Hyperbanana Graphs

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Abstract

A *bar-and-joint* framework is a finite set of points together with specified distances between selected pairs. In rigidity theory we seek to understand when the remaining pairwise distances are also fixed. If there exists a pair of points which move relative to one another while maintaining the given distance constraints, the framework is *flexible*; otherwise, it is *rigid*.

Counting conditions due to Maxwell give a necessary combinatorial criterion for generic minimal bar-and-joint rigidity in all dimensions. Laman showed that these conditions are also sufficient for frameworks in $\mathbb{R}^2$. However, the flexible "double banana" shows that Maxwell's conditions are not sufficient to guarantee rigidity in $\mathbb{R}^3$. We present a generalization of the double banana to a family of *hyperbananas*. In dimensions 3 and higher, these are (infinitesimally) flexible, providing counterexamples to the natural generalization of Laman's theorem.

1 Introduction

A *bar-and-joint framework* is composed of universal *joints* whose relative positions are constrained by fixed-length *bars*. An *embedding* of such a framework in $\mathbb{R}^d$ associates a point in $\mathbb{R}^d$ to each joint with the property that the distance between joints connected by a bar is satisfied by the embedding. Bar-and-joint frameworks can be used to model structures arising in many applications, including sensor networks, proteins, and Computer Aided Design (CAD) systems. In combinatorial rigidity theory we seek an understanding of the structural properties of such a framework, and ask whether it is flexible (i.e., admits an internal motion that respects the constraints) or rigid.

In this paper, we assume that we are given an embedding of a bar-and-joint framework from which the lengths of bars can be inferred.

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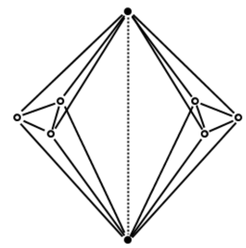


Figure 1: The double banana is a Maxwell graph in $\mathbb{R}^3$, but is flexible. Each "banana" can rotate about the *implied hinge* (dotted).

Definition 1 *A* bar-and-joint framework $F = (G, \mathbf{p})$ *embedded in* $\mathbb{R}^d$ *is composed of a graph* $G = (V, E)$ *with* $|V| = n$ *and* $|E| = m$ *and an embedding* $\mathbf{p} : V \to \mathbb{R}^d$*, which assigns a position vector* $\mathbf{p}_i$ *to each vertex* v_i.

We only concern ourselves with *generic embeddings* of these frameworks, which can be thought of as embeddings with the properties we would expect if we chose an embedding at random. To formally define genericity we require the notion of a **rigidity matrix**, which encodes the infinitesimal behavior of the framework.

Definition 2 *For a framework* $F = (G, \mathbf{p})$ *embedded in* $\mathbb{R}^d$ *we define a* rigidity matrix M_F *to be an* $m \times dn$ *matrix in which the columns are grouped into* n *sets of* d *coordinates for each vertex. Each row of the rigidity matrix corresponds to an edge* ij *and has the following pattern.*

$$\begin{array}{cccccccc} & v_1 & \dots & v_i & \dots & v_j & \dots & v_n \\ ij & [\,0 & \cdots 0 \cdots & \mathbf{p}_i - \mathbf{p}_j & \cdots 0 \cdots & \mathbf{p}_j - \mathbf{p}_i & \cdots 0 \cdots & 0\,] \end{array}$$

If F is a framework, M_F determines if it is *infinitesimally* flexible or rigid; for brevity, we omit "infinitesimally" for the remainder of this paper. We say that F is *rigid* if the insertion of any new bar between vertices does not change the rank of M_F; otherwise it is *flexible*. A rigid framework is *minimally rigid* if the rows of M_F are independent.

The *infinitesimal motions* of F can be encoded by assigning a velocity vector $\mathbf{p}'_i \in \mathbb{R}^d$ to each vertex v_i so that $(\mathbf{p}'_1, \dots, \mathbf{p}'_n)$ is nonzero and is in the null space of M_F (intuitively, these are instantaneous velocities that do not shrink or stretch the bar constraints). There is always a set of *trivial motions* corresponding to rigid body motions of $\mathbb{R}^d$; the space of rigid motions of $\mathbb{R}^d$ has dimension $\binom{d+1}{2}$ and is generated by rotations about $(d-2)$-dimensional affine linear subspaces and translations. In general, then, a framework on at least d vertices is minimally rigid if and only if its rigidity matrix has nullity $\binom{d+1}{2}$. However, if a framework F is contained in an affine subspace $H \subset \mathbb{R}^d$ where $\dim H \leq d-2$, then there is a rigid motion of $\mathbb{R}^d$ that fixes F; hence, the null space of M_F has dimension less than $\binom{d+1}{2}$.

