Face-Guarding Polyhedra

Giovanni Viglietta[*]

Abstract

We study the Art Gallery Problem for face guards in polyhedral environments. The problem can be informally stated as: *how many (not necessarily convex) windows should we place on the external walls of a dark building, in order to completely illuminate it?*

We consider both closed and open face guards (i.e., faces with or without their boundary), and we give some upper and lower bounds on the minimum number of faces required to guard a given polyhedron, in terms of the total number of its faces, f. In some notable cases, our bounds are tight: $\lfloor f/6 \rfloor$ *open* face guards for *orthogonal* polyhedra, and $\lfloor f/4 \rfloor$ *open* face guards for 4-*oriented* polyhedra (i.e., polyhedra whose faces have only four different orientations).

Then we show that it is **NP**-hard to approximate the minimum number of (closed or open) face guards within a factor of $\Omega(\log f)$, even for polyhedra that are orthogonal and simply connected.

Along the way we discuss some applications, arguing that face guards are *not* a reasonable model for guards *patrolling* on the surface of a polyhedron.

1 Introduction

Previous work. Art Gallery Problems have been studied in computational geometry for decades: given an *enclosure*, place a (preferably small) set of *guards* such that every location in the enclosure is seen by some guard. Most of the early research on the Art Gallery Problem focused on guarding 2-dimensional polygons with either point guards or segment guards [9, 10, 12].

Gradually, some of the attention started shifting to 3-dimensional settings, as well. Several authors have considered edge guards in 3-dimensional polyhedra, either in relation to the classical Art Gallery Problem or to its variations [2, 3, 4, 13, 14].

Recently, Souvaine et al. [11] introduced the model with *face guards* in 3-dimensional polyhedra. Ideally, each guard is free to roam over an entire face of a polyhedron, including the face's boundary. They gave lower and upper bounds on g, the number of face guards that are required to guard a given polyhedron, in terms of f, the total number of its faces. For general polyhedra, they showed that $\lfloor f/5 \rfloor \leqslant g \leqslant \lfloor f/2 \rfloor$ and, for

---
[*]School of Computer Science, Carleton University, Ottawa ON, Canada, `viglietta@gmail.com`.

the special case of orthogonal polyhedra (i.e., polyhedra whose faces meet at right angles), they showed that $\lfloor f/7 \rfloor \leqslant g \leqslant \lfloor f/6 \rfloor$. They also suggested several open problems, such as studying *open* face guards (i.e., face guards whose boundary is omitted), and the computational complexity of minimizing the number of face guards.

Subsequently, face guards have been studied to some extent also in the case of polyhedral terrains. In [8], a lower bound is obtained, and in [7] it is proved that minimizing face guards in terrains is **NP**-hard.

Our contribution. In this paper we solve some of the problems left open in [11], and we also expand our research in some new directions.

In Section 2 we discuss the face guard model, arguing that a face guard fails to meaningfully represent a guard "patrolling" on a face of a polyhedron. Essentially, there are cases in which the path that such a patrolling guard ought to follow is so complex (in terms of the number of turns, if it is a polygonal chain) that a much simpler path, striving from the face, would guard not only the region visible from that face, but the entire polyhedron. However, face guards are still a good model for illumination-related problems, such as placing (possibly non-convex) windows in a dark building.

In Section 3 we obtain some new bounds on g, for both closed and open face guards. Namely, we generalize the upper bounds given in [11] by showing that, for c-oriented polyhedra (i.e., whose faces have c distinct orientations), $g \leqslant \lfloor f/2 - f/c \rfloor$. We also provide some new lower bound constructions, which meet our upper bounds in two notable cases: orthogonal polyhedra with open face guards ($g = \lfloor f/6 \rfloor$), and 4-oriented polyhedra with open face guards ($g = \lfloor f/4 \rfloor$).

In Section 4 we provide an approximation-preserving reduction from SET COVER to the problem of minimizing the number of (closed or open) face guards in simply connected orthogonal polyhedra. It follows that the minimum number of face guards is **NP**-hard to approximate within a factor of $\Omega(\log f)$. We also discuss the membership in **NP** of the minimization problem.

