T_2 = \triangle p_2^2p_3^2p_4^2, \ldots,$ and $T_n = \triangle p_n^np_1^np_2^n$. We also set $T_{n+1} = \triangle p_1^{n+1}p_2^{n+1}p_{n-2}^{n+1}$, $T_{n+2} = \triangle p_2^{n+2}p_3^{n+2}p_{n-1}^{n+2}, \ldots,$ and $T_{2n} = \triangle p_n^{2n}p_1^{2n}p_{n-3}^{2n}$. We assume that n is a multiple of 4, that the regular n-gon is oriented so that two of its sides are horizontal, and that p_2p_3 is the bottom horizontal side.

Observation 1 *Let $p \in S \subseteq S'$. If $p \notin CH(S)$, then $p \notin CH(S')$.*

Lemma 4 *Let $T = \{T_1, T_2\}$. Suppose that S_{T_λ} satisfies the convex condition, and $p_2^1p_3^1$ is not collinear with $p_2^2p_3^2$. Then the counterclockwise order of S_{T_λ} in $CH(S_{T_\lambda})$ is either $p_1^1p_2^1p_2^2p_3^2p_4^2p_3^1$ or $p_1^1p_2^1p_3^1p_3^2p_4^2p_2^2$.*

Proof. Suppose that we fix the position of T_1. Then if we extend the line segments of T_1 to lines, the plane is decomposed into four open regions $F_1, \ldots, F_4$ and three closed regions $R_1, \ldots, R_3$, as shown in Figure 5. It is easy to verify that in order to satisfy the convex condition, no point of T_2 can lie in any of the regions F_1, F_2, F_3, and no vertex of T_2 can lie in F_4. In particular, p_2^2, p_3^2, and p_4^2 can lie only in $R_1 \cup R_2 \cup R_3$. In principle, there are 27 cases to consider, based on all possible combinations. Fortunately, we only need to distinguish a few situations.

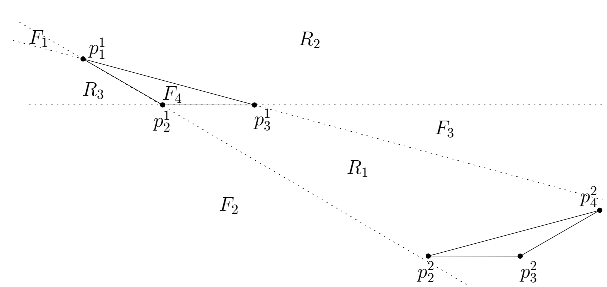

Figure 5: A valid combinatorial placement of T_1 and T_2.

We first suppose that $p_2^2 \in R_1$. Notice that in this case it is not possible that $p_3^2 \in R_2$ or $p_3^2 \in R_3$. So we can assume that $p_3^2 \in R_1$. Now we can easily rule out the case $p_4^2 \in R_3$. Point p_4^2 cannot lie in R_2 either, because in this case the side $p_3^2p_4^2$ would intersect F_3. Therefore p_4^2 can lie only in R_1. This gives a valid combinatorial placement of T_1 and T_2 where the counterclockwise order of S_{T_λ} in $CH(S_{T_\lambda})$ is $p_1^1p_2^1p_2^2p_3^2p_4^2p_3^1$ (see Figure 5).

Next suppose that $p_2^2 \in R_2$. Then, $p_3^2 \in R_2$ and $p_4^2 \in R_2$. This gives another valid combinatorial placement of T_1 and T_2 where the counterclockwise order of S_{T_λ} in $CH(S_{T_\lambda})$ is $p_1^1p_2^1p_3^1p_3^2p_4^2p_2^2$ (see Figure 6). Note that there are no other possible combinatorial placements since we assume that $p_2^1p_3^1$ and $p_2^2p_3^2$ are not collinear.

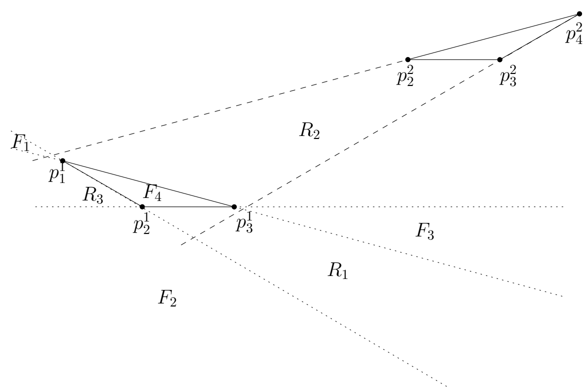

Figure 6: A valid combinatorial placement of T_1 and T_2.