Combinatorial counting conditions, first observed by Maxwell [5], give a necessary condition for minimal bar-and-joint rigidity. Throughout this paper, we will use the convention that, if V' is a subset of the vertices of a graph G and $\mathcal{E}$ is a subset of the edges of G, then $\mathcal{E}(V')$ is the set of edges in $\mathcal{E}$ induced by the vertices in V'.

Definition 3 *A* Maxwell graph $G = (V, E)$ *in dimension d satisfies*

1. $|E| = d|V| - \binom{d+1}{2}$
2. $|E(V')| \leq d|V'| - \binom{d+1}{2}$, *for all* $V' \subseteq V$ *where* $|V'| \geq d$.

For almost all frameworks $F = (G, \mathbf{p})$ on a fixed graph G, the rank of M_F is constant, as the set of special embeddings for which M_F drops rank is parameterized by a closed subset of $\mathbb{R}^{dn}$. We formally define genericity as follows.

Definition 4 *A framework* $(G, \mathbf{p})$ *is* generic *if its rigidity matrix achieves the maximum rank over all frameworks* $(G, \mathbf{q})$.

We call a framework *generically minimally rigid* if there exists a generic framework with the same underlying graph that is minimally rigid. We analyze the generic behavior of a framework purely by the combinatorial structure of the graph. Therefore, from here on we will write M_G to denote the rigidity matrix associated to a generic embedding of G.

In $\mathbb{R}^2$, Laman proved that the Maxwell conditions are sufficient for generic minimal rigidity.

Theorem 5 (Laman [3]) *A bar-and-joint framework, with underlying graph* $G = (V, E)$, *embedded in* $\mathbb{R}^2$ *is generically minimally rigid if and only if it satisfies the following conditions:*

1. $|E| = 2|V| - 3$
2. $|E(V')| \leq 2|V'| - 3$, *for all* $V' \subseteq V$ *where* $|V'| \geq 2$

However, the sufficiency of the Maxwell counting conditions for rigidity does not generalize to higher dimensions. In $\mathbb{R}^3$, the well-known "double banana" is a Maxwell graph that is flexible [2]. This structure is composed of two "bananas" joined on a pair of vertices (refer to Figure 1) and exhibits a hinge motion about the dotted line. This denotes the existence of an *implied edge* between two vertices that are not incident to each other, yet whose distance is fixed as a consequence of the other constraints. Since a rotation is allowed about the edge, it is called an *implied hinge*.

Counterexamples like the double banana can provide insight into the challenges presented in dimension 3 and higher for which no combinatorial characterization of bar-and-joint rigidity is known.

Contributions. In this paper, we describe a class of graphs called *hyperbananas* that generalize the double banana to higher dimensions. We present hyperbananas that are Maxwell graphs and show these to be (infinitesimally) flexible. To the best of our knowledge, this is the first family of counterexamples to the sufficiency of the Maxwell conditions for minimal bar-and-joint rigidity addressing all dimensions of 3 and higher.

Related work. Other generalizations of the double banana include the banana spider graphs of Mantler and Snoeyink [4]. These were developed to address an attempt at classifying 3D bar-and-joint rigidity by vertex connectivity, as it was conjectured that all graphs with implied hinges must be 2-connected (like the double banana). The banana spider graphs provide examples with higher vertex connectivity, answering this conjecture in the negative. The key idea was to add "spider" components to the double banana, increasing vertex connectivity while maintaining flexibility about the implied hinge.