2 Model and motivations

Definitions. Given a polyhedron in $\mathbb{R}^3$, we say that a point x is *visible* to a point y if no point in the straight

line segment xy lies in the exterior of the polyhedron. For any point x, we denote by $\mathcal{V}(x)$ the *visible region* of x, i.e., the set of points that are visible to x. In general, for any set $S \subset \mathbb{R}^3$, we let $\mathcal{V}(S) = \bigcup_{x \in S} \mathcal{V}(x)$. A set is said to *guard* a polyhedron if its visible region coincides with the entire polyhedron (including its boundary). The Art Gallery Problem for face guards in polyhedra consists in finding a (preferably small) set of faces whose union guards a given polyhedron. If such faces include their relative boundary, they are called *closed* face guards; if their boundary is omitted, they are called *open* face guards.

A polyhedron is *c-oriented* if there exist c unit vectors such that each face is orthogonal to one of the vectors. If these unit vectors form an orthonormal basis of $\mathbb{R}^3$, the polyhedron is said to be *orthogonal*. Hence, a cube is orthogonal, a tetrahedron and a regular octahedron are both 4-oriented, etc.

Motivations. There is a straightforward analogy between *guarding* problems and *illumination* problems: placing guards in a polyhedron corresponds to placing light sources in a dark building, in order to illuminate it completely. For instance, a point guard would model a *light bulb* and a segment guard could be a *fluorescent tube*. Because face guards are 2-dimensional and lie on the boundary of the polyhedron, we may think of them as *windows*. A window may have any shape, but should be flat, and hence it should lie on a single face. It follows that, if our purpose is to illuminate as big a region as possible, we may assume without loss of generality that a window always coincides with some face.

Face guards were introduced in [11] to represent guards *roaming* over a face. This is in accordance with the traditional usage of segment guards as a model for guards that *patrol* on a line [9]. While this is perfectly sound in the case of segment guards, face guards pose additional problems, as explained next.
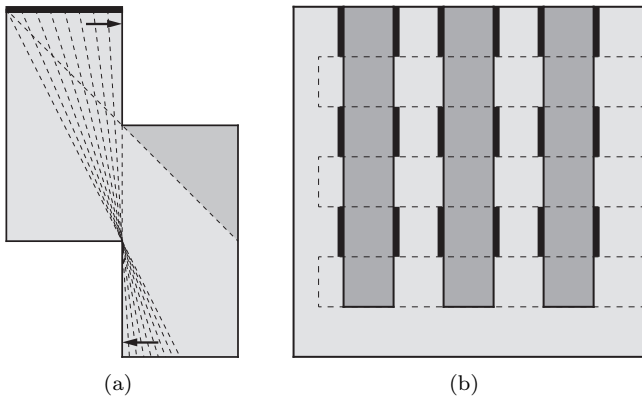

Figure 1: Constructing the polyhedron in Figure 2

We begin by observing that, even in 2-dimensional

polygons, there may be edge guards that cannot be locally "replaced" by finitely many point guards. Figure 1(a) shows an example: if a subset G of the top edge ℓ is such that $\mathcal{V}(G) = \mathcal{V}(\ell)$, then the right endpoint of ℓ must be a limit point of G.

We can exploit this fact to construct the class of polyhedra sketched in Figure 2. We cut long parallel *dents* on opposite faces of a cuboid, in such a way that the resulting polyhedron looks like an extruded "iteration" of the polygon in Figure 1(a). Then we stab this construction with a row of *girders* running orthogonally with respect to the dents.
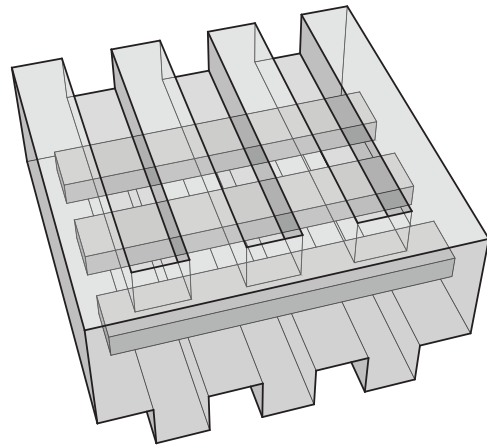

Figure 2: A guard patrolling on the top face must follow a path of quadratic complexity

Suppose that a guard has to patrol the top face of this construction, eventually seeing every point that is visible from that face. The situation is represented in Figure 1(b), where the light-shaded region is the top face, and the dashed lines mark the underlying girders. By the above observation and by the presence of the girders, each thick vertical segment must be approached by the patrolling guard from the interior of the face.

