One-Round Discrete Voronoi Game in $\mathbb{R}^2$ in Presence of Existing Facilities

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Abstract

In this paper we consider a simplified variant of the discrete Voronoi Game in $\mathbb{R}^2$, which is also of independent interest in competitive facility location. The game consists of two players P1 and P2, and a finite set U of users in the plane. The players have already placed two sets of facilities F and S, respectively in the plane. The game begins by P1 placing a new facility followed by P2 placing another facility, and the objective of both the players is to maximize their own total payoffs. When $|F| = |S| = m$, this corresponds to the last round of the $(m + 1)$-round discrete Voronoi Game in $\mathbb{R}^2$. In this paper we propose polynomial time algorithms for obtaining optimal strategies of both the players under arbitrary locations of the existing facilities F and S. We show that the optimal strategy of P2, given any placement of P1, can be found in $O(n^2)$ time, and the optimal strategy of P1 can be found in $O(n^8)$ time.

1 Introduction

The main objective in any facility location problem is to judiciously place a set of facilities serving a set of users such that certain optimality criteria are satisfied. Facilities and users are generally modeled as points in the plane. The set of users (demands) is either *discrete*, consisting of finitely many points, or *continuous*, that is, a region where every point is considered to be a user. We assume that the facilities are equally equipped in all respects, and a user always avails the service from its nearest facility. Consequently, each facility has its *service zone*, consisting of the set of users that are served by it. For a set U of users, finite or infinite, and a set F of facilities, define for every $f \in F$, $U(f, F)$ as the set of users in U that are served by the facility f. In such a scenario, when the users choose the facilities based on the nearest-neighbor rule, the optimization criteria is to maximize the cardinality or the area of the service zone depending on whether the demand region is discrete or continuous, respectively.

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The game-theoretic analogue of such competitive problems for continuous demand regions is a situation where two players place two disjoint sets of facilities in the demand region. A player p is said to own a part of the demand region that is closer to the facilities owned by p than to the other player, and the player which finally owns the larger area is the winner of the game. The area a player owns at the end of the game is called the payoff of the player. In the *one-round game* the first player places m facilities following which the second player places another m facilities in the demand region. In the *m-round game* the two players place one facility each alternately for m rounds in the demand region.

Ahn et al. [1] studied a one-dimensional Voronoi Game, where the demand region is a line segment. They showed that when the game takes m rounds, the second player always has a winning strategy that guarantees a payoff of $1/2 + \varepsilon$, with $\varepsilon > 0$. However, the first player can force ε to be arbitrarily small. On the other hand, in the one-round game with m facilities, the first player always has a winning strategy. The one-round Voronoi Game in $\mathbb{R}^2$ was studied by Cheong et al. [7], for a square-shaped demand region. They proved that for any placement W of the first player, with $|W| = m$, there is a placement B of the second player $|B| = m$ such that the payoff of the second player is at least $1/2 + \alpha$, where $\alpha > 0$ is an absolute constant and m large enough. Fekete and Meijer [9] studied the two-dimensional one-round game played on a rectangular demand region with aspect ratio ρ. The Voronoi Game, for which the underlying space is a graph, was considered by Bandyapadhyay et al. [3].

In the discrete regime, the possible demand set is generally modeled as a finite graph, and users and facilities are restricted to lie on the nodes of the graph. As before, the players alternately chose nodes (facilities) from the graph, and all vertices (customers) are then assigned to closest facilities based on the graph distance. The payoff of a player is the number of customers assigned to it. Dürr and Thang [8] showed that deciding the existence of a Nash equilibrium for a given graph is NP-hard. Recently, Teramoto et al. [13] studied the same problem and considered following very restricted case: the game arena is an arbitrary graph, the first player occupies just one vertex which is predetermined, and the second player occupies m vertices in any way. They proved that in this strongly restricted discrete Voronoi Game it is NP-hard to decide whether the second player has a winning strategy. They also proved that for a given graph G and the number r of rounds determining whether the first player has a winning strategy on G is PSPACE-complete. The discrete Voronoi Game for path graphs was studied by Kiyomi et al. [10].