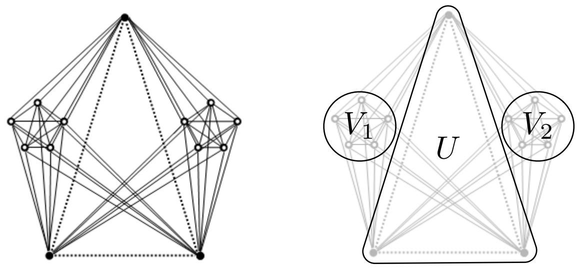
Another class of counterexamples to Maxwell's conditions in 3D was developed by Cheng et al. [1]. These "ring of roofs" frameworks, first described by Tay [7], provide examples of flexible Maxwell graphs that admit no non-trivial rigid subgraphs, i.e., rigid subgraphs larger than a tetrahedron. This countered an earlier attempt by Sitharam and Zhou [6] to characterize 3D bar-and-joint rigidity by detecting rigid components and adding the resulting implied edges.

2 Maxwell hyperbananas

We now present a family of graphs called hyperbanana graphs; under certain conditions, hyperbananas are Maxwell graphs. We generalize the double banana, which consists of two minimally rigid "bananas" glued together on a pair of vertices. Each banana can be built using the following inductive construction.

Definition 6 *Fix a positive integer d. A* d-Henneberg 0-extension *on a graph G results in a new graph by adding a single vertex and connecting it to d distinct vertices in G.*

When a d-Henneberg 0-extension is applied to a minimally rigid framework in $\mathbb{R}^d$, minimal rigidity is preserved, and hence so are the Maxwell conditions [8]. In



(a) (b)

Figure 2: The hyperbanana $H_{5,3}$ is a flexible Maxwell graph $\mathbb{R}^5$; there are 3 implied edges (dotted) among the vertices in U.

the double banana, each individual banana is created by two 3-Henneberg 0-extensions on a triangle (which is minimally rigid in $\mathbb{R}^3$), connecting each new vertex to the 3 vertices of the triangle.

Before generalizing the banana construction, we give some additional notation. If U and W are finite sets, let K_U denote the complete graph with vertex set U and $K_{U,W}$ be the complete bipartite graph on the two disjoint sets U and W.

Definition 7 *A* banana bunch *is a graph $B_{d,b}$ obtained by performing b d-Henneberg 0-extensions on a K_d. The b vertices added by the Henneberg extensions are called* banana vertices*.*

Since K_d embedded in $\mathbb{R}^d$ is minimally rigid for any d, $B_{d,b}$ is generically minimally rigid in dimension d.

Hyperbananas are composed of two banana bunches glued together along the banana vertices.

Definition 8 *For $i = 1, 2$, let G_i be a copy of $B_{d,b}$ with vertex set partitioned into $V_i \cup U_i$, where the K_d has vertex set V_i and the set U_i consists of banana vertices. We define the* hyperbanana *$H_{d,b}$ to be $G_1 \cup G_2/\sim$, where $\sim$ identifies banana vertices based on some fixed bijection from U_1 to U_2. The vertex set of $H_{d,b}$ is the set $V = V_1 \cup V_2 \cup U$, where U is the set of banana vertices.*

The double banana is simply $H_{3,2}$. An example of a higher dimensional hyperbanana, $H_{5,3}$, is pictured in Figure 2. While this is a Maxwell graph, not all choices of b and d satisfy the counting conditions. For example, simply checking the counts on the total number of edges for the hyperbanana $H_{4,3}$ confirms that this graph has too many edges to be Maxwell. In fact, it is rigid in $\mathbb{R}^4$, but *overconstrained*. Therefore, it is not minimally rigid as its rigidity matrix contains dependencies. Checking the counts on the total number of edges for the hyperbanana $H_{6,3}$ shows that it is *underconstrained* and therefore flexible in $\mathbb{R}^6$.