Suppose that the polyhedron has n dents and n girders. Then, the number of its vertices, edges, or faces is $\Theta(n)$. Now, if the guard moves along a polygonal chain lying on the top face, such a chain must have at least a vertex on each thick segment, which amounts to $\Omega(n^2)$ vertices. Similarly, if the face guard has to be substituted with segment guards lying on it, quadratically many guards are needed.

On the other hand, it is easy to show that a path of linear complexity is sufficient to guard any polyhedron: triangulate every face (thus adding linearly many new "edges") and traverse the resulting 1-skeleton in depth-first order starting from any vertex, thus covering all edges. Because the set of edges is a guarding set for any polyhedron [13], the claim follows.

This defeats the purpose of having faces model guards patrolling on segments, as it makes little sense for a

face of "unit weight" to represent quadratically many guards. Analogously, a roaming guard represented by a face may have to follow a path that is overly complex compared to the guarding problem's optimal solution.

Even if we are allowed to replace a face guard with guards patrolling any segment in the polyhedron (i.e, not necessarily constrained to live on that face), a linear number of them may be required. Indeed, consider a cuboid with very small height, and arrange n thin and long *chimneys* on its top, in such a way that no straight line intersects more than two chimneys. The complexity of the polyhedron is $\Theta(n)$, and a face guard lying on the bottom face must be replaced by $\Omega(n)$ segment guards. On the other hand, we know that a linear amount of segment guards is enough not only to "dominate" a single face, but to entirely guard any polyhedron.

Summarizing, a face guard appropriately models an entity that is naturally constrained to live on a single face, like a flat window, and unlike a team of patrolling guards. In the case of a single roaming guard, the model is insensitive to the complexity of the guard's path.

3 Bounds on face guard numbers

Upper bounds. By generalizing the approach used in [11, Lemmas 2.1, 3.1], we provide an upper bound on face guard numbers, which becomes tight for open face guards in orthogonal polyhedra and open face guards in 4-oriented polyhedra. We emphasize that our upper bound holds for both closed and open face guards, and for polyhedra of any genus.

Theorem 1 *Any c-oriented polyhedron with f faces is guardable by*

$$\left\lfloor \frac{f}{2} - \frac{f}{c} \right\rfloor$$

closed or open face guards.

Proof. Let $\mathcal{P}$ be a polyhedron whose faces are orthogonal to $c \geqslant 3$ distinct vectors. Let f_i be the number of faces orthogonal to the i-th vector v_i. We may assume that $i < j$ implies $f_i \geqslant f_j$. Then,

$$f_1 + f_2 \geqslant \left\lfloor \frac{2f}{c} \right\rfloor.$$

Let us stipulate that the direction of the cross product $v_1 \times v_2$ is *vertical*. Thus, there are at most

$$f - \left\lfloor \frac{2f}{c} \right\rfloor$$

non-vertical faces. Some of these are facing up, the others are facing down. Without loss of generality, at most half of them are facing down, and we assign a face guard to each of them. Therefore, at most

$$\left\lfloor \frac{f}{2} - \frac{f}{c} \right\rfloor$$

face guards have been assigned.

Let x be any point in $\mathcal{P}$. If x belongs to a face with a guard, x is guarded. Otherwise, consider an infinite circular cone $\mathcal{C}$ with apex x and axis directed upward. Let $\mathcal{G}$ be the intersection of $\mathcal{V}(x)$, $\mathcal{C}$, and the boundary of $\mathcal{P}$. If the aperture of $\mathcal{C}$ is small enough, the relative interior of $\mathcal{G}$ belongs entirely to faces containing guards and to at most two vertical faces containing x. Because these vertical faces obstruct at most one dihedral angle from x's view, the portion of $\mathcal{G}$ not belonging to them has non-empty interior. If we remove from this portion the (finitely many) edges of $\mathcal{P}$, we still have a non-empty region. By construction, this region belongs to the interiors of faces containing a guard; hence x is guarded. □

Our guarding strategy becomes less efficient as c grows. In general, if no two faces are parallel (i.e., $c = f$), we get an upper bound of $\lfloor f/2 \rfloor - 1$ face guards, which improves on the one in [11] by just one unit.