Recently, Banik et al. [4] considered the discrete Voronoi Game where the universe is modeled as $\mathbb{R}$, and the distance between the users and the facilities are measured by their Euclidean distance. The problem consists of a finite user set $U \subset \mathbb{R}$, with $|U| = n$, and two players Player 1 (P1) and Player 2 (P2) each having $m = O(1)$ facilities. At first, P1 chooses a set $\mathcal{F}_1 \subset \mathbb{R}$ of k facilities following which P2 chooses another set $\mathcal{F}_2 \subset \mathbb{R}$ of m facilities, disjoint from $\mathcal{F}_1$. The payoff of P2 is defined as the cardinality of the set of points in U which are closer to a facility owned by P2 than to every facility owned by P1. The payoff of P1 is the number of users in U minus the payoff of P2. The objective of both the players is to maximize their respective payoffs. The authors showed that if the sorted order of the points in U along the line is known, then the optimal strategy of P2, given any placement of facilities by P1, can be computed in $O(n)$ time. Also, for $m \geq 2$ the optimal strategy of P1 can be computed in $O(n^{m-\lambda_m})$ time, where $0 < \lambda_m < 1$, is a constant depending only on m. The discrete Voronoi Game for polygonal domains were considered by Banik et al. [5].

The discrete Voronoi Game when the user set consists of a finite set of points in $\mathbb{R}^2$ poses a major challenge. To the best of our knowledge, this problem has never been addressed before, and answering rather simple questions about this game is rather difficult. In this paper we consider a simplified variant of the discrete Voronoi Game in $\mathbb{R}^2$, which is also of independent interest in competitive facility location. The game consists of two players P1 and P2 and a finite set U of users in the plane. Moreover, the two players have already placed a set of facilities F and S, respectively, in the plane. The game begins by P1 placing a new facility followed by P2 placing another facility. The objective of both the players is to maximize their respective payoffs.

For any placement of facilities A by P1 and B by P2, the payoff of P2, $\mathcal{P}_2(A, B)$ is defined as the cardinality of the set of points in U which are closer to a facility owned by P2 than to every facility owned by P1, that is, $\mathcal{P}_2(A, B) = |\bigcup_{f \in B} U(f, A \cup B)|$. Similarly, the payoff of P1, $\mathcal{P}_1(A, B)$ is $|\bigcup_{f \in A} U(f, A \cup B)|$, $|U \backslash \mathcal{P}_2(A, B)|$. Now, the One Round Discrete Voronoi Game in $\mathbb{R}^2$ in Presence of Existing Facilities can be formally stated as follows:

One Round Discrete Voronoi Game in $\mathbb{R}^2$ in Presence of Existing Facilities: Given a set U of n users, and two sets of facilities F and S owned by two competing players P1 and P2, respectively, at first P1 chooses a facility f_1 following which P2 chooses another facility f_2 such that

(a) $\max_{f_2' \in \mathbb{R}^2} |\mathcal{P}_2(F \cup \{f_1\}, S \cup \{f_2'\})|$ is attained at the point f_2.

(b) $\max_{f \in \mathbb{R}^2} \nu(f)$ is attained at the point f_1, where $\nu(f) = n - \max_{f_2' \in \mathcal{R}} |\mathcal{P}_2(F \cup \{f\}, S \cup \{f_2\})|$.

The quantity $\nu(f_1)$ is called the optimal payoff of P1 and f_1 is the optimal strategy of P1.

In this paper we develop algorithms for the optimal strategies of the two players in the above game. Hereafter, we shall refer to this version of the Voronoi Game as $G_n(F, S)$. Note that when $|F| = |S| = m$ the situation described in the $G_n(F, S)$ game is identical to the last round of the $(m + 1)$-round discrete Voronoi Game in $\mathbb{R}^2$. Therefore, this problem takes the first nontrivial step towards solving the discrete Voronoi Game problem in $\mathbb{R}^2$. Moreover, as mentioned before, this problem is of independent interest in competitive facility location. In any growing economy the expansion of the service zone is of utmost importance. However, because of some implied constraint it is never possible to place all your facilities at once. So it is of utmost importance to find a strategy which will guide how to place a set of facilities in a sequential manner, as the market grows. The $G_n(F, S)$ game is an instance of such a problem. Imagine there are 2 competing companies are providing a service to a set of users in a city. Suppose both these companies already have their respective service centers located in different parts of the city. Now, if both of them wish to open a new service center with the individual goal to maximize their total payoff, then the problem is an instance of the Voronoi Game described above. Though the $G_n(F, S)$ game, as described above, has never been studied before, if both F and S are empty, then it is a well-known fact that optimal strategy of P1 in the $G_n(F, S)$ game is at the halfspace median of U [12], which can be computed in $O(n \log^3 n)$ time [11]. However when the sets F and S are non-empty the problem becomes immensely more complicated. In this paper we propose polynomial time algorithms for obtaining optimal strategies of both the players in the $G_n(F, S)$ game.