2.1 Odd-dimensional hyperbananas

When d is odd and equal to $2b-1$, we obtain hyperbananas that are Maxwell graphs. We begin with a more general lemma that will be used in proving the counting conditions. In the proofs that follow, we define $V_i' = V' \cap V_i$ and $U' = V' \cap U$ for a subset V' of the vertex set of $H_{d,b}$,

Lemma 9 *If $H_{d,b} = (V, E)$ and $V' \subseteq V$, and $|V_i' \cup U'| \geq d$ for $i = 1, 2$, then*
$$|E(V')| \leq d|V'| - 2\binom{d+1}{2} + d|U'|.$$

Proof. As each banana bunch is minimally rigid we have
$$|E(V_i' \cup U')| \leq d|V_i' \cup U'| - \binom{d+1}{2} \qquad (1)$$
for each i. Adding the inequalities yields
$$\begin{aligned} |E(V')| &\leq d(|V_1'| + |V_2'| + 2|U'|) - 2\binom{d+1}{2} \\ &= d(|V_1'| + |V_2'| + |U'|) - 2\binom{d+1}{2} + d|U'| \qquad (2) \\ &= d|V'| - 2\binom{d+1}{2} + d|U'|. \end{aligned}$$
□

We can now show that the specific class of hyperbananas in odd-dimensional spaces are Maxwell graphs.

Theorem 10 *The hyperbanana $H_{d,b}$ embedded in $\mathbb{R}^d$ with $d = 2b-1$ is a Maxwell graph.*

Proof. We check condition 1 of Definition 3 by vertex and edge counts. The graph $H_{d,b}$ has d vertices from each complete K_d graph and b banana vertices, totaling $2d + b$ vertices. Since $d = 2b - 1$, there are $\frac{5d+1}{2}$ vertices. Each K_d has $\binom{d}{2}$ edges, and each banana vertex is incident to $2d$ edges. This sums to an edge count of $2\binom{d}{2} + 2d(\frac{d+1}{2})$. Simplifying, we can verify that the edge count is $|E| = 2d^2$. Substituting the vertex count $|V| = \frac{5d+1}{2}$, we see that Maxwell condition 1 is satisfied:
$$d|V| - \binom{d+1}{2} = d\left(\frac{5d+1}{2}\right) - \binom{d+1}{2} = |E|$$
Now we check Maxwell condition 2. If V' is contained within a single banana bunch, the condition is satisfied as $B_{d,b}$ is minimally rigid and therefore Maxwell. If V' intersects both banana bunches non-trivially, then there are three cases which depend on whether the intersection with each banana bunch contains at least d vertices.

If $|V_i' \cup U'| \geq d$ for both i, then combining $|U'| \leq b = \frac{d+1}{2}$ with Lemma 9 gives the result.

Now suppose, without loss of generality, that $|V_1' \cup U'| \geq d$, but $|V_2' \cup U'| < d$. We know that
$$\begin{aligned} |E(V_2' \cup U')| &= \binom{|V_2'|}{2} + |U'||V_2'| \qquad (3) \\ &= \frac{(|V_2'|-1)|V_2'|}{2} + |U'||V_2'| \qquad (4) \\ &\leq \frac{(d-2)|V_2'|}{2} + b|V_2'| \qquad (5) \end{aligned}$$

Since $b = \frac{d+1}{2}$, we obtain $|E(V_2' \cup U')| \leq d|V_2'|$. Combining this with Inequality 1 gives the desired inequality in the second case.

Finally, suppose that both $|V_i' \cup U'| < d$ and $|V_1'| \geq |V_2'|$. As $|V_1' \cup V_2' \cup U'| \geq d$, there exists a subset $W \subseteq V_2'$ so that $|V_1' \cup W \cup U'| = d$. Let $W' = V_2' \backslash W$. The set $|E(V_2')|$ consists of the edges of K_W, the edges of $K_{W'}$ and the edges of $K_{W,W'}$.

Now suppose we had a set V_1'' satisfying $V_1 \supseteq V_1'' \supset V_1'$ and $|V_1'' \cup U'| = d$. Then

$$|E(V_1' \cup W \cup U')| + |E(K_{V_1',W})| = |E(V_1'' \cup U')|,$$

and by Inequality 1,

$$|E(V_1' \cup W \cup U')| + |E(K_{V_1',W})| \leq d|V_1' \cup W \cup U'| - \binom{d+1}{2}. \tag{6}$$

Applying the argument in the second case to the set $W' \cup U'$ and adding the inequality to 6, gives the result in this final case as $|E(K_{W,W'})| < |E(K_{V_1',W})|$.

□

2.2 Even-dimensional hyperbananas

We observed earlier that hyperbananas may be either overconstrained or underconstrained in even-dimensional spaces and are not Maxwell graphs. However, by making a small modification to our definition, we obtain Maxwell graphs for even-dimensional spaces.