Lower bounds. In [11], Souvaine et al. construct a class of orthogonal polyhedra with f faces that need $\lfloor f/7 \rfloor$ *closed* face guards. In Figure 3 we give an alternative construction, with the additional property of having a 3-regular 1-skeleton. Indeed, each small L-shaped polyhedron that is attached to the big cuboid adds seven faces to the construction, of which at least one must be selected.
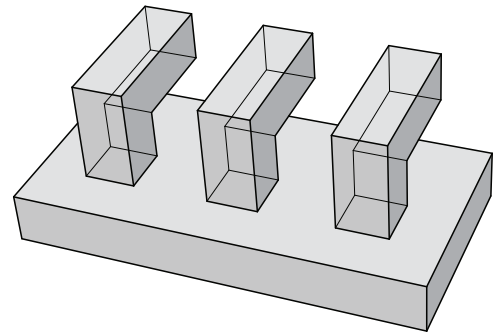

Figure 3: Orthogonal polyhedron that needs $\lfloor f/7 \rfloor$ closed face guards

For *open* face guards, we modify our previous construction by moving all the L-shaped pieces to the boundary of the top face, as in Figure 4. Thus, each piece adds just six faces to the construction (one face is shared by all of them), of which at least one must be selected. Moreover, no matter how these faces are selected, some portion of the big cuboid below remains unguarded, and needs one more face guard. This amounts to $\lfloor f/6 \rfloor$ open face guards in total.

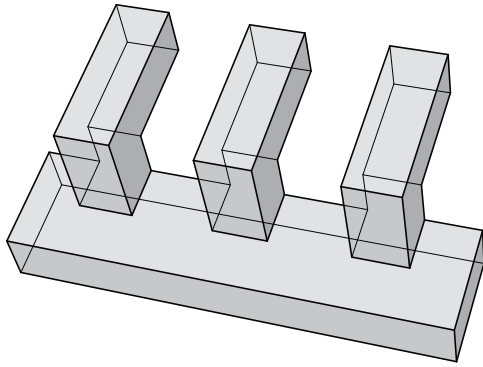Plugging $c = 3$ in Theorem 1 reveals that our lower bound is also tight.

Figure 4: Orthogonal polyhedron that needs $\lfloor f/6 \rfloor$ open face guards

Theorem 2 *To guard an orthogonal polyhedron having f faces, $\lfloor f/6 \rfloor$ open face guards are always sufficient and occasionally necessary.* □

Moving on to *closed* face guards in 4-oriented polyhedra, we propose the construction in Figure 5. Each closed face sees the tip of at most one of the k tetrahedral *spikes*, hence k guards are needed. Because there are $5k + 2$ faces in total, a lower bound of $\lfloor f/5 \rfloor$ closed face guards follows.
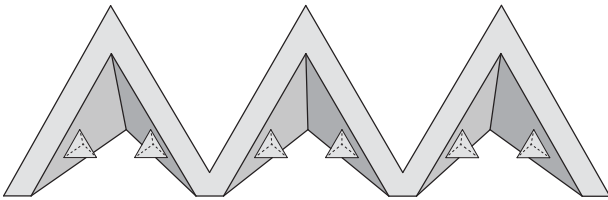

Figure 5: 4-oriented polyhedron that needs $\lfloor f/5 \rfloor$ closed face guards

For *open* face guards in 4-oriented polyhedra, we modify the previous example by carefully placing additional spikes on the other side of the construction, as Figure 6 illustrates. Once again, since each open face sees the tip of at most one of the k spikes and there are $4k + 2$ faces in total, a lower bound of $\lfloor f/4 \rfloor$ open face guards follows.
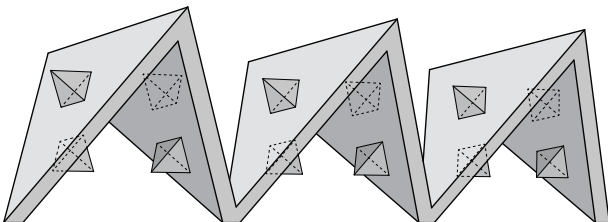

Figure 6: 4-oriented polyhedron that needs $\lfloor f/4 \rfloor$ open face guards

This bound is also tight, as easily seen by plugging $c = 4$ in Theorem 1.

Theorem 3 *To guard a 4-oriented polyhedron having f faces, $\lfloor f/4 \rfloor$ open face guards are always sufficient and occasionally necessary.* □

4 Minimizing face guards

Hardness of approximation. In [11], Souvaine et al. ask for the complexity of minimizing face guards in a given polyhedron. We show that this problem is at least as hard as SET COVER, and we infer that approximating the minimum number of face guards within a factor of $\Omega(\log f)$ is **NP**-hard.