Finally suppose that $p_2^2 \in R_3$. Then, $p_3^2 \notin R_1$ and $p_4^2 \notin R_1$. Recall that p_2^2 is on the same horizontal line as p_3^2 and to its left, and p_4^2 is above and to the right of p_3^2. If $p_3^2 \in R_2$ and $p_4^2 \in R_2$, then we have that p_3^1 is below and to the right of p_3^2. Consequently, p_3^2 lies inside the triangle $\triangle p_2^2p_4^2p_3^1$ and violates the convex condition. The combination $p_3^2 \in R_2$ and $p_4^2 \in R_3$ is not possible. If $p_3^2 \in R_3$, then we have that p_2^1 is below and to the right of p_3^2 (since we are assuming that $p_2^1p_3^1$ is not collinear with $p_2^2p_3^2$). Consequently, p_3^2 lies inside the triangle $\triangle p_2^2p_4^2p_2^1$ and violates the convex condition. $\square$

Lemma 5 *Let $T = \{T_1, T_2, T_{\frac{n}{2}+1}, T_{\frac{n}{2}+2}\}$. If S_{T_λ} satisfies the convex condition, then $p_2^1p_3^1$ is collinear with $p_2^2p_3^2$.*

Proof. [Sketch] Due to space limitations, we only sketch the proof here. It suffices to prove that the combinatorial placements for $\triangle p_1^1p_2^1p_3^1$ and $\triangle p_2^2p_3^2p_4^2$ of Lemma 4 are no longer valid. To show that the combinatorial placement in which the counterclockwise order around the convex hull is $p_1^1p_2^1p_2^2p_3^2p_4^2p_3^1$ is no longer valid, we try to add $T_{\frac{n}{2}+1}$ to the placement, and prove that this is not possible. The other combinatorial placement from Lemma 4 can be ruled out analogously. $\square$

By symmetry, we have the following corollary:

Corollary 6 *If S_{T_λ} satisfies the convex condition, then $p_2^1p_3^1$ is collinear with $p_2^2p_3^2$, $p_3^2p_4^2$ is collinear with $p_3^3p_4^3, \ldots,$ and $p_1^np_2^n$ is collinear with $p_1^1p_2^1$.*

Lemma 7 *If S_{T_λ} satisfies the convex condition, then $p_2^1p_3^1$ is collinear with $p_2^2p_3^2$, and p_2^1 is not right of p_2^2.*

We omit the proof due to space considerations.

Suppose that, in some placement of T_{i-1} and T_i, $p_i^{i-1}p_{i+1}^{i-1}$ is collinear with $p_i^ip_{i+1}^i$. Suppose that we rotate the triangles so that T_{i-1} has the same orientation as T_1 and T_i has the same orientation as T_2. We say that T_{i-1} and T_i *cross* if, after this rotation, p_i^{i-1} is to the right of p_i^i. By symmetry, we have the following corollary, which subsumes Corollary 6 and Lemma 7:

Corollary 8 *Suppose that S_{T_λ} satisfies the convex condition. Then $p_2^1p_3^1$ is collinear with $p_2^2p_3^2$, $p_3^2p_4^2$ is collinear with $p_3^3p_4^3, \ldots,$ and $p_1^np_2^n$ is collinear with $p_1^1p_2^1$. Furthermore, none of the pairs $\{T_i, T_{i+1}\}$ cross.*

Lemma 9 *Suppose that S_{T_λ} satisfies the convex condition. Then $p_2^1p_3^1$ is collinear with $p_2^2p_3^2$, and p_2^1 is not to the left of p_2^2.*

Proof. We proceed by contradiction. Consider the regular n-gon having one side at $p_2^2p_3^2$. Since none of the pairs $\{T_i, T_{i+1}\}$ cross, this polygon lies inside the polygon formed by the placement of $T_1, \ldots, T_n$ (see Figure 7 for an example). Since we are assuming that p_2^1 is to the left of p_2^2, we also have that p_n of the polygon lies strictly inside (that is, not on the boundary) the polygon formed by the placement of $T_1, \ldots, T_n$. We now try to place T_{n+3}. In order to maintain the convex condition, the vertices of T_{n+3} must be placed on the boundary of the polygon formed by the placement of $T_1, \ldots, T_n$, or on the triangles bounded by an edge of the polygon and two dotted segments shown in Figure 7. Such a placement of T_{n+3} is not possible. $\square$