Definition 11 *For even d, we define the* even hyperbanana *to be a graph $H^+_{d,b}$ consisting of a hyperbanana $H_{d,b}$ together with an additional $\frac{d}{2}$ edges connecting distinct vertices of the complete graphs in the two banana bunches.*

This addition of $\frac{d}{2}$ edges between the complete graphs in $H_{d,b}$ results in $H^+_{d,b}$ being a Maxwell graph for the even-dimensional spaces for certain values of d relative to b. One example of an even hyperbanana, $H^+_{4,2}$, is shown in Figure 3. Note that $H^+_{d,b} = (V, F)$, is built from $H_{d,b} = (V, E)$; let E^+ be the additional $\frac{d}{2}$ edges so that $F = E \cup E^+$. In Figure 3, for example, E^+ is composed of the 2 dashed edges.

Theorem 12 *The even hyperbanana $H^+_{d,b} = (V, F)$ embedded in $\mathbb{R}^d$ with $d = 2b$ is a Maxwell graph.*

Proof. Since $d = 2b$, the number of vertices in $H^+_{d,b}$ is $|V| = 2d + b = \frac{5}{2}d$, as there are two K_d graphs and b banana vertices. There are 2 complete graphs with $\binom{d}{2}$ edges, b banana vertices connecting to the $2d$ complete graph vertices, and $\frac{d}{2}$ edges between the complete graphs, resulting in $|F| = 2d^2 - \frac{d}{2}$. By substituting the vertex count, we can verify Maxwell condition 1.

$$d|V| - \binom{d+1}{2} = 2d^2 - \frac{d}{2} = |F|.$$

Now let $V' \subseteq V$ with $|V'| \geq d$. If V' is completely contained in a banana bunch, Maxwell condition 2 is satisfied as $B_{d,b}$ is Maxwell. Assume, then, that V' non-trivially intersects both vertex sets V_1 and V_2.

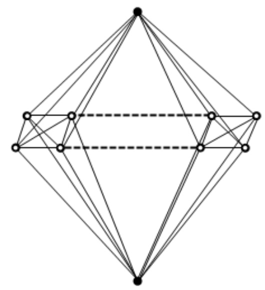


Figure 3: The even hyperbanana $H^+_{4,2}$ is a flexible Maxwell graph; it is built from the hyperbanana $H_{4,2}$ by an additional 2 (dashed) edges.

If $|V_i' \cup U'| \geq d$ for both $i = 1, 2$, then by Lemma 9,

$$|E(V')| \leq d|V'| - 2\binom{d+1}{2} + d|U'|).$$

The number of banana vertices is $b = \frac{d}{2}$, so $|U'| \leq \frac{d}{2}$. Therefore,

$$\begin{aligned}|E(V')| &\leq d|V'| - 2\binom{d+1}{2} + \frac{d^2}{2} \\ &= d|V'| - \binom{d+1}{2} - \frac{d^2+d}{2} + \frac{d^2}{2} \\ &= d|V'| - \binom{d+1}{2} - \frac{d}{2}.\end{aligned}$$

Since $F = E \cup E^+$, $|F(V')| = |E(V')| + |E^+(V')|$. By adding $|E^+(V')|$ to both sides of the previous inequality we obtain

$$|F(V')| \leq d|V'| - \binom{d+1}{2} - \frac{d}{2} + |E^+(V')|.$$

By definition, $|E^+| = \frac{d}{2}$, implying $|E^+(V')| \leq \frac{d}{2}$. Therefore, we can conclude that Maxwell condition 2,

$$|F(V')| \leq d(|V'|) - \binom{d+1}{2},$$

holds in this case.

Now suppose, without loss of generality, that $|V_1' \cup U'| \geq d$, but $|V_2' \cup U'| < d$. Since $b = \frac{d}{2}$, Inequality 5 implies

$$|E(V_2' \cup U')| \leq (d-1)|V_2'|. \tag{7}$$

We can combine this with

$$|E(V_1' \cup U')| \leq d|V_1' \cup U'| - \binom{d+1}{2}$$

and the edges in $E^+(V')$ to obtain

$$\begin{aligned}|F(V')| &\leq d|V_1' \cup U'| - \binom{d+1}{2} + (d-1)|V_2'| + |E^+(V')| \\ &= d|V'| - \binom{d+1}{2} - |V_2'| + |E^+(V')| \\ &\leq d|V'| - \binom{d+1}{2}\end{aligned}$$

as $|E^+(V')| \leq |V_2'|$.