Theorem 4 SET COVER *is L-reducible to the problem of minimizing (closed or open) face guards in a simply connected orthogonal polyhedron.*

Proof. Let an instance of SET COVER be given, i.e., a *universe* $U = \{1, \cdots, n\}$, and a collection $S \subseteq \mathcal{P}(U)$ of $m \geqslant 1$ subsets of U. We will construct a simply connected orthogonal polyhedron with $f = O(mn)$ faces that can be guarded by k (closed or open) faces if and only if U is the union of $k - 1$ elements of S.

Figure 7 shows our construction for $U = \{1, 2, 3, 4\}$ and $S = \{\{2, 4\}, \{1, 3\}, \{2\}\}$. Figure 8 illustrates the side view of a generic case in which $m = 4$.
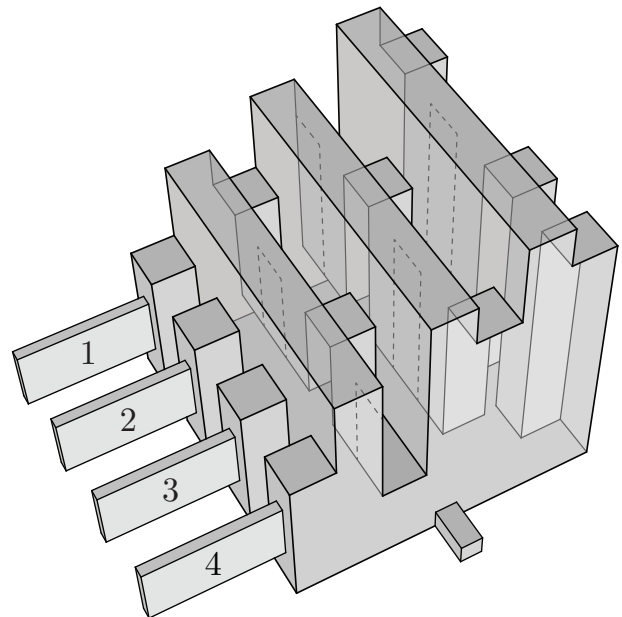

Figure 7: SET COVER reduction, 3D view

Each of the thin cuboids on the far left is called a *fissure*, and represents an element of U. In front of the fissures there is a row of m *mountains* of increasing height, separated by *valleys* of increasing depth. The
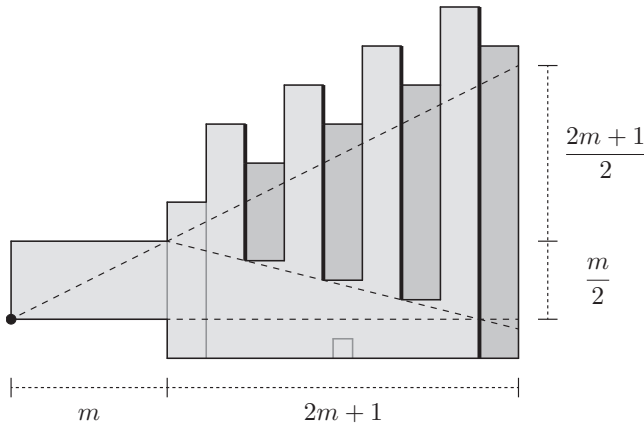
Figure 8: SET COVER reduction, side view

m vertical walls that are facing the fissures (drawn as thick lines in Figure 8) are called *set faces*, and each of them represents an element of S.

For each $S_i \in S$, we dig a narrow rectangular *dent* in the i-th set face in front of the j-th fissure, if and only if $j \notin S_i$. Each dent reaches the bottom of its set face, and almost reaches the top, so that it does not separate the set face into two distinct faces. Moreover, every dent (except those in the rightmost set face) is so deep that it connects two neighboring valleys. In Figure 8, dents are depicted as darker regions; in Figure 7, the dashed lines mark the areas where dents are *not* placed.