By symmetry, we obtain the following corollary, which subsumes Lemma 9 and Corollary 8:

Corollary 10 *Suppose that S_{T_λ} satisfies the convex condition. Then p_2^1 is on the same position as p_2^2, p_3^2 is on the same position as $p_3^3, \ldots,$ and p_1^n is on the same position as p_1^1. Equivalently, the vertices of $\triangle p_1^1p_2^1p_3^1, \triangle p_2^2p_3^2p_4^2, \ldots, \triangle p_n^np_1^np_2^n$ form a regular n-gon.*

To complete the proof of Proposition 3, it only remains to see that the triangles $T_{n+1}, T_{n+2}, \ldots, T_{2n}$ are also placed in the "natural" way. We prove it in the next lemma for T_{n+2} and, by symmetry, the result also holds for the other triangles.

Lemma 11 *Suppose that S_{T_λ} satisfies the convex condition. Then, $\triangle p_2^{n+2}p_3^{n+2}p_{n-1}^{n+2}$ is placed so that p_2^{n+2} is on the same position as p_2^2.*

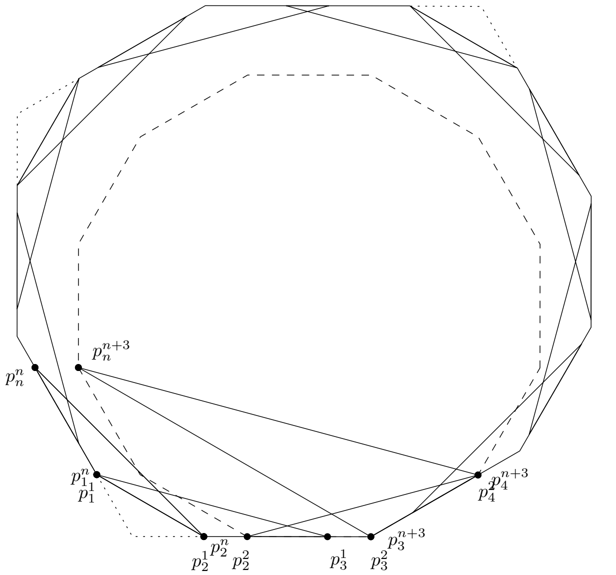

Figure 7: The vertex p_n^{n+3} cannot be placed in Lemma 9 if p_2^1 is to the left of p_2^2 (case 1).

Proof. We know that the vertices of $T_1, T_2, \ldots, T_n$ form a regular n-gon. In order to maintain the convex condition, the vertices of T_{n+2} must be placed on the boundary of the regular n-gon or on the triangles bounded by an edge of the polygon and two dotted segments shown in Figure 8. It is clear that the only way to do it consists in placing p_2^{n+2} on the same position as p_2^2. $\square$

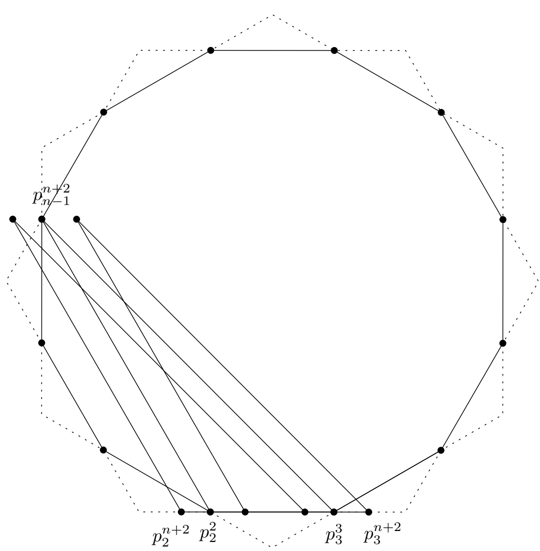

Figure 8: Points with the same superscript must again coincide after the translations.

4 Concluding remarks

If we allow scaling in addition to translation (Problem C from the introduction), then it is not known if the problem is still NP-hard. In addition, it is not clear if our problem is in NP. The obvious certificate would be the sequence of translations, but one must argue that the representation of these translations is not too large in terms of the input representation. In fact, our problem has some similarities to the order type realizability problem, which is known to be complete for the existential theory of the reals [13], and the coordinate representation of some order types requires exponential storage [9].

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