Finally, suppose that both $|V_i' \cup U'| < d$. Assume that $|V_1'| \geq |V_2'|$ and define W and W' as in the proof of Theorem 10. Adding Inequalities 6 and 7 (with W'

replacing V_2'),

$$\begin{aligned}&|E(V_1'\cup W\cup U')| + |E(K_{V_1',W})| + |E(W'\cup U')|\\ &\le d|V_1'\cup W\cup U'| - \tbinom{d+1}{2} + (d-1)|W'|\\ &= d|V'| - \tbinom{d+1}{2} - |W'|,\end{aligned}$$

and hence

$$\begin{aligned}&|E(V_1'\cup W\cup U')| + |E(K_{V_1',W})| + |E(W'\cup U')| + |W'|\\ &\le d|V'| - \tbinom{d+1}{2}.\end{aligned}$$

Since $|F(V')|$ is equal to
$|E(V_1'\cup W\cup U')|+|E(K_{W,W'})|+|E(W'\cup U')|+|E^+(V')|$, it will suffice to show that

$$|E(K_{W,W'})| + |E^+(V')| \le |E(K_{V_1',W})| + |W'|,$$

or that

$$|W|\cdot|W'| + |E^+(V')| \le |V_1'|\cdot|W| + |W'| \qquad (8)$$

Now let $t = |V_1'| - |W'|$. Since $|V_1'| \ge |V_2'|$ and $|V_1'| < d$, $|W| > 0$, which implies that $|V_1'| > |W'|$ and hence that $t \ge 1$. Setting $|W'| = |V_1'| - t$, we have

$$\begin{aligned}&|W|\cdot|W'| + |E^+(V')|\\ &= |W|\cdot(|V_1'| - t) + |E^+(V')|\\ &= |V_1'|\cdot|W| - t|W| + |E^+(V')|\\ &\le |V_1'|\cdot|W| - |W| + |E^+(V')|,\end{aligned}$$

as $t \ge 1$. Then

$$|V_1'|\cdot|W| - |W| + |E^+(V')| \le |V_1'|\cdot|W| + |W'|$$

if and only if

$$|V_1'|\cdot|W| + |E^+(V')| \le |V_1'|\cdot|W| + |W'| + |W|.$$

Indeed, since $|W'| + |W| = |V_2'| \ge |E^+(V')|$, this inequality holds, completing the proof. □

3 Flexible hyperbananas

In this section, we prove that the Maxwell hyperbananas are flexible.

We begin by considering the rigidity matrix $M_{B_{d,b}}$ for a generic framework on the banana bunch $B_{d,b}$ in dimension d, which has $d(d+b)$ columns and $\binom{d}{2} + db$ rows. Since the banana bunch is minimally rigid, the rank of its rigidity matrix is maximal and equal to the number of rows $\binom{d}{2} + db$. Let the vertex set of $B_{d,b}$ be partitioned into sets V_1 and U, where the set U consists of banana vertices. Assume that the columns of $M_{B_{d,b}}$ are arranged so that the columns corresponding to the vertices in V_1 come first, followed by the columns for U.

Lemma 13 *Each row of the block matrix*

$$\left[\begin{array}{c|c} 0 & M_{K_U}\end{array}\right]$$

with d^2 columns of zeros (d columns for each vertex in the V_1), is in the row space of $M_{B_{d,b}}$.

Proof. Since the banana bunch is minimally rigid and spans $\mathbb{R}^d$, $M_{B_{d,b}}$ has nullity $\binom{d+1}{2}$. If we add an edge from K_U, the new rigidity matrix will still have nullity $\binom{d+1}{2}$. Thus, each such row must be a linear combination of the rows of $M_{B_{d,b}}$. □

Proposition 14 *If $B_{d,b} = (V_1 \cup U, E)$ is embedded in $\mathbb{R}^d$, and the rank of M_{K_U} is $\binom{b}{2}$, then $M_{B_{d,b}}$ is row-equivalent to a matrix of the form*

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{M^*_{B_{d,b}}} \\ \hline 0 & M_{K_U}\end{array}\right],$$

where $M^*_{B_{d,b}}$ consists of $|E| - \binom{b}{2}$ rows of the original matrix $M_{B_{d,b}}$.