We want to fix the width of the fissures in such a way that only a restricted number of faces can see their bottom. Specifically, consider n *distinguished points*, located in the middle of the lower-left edges of the fissures (indicated by the thick dot in Figure 8). The j-th distinguished point definitely sees some portions of the i-th set face, provided that $j \in S_i$. If this is the case, and $i < m$, it also sees portions of two other faces (one horizontal, one vertical) surrounding the same valley. Moreover, if $j \notin S_m$, the j-th distinguished point also sees the bottom of a dent in the rightmost set face. We want no face to be able to see any distinguished point, except the faces listed above (plus of course the faces belonging to fissures or surrounding their openings). To this end, assuming that the dents have unit width, we set the width of the fissures to be slightly less than $1/4$. Indeed, referring to Figure 8, the width of the visible region of a distinguished point, as it reaches the far right of the construction, is strictly less than

$$\frac{(m) + (2m+1)}{m} \cdot \frac{1}{4} = \left(3 + \frac{1}{m}\right) \cdot \frac{1}{4} \leqslant 4 \cdot \frac{1}{4} = 1,$$

because $m \geqslant 1$.

Finally, a small *niche* is added in the lower part of the construction. Its purpose is to enforce the selection of a "dedicated" face guard, as no face can see both a distinguished point and the bottom of the niche.

Let a guarding set for our polyhedron be given, consisting of k face guards. We will show how to compute in polynomial time a solution of size at most $k-1$ for the given SET COVER instance, provided that it is solvable at all.

We first discard every face guard that is not guarding any distinguished point. Because at least one face must guard the niche, we are left with at most $k - 1$ guards. Then, if any of the remaining face guards borders the i-th valley, with $i < m$, we replace it with the i-th set face. Indeed, it is easy to observe that such set face can see the same distinguished points, plus possibly some more. By construction, all the remaining guards can see exactly one distinguished point (they are either faces belonging to some fissure, or surrounding its opening, or bottom faces of the rightmost dents). We replace each of these face guards with any set face that guards the same distinguished point (which exists, otherwise the SET COVER instance would be unsolvable). As a result, we have at most $k - 1$ set faces guarding all the distinguished points. These immediately determine a solution of equal size to the given SET COVER instance.

Conversely, if the SET COVER instance has a solution of size k, it is easy to see that our polyhedron has a guarding set of $k + 1$ guards: all the set faces corresponding to the SET COVER's solution, plus the bottom face. □

Corollary 5 *Given a simply connected orthogonal polyhedron with f faces, it is **NP**-hard to approximate the minimum number of (closed or open) face guards within a factor of $\Omega(\log f)$.*

Proof. The polyhedra constructed in the L-reduction of Theorem 4 have $f = O(mn)$ faces. It was proved in [1] that SET COVER is **NP**-hard to approximate within a ratio of $\Omega(\log n)$ and, by inspecting the reduction employed, it is apparent that all the hard SET COVER instances generated are such that $m = O(n^c)$, for some constant $c \geqslant 1$. As a consequence, we may assume that $\Omega(\log n) = \Omega(\log n^{c+1}) = \Omega(\log(mn)) = \Omega(\log f)$, and our claim follows. □

Computing visible regions. The next natural question is whether the minimum number of face guards can be computed in **NP**, and possibly approximated within a factor of $\Theta(\log f)$. Usually, when finitely many possible guard locations are allowed (such as with vertex guards and edge guards), this is established by showing that the visible region of any guard can be computed efficiently, as well as the intersection of two visible regions, etc. As a result, the environment is partitioned into polynomially many regions such that, for every region R and every guard g, either $R \subseteq \mathcal{V}(g)$ or $R \cap \mathcal{V}(g) = \varnothing$. This immediately leads to a reduction to SET COVER,

which implies an approximation algorithm with logarithmic ratio, via a well-known greedy heuristic [6].

With face guards (and also with edge guards in polyhedra) the situation is complicated by the fact that the visible region of a guard may not be a polyhedron, but in general its boundary is a piecewise quadric surface.

For example, consider the orthogonal polyhedron in Figure 9. It is easy to see that the visible region of the bottom face (and also the visible region of edge a) is the whole polyhedron, except for a small region bordered by the thick dashed lines.
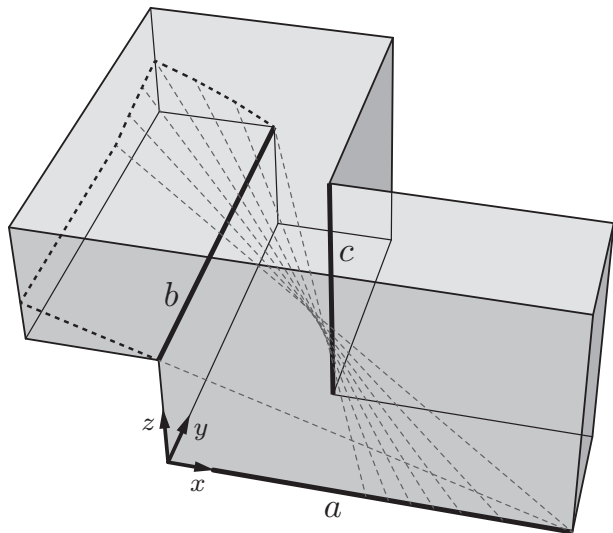

Figure 9: The visible region of the bottom face is bounded by a hyperboloid of one sheet.

The surface separating the visible and invisible regions consists of a right trapezoid plus a bundle of mutually skew segments whose extensions pass through the edges a, b, and c. These edges lie on three lines having equations

$$y^2 + z^2 = 0,$$
$$x^2 + (z-1)^2 = 0,$$
$$(x-1)^2 + (y-1)^2 = 0,$$

respectively. A straightforward computation reveals that the bundle of lines passing through these three lines has equation

$$xy - xz + yz - y = 0,$$

which defines a hyperboloid of one sheet.

In general, the boundary of the visible area of a face (or an edge) is determined by lines passing through pairs or triplets of edges of the polyhedron. If three edges are all parallel to a common plane, the surface they determine is a hyperbolic paraboloid (degenerating into a plane if two of the edges are parallel to each other), otherwise they determine a hyperboloid of one sheet, as in the above example.

There exists an extensive literature of purely algebraic methods to compute intersections of quadric surfaces (see for instance [5]), but the parameterizations involved may yield coefficients containing radicals. At this stage in our understanding, it is not clear whether any of these methods can be effectively applied to reduce the minimization problem of face-guarding polyhedra (or even edge-guarding polyhedra) to SET COVER.

References

[1] N. Alon, D. Moshkovitz, and S. Safra. Algorithmic construction of sets for k-restrictions. *ACM Transactions on Algorithms*, vol. 2, pp. 153–177, 2006.

[2] N. Benbernou, E. D. Demaine, M. L. Demaine, A. Kurdia, J. O'Rourke, G. T. Toussaint, J. Urrutia, and G. Viglietta. Edge-guarding orthogonal polyhedra. In *Proceedings of the 23rd Canadian Conference on Computational Geometry*, pp. 461–466, 2011.

[3] J. Cano, C. D. Tóth, and J. Urrutia. Edge guards for polyhedra in 3-space. In *Proceedings of the 24th Canadian Conference on Computational Geometry*, pp. 155–160, 2012.

[4] T. Christ and M. Hoffmann. Wireless localization within orthogonal polyhedra. In *Proceedings of the 23rd Canadian Conference on Computational Geometry*, pp. 467–472, 2011.

[5] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. A new algorithm for the robust intersection of two general quadrics. In *Proceedings of the 19th Annual ACM Symposium on Computational Geometry*, pp. 246–255, 2003.

[6] S. K. Ghosh. Approximation algorithms for art gallery problems. In *Proceedings of the Canadian Information Processing Society Congress*, pp. 429–434, 1987.

[7] C. Iwamoto, Y. Kitagaki, and K. Morita. Finding the minimum number of face guards is NP-hard. *IEICE Transactions on Information and Systems*, vol. E95-D, pp. 2716–2719, 2012.

[8] C. Iwamoto, J. Kishi, and K. Morita. Lower bound of face guards of polyhedral terrains. *Journal of Information Processing*, vol. 20, pp. 435-437, 2012.

[9] J. O'Rourke. *Art gallery theorems and algorithms*. Oxford University Press, New York, 1987.

[10] T. Shermer. Recent results in art galleries. In *Proceedings of the IEEE*, vol. 80, pp. 1384–1399, 1992.

[11] D. L. Souvaine, R. Veroy, and A. Winslow. Face guards for art galleries. In *Proceedings of the XIV Spanish Meeting on Computational Geometry*, pp. 39–42, 2011.

[12] J. Urrutia. Art gallery and illumination problems. In J.-R. Sack and J. Urrutia, editors, *Handbook of Computational Geometry*, pp. 973–1027, North-Holland, 2000.

[13] G. Viglietta. *Guarding and searching polyhedra*. Ph.D. Thesis, University of Pisa, 2012.

[14] G. Viglietta. Searching polyhedra by rotating half-planes. *International Journal of Computational Geometry and Applications*, vol. 22, pp. 243–275, 2012.