Proof. Let R be a row in $[\ 0 \mid M_{K_U}\ ]$. By Lemma 13, R may be written as a linear combination of rows of $M_{B_{d,b}}$. Any row of $M_{B_{d,b}}$ appearing in such a linear combination with a nonzero coefficient may be replaced by R through a sequence of elementary row operations. Any subsequent row R' of $[\ 0 \mid M_{K_U}\ ]$ will remain dependent on the rows of the modified matrix. Moreover, when we express R' as a linear combination of the current set of rows, some remaining row of the original matrix $M_{B_{d,b}}$ must appear with a nonzero coefficient as the rows of M_{K_U} are independent. Thus, we can insert each row of $[\ 0 \mid M_{K_U}\ ]$ in this way. □

With this we can prove the following theorem.

Theorem 15 *If G is the hyperbanana $H_{d,b} \subset \mathbb{R}^d$ where $d = 2b-1$ or $H^+_{d,b} \subset \mathbb{R}^d$ where $d = 2b$ and $b \ge 2$, then G is flexible.*

Proof. Consider the hyperbanana $H_{d,b}$ partitioned into two bunches $B_{d,b}(1)$ and $B_{d,b}(2)$. Let $M_{B_{d,b}}(1)$ be the rigidity matrix for $B_{d,b}(1)$, $M_{B_{d,b}}(2)$ be the rigidity matrix for $B_{d,b}(2)$ and M be the rigidity matrix for $H_{d,b}$. If we put the vertices in an order with (V_1, U, V_2) and order the columns of M accordingly, then M is a block matrix of the form

$$\begin{array}{c} \\ B_{d,b}(1) \\ \hline B_{d,b}(2)\end{array} \begin{array}{ccc} V_1 & U & V_2 \\ \multicolumn{2}{c|}{M_{B_{d,b}}(1)} & 0 \\ 0 & \multicolumn{2}{|c}{M_{B_{d,b}}(2)}\end{array}$$

By Proposition 14 M is row equivalent to

$$\begin{array}{c|ccc} & V_1 & U & V_2 \\ \hline B_{d,b}(1) & \multicolumn{2}{c|}{M_{B_{d,b}}(1)^*} & 0 \\ & 0 & M_{K_U} & 0 \\ \hline B_{d,b}(2) & 0 & \multicolumn{2}{|c}{M_{B_{d,b}}(2)^*} \\ & 0 & M_{K_U} & 0 \end{array}$$

We can see that there are at least $\binom{b}{2}$ dependencies in M, since the $[0|M_{K_U}|0]$ is seen twice in the matrix. Therefore, since the number of columns is $d|V|$ and the number of rows is $|E|$, the nullity of M is at least $\binom{d+1}{2} + \binom{b}{2}$. Thus, since a framework with at least d vertices is minimally rigid in $\mathbb{R}^d$ if and only if it has nullity $\binom{d+1}{2}$, $H_{d,b}$ is flexible. Moreover, since M is a submatrix of the rigidity matrix of $H^+_{d,b}$, which satisfies the Maxwell counts, we see that $H^+_{d,b}$ is also flexible. □

For odd-dimensional bananas, we can show this bound is tight using the following proposition.

Proposition 16 *Any linear combination of rows of $M^*_{B_{d,b}}$ of the form*

$$\begin{matrix} V_1 & U \\ [\,0 & \mid \; *\,] \end{matrix},$$

must be trivial, where the $*$ represents potentially nonzero entries.

Proof. Suppose for contradiction that there is a linear combination of rows of $M^*_{B_{d,b}}$ equal to R where R has nonzero entries only in columns corresponding to U. Let $\overline{R}$ be the projection of R to the columns corresponding to U.

If $\overline{R}$ is dependent on the rows in M_{K_U}, then the rank of $M_{B_{d,b}}$ is not maximal, which is a contradiction. So, we must assume that $\overline{R}$ is independent of these rows. Thus, the nullspace of M_{K_U} augmented by the row $\overline{R}$ is smaller than the nullspace of M_{K_U}. But all of the elements of the nullspace of M_{K_U} are obtained from rigid motions of $\mathbb{R}^d$. So there is a nonzero vector $\mathbf{p}' \in \mathbb{R}^{db}$ in the null space of M_{K_U} which assigns velocities to vertices in U and has the property that $\overline{R} \cdot \mathbf{p}' \neq 0$.

Since K_U is rigid, $\mathbf{p}'$ must be obtained by restricting a rigid motion of $\mathbb{R}^d$ to K_U. Applying this rigid motion to all of $B_{d,b}$ gives a vector $\mathbf{q}'$ that assigns velocities to all vertices in $B_{d,b}$ and is equal to $\mathbf{p}'$ for the vertices in U. As R has nonzero entries only in columns corresponding to U, $\mathbf{q}' \cdot R = \mathbf{p}' \cdot R \neq 0$. R is in the row space of $M_{B_{d,b}}$, so this implies that the nullspace of $M_{B_{d,b}}$ is missing one of the rigid motions of $\mathbb{R}^d$. This is a contradiction because $M_{B_{d,b}}$ is a rigidity matrix. □

Theorem 17 *The hyperbanana $H_{d,b} \subset \mathbb{R}^d$ where $d = 2b-1$ has rigidity matrix $M_{H_{d,b}}$ with nullity exactly $\binom{d+1}{2} + \binom{b}{2}$.*

Proof. We will show that

$$M' = \begin{array}{c|c} & \begin{matrix} V_1 & U & V_2 \end{matrix} \\ B_{d,b}(1) & \left[\begin{array}{cc|c} \multicolumn{2}{c|}{M_{B_{d,b}}(1)^*} & 0 \\ \hline 0 & M_{K_U} & \\ \hline 0 & \multicolumn{2}{c}{M_{B_{d,b}}(2)^*} \end{array}\right] \\ B_{d,b}(2) & \end{array}$$

has full rank and hence nullity $\binom{d+1}{2} + \binom{b}{2}$.

Since $M_{B_{d,b}}(1)$ has full rank, we know that the top block of M' has linearly independent rows. Similarly, the rows in $[0|M_{B_{d,b}}(2)^*]$ are also an independent set.

Now suppose there is a row $R \in [0|M_{B_{d,b}}(2)^*]$ that is dependent on the upper block of M'; then R is a linear combination of the rows of $[M_{B_{d,b}}(1)^*|0]$ and $[0|M_{K_d}|0]$. There must be at least one row of $[M_{B_{d,b}}(1)^*|0]$ with a nonzero coefficient or we would contradict the independence of $[0|M_{B_{d,b}}(2)]$. Since R is zero in the columns corresponding to vertices in V_1, this implies that there is a linear combination of rows of $[M_{B_{d,b}}(1)^*|0]$ that is nonzero only in the banana vertex columns, which contradicts Proposition 16.

□

4 Conclusions and Future Work

We presented a family of hyperbanana graphs and showed that they are Maxwell graphs under certain conditions. We further proved that they are flexible, providing counterexamples to the sufficiency of the Maxwell counts for bar-and-joint rigidity in dimensions 3 and higher.

For hyperbananas embedded in odd-dimensional spaces, we gave a precise analysis of the space of infinitesimal motions. However, it remains an open problem to give an exact analysis for the even hyperbananas, as the addition of the $\frac{d}{2}$ edges prevents us from extending our proof. Based on Mathematica calculations on randomized embeddings of even hyperbananas, we conjecture the following:

Conjecture 1 *The even hyperbanana $H^+_{d,b} \in \mathbb{R}^d$ where $d = 2b$ and $b \geq 2$ has a rigidity matrix with nullity exactly $\binom{d+1}{2} + \binom{b}{2}$.*

Since counterexamples provide an increased understanding of barriers to finding combinatorial characterizations of higher-dimensional bar-and-joint rigidity, it would also be interesting to further generalize the hyperbananas by parametrizing the number of banana bunches instead of always gluing two.

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