The optimal strategy of P2, given any placement of P1, is identical to the solution of the MaxCov problem studied by Cabello et al. [6]. Suppose we are given a set of users U, existing facilities F and S, and any placement of a new facility f by P1. Let $U_1 \subseteq U$ denote the subset of users that are served by P1, in presence of F, S, and f. For every point $u \in U_1$, consider the *nearest facility disk* C_u centered at u and passes through the facility in $F \cup \{f\}$ which is closet to u. Note that a new facility s placed by P2 will serve any user $u \in \mathcal{U}_1$ if and only if $s \in C_u$. If $\mathcal{C} = \{C_u | u \in U_1\}$, the optimal strategy for P2, given any placement f of P1, is to place the new facility at a point where maximum number of disks in $\mathcal{C}$ overlap. This is the problem of finding the maximum depth in an arrangement of n disks, and can be computed in $O(n^2)$ time [2].

Therefore, the main challenge in the $G_n(F,S)$ game lies in finding the optimal strategy of P1. In this paper, we provide a complete characterization of the event points and obtain a polynomial time algorithm for obtaining an optimal placement of P1:

Theorem 1 *Given a set U of n users, two sets of facilities F and S owned by two competing players P1 and P2, respectively, the optimal strategy of P1 in the $G_n(F,S)$ game can be found in $O(n^8)$ time.*

2 Understanding the Optimal Strategy of P1

In the $G_n(F,S)$ game, we are given a set U of n users, two sets of facilities F and S owned by two competing players P1 and P2 respectively. Observe that the set of facilities F and S will divide the set of users U into two groups U_F and U_S where U_F is the set of users served by the facilities placed by P1 and U_S is the set of users served by the facilities placed by P2.

Let f be any new placement by P1. Denote the set of users served by f, by $U_{FS}(f)$. More formally,

$$U_{FS}(f) = \{u_i | d(u_i, f) < d(u_i, f'), \forall f' \in F \cup S)\}$$

Further let the users served by the set of facilities F and S after placement of f be $U_{F\setminus f}$ and $U_{S\setminus f}$, that is,

$$U_{F\setminus f} = \bigcup_{f' \in F} U(f', F \cup S \cup \{f\})$$

and

$$U_{S\setminus f} = \bigcup_{f' \in S} U(f', F \cup S \cup \{f\}).$$

Hence, any facility f by P1 will divide the set of users into three disjoint sets $U_{FS}(f)$, $U_{F\setminus f}$ and $U_{S\setminus f}$. Now any new placement s by P2 can serve a subset of users from all these three sets. For any placement of facility s by P2, let $U_f(s) \subset U_{FS}(f)$ be the set of users such that for all $u_i \in U_f(s)$, $d(u_i, s) < d(u_i, f)$. Similarly define the set of users $U_{F\setminus f}(s) \subset U_{F\setminus f}$ such that for all $u_j \in U_{F\setminus f}(s)$, $d(u_j, s) < d(u_j, f_k)$ for all $f_k \in F$.

Observe that for any placement f and s by P1 and P2 respectively the payoff of P2 will be equal to

$$\mathcal{P}_2(F \cup \{f\}, S \cup \{s\}) = |U_{S\setminus f}| + |U_f(s)| + |U_{F\setminus f}(s)|$$

For any placement of facility f by P1 define the *effective depth* of f, $\delta(f)$ as

$$\delta(f) = |U_{S\setminus f}| + \max_{s \in \mathbb{R}^2} (|U_f(s)| + |U_{F\setminus f}(s)|)$$

The optimal strategy of P1 is to find a point f_1 such that $\delta(f_1) = \arg\min_{f \in \mathbb{R}^2} \delta(f)$, that is the point of minimum effective depth. In order to do that we will subdivide $\mathbb{R}^2$ into a polynomial many cells such that the effective depth of all points in each cell is the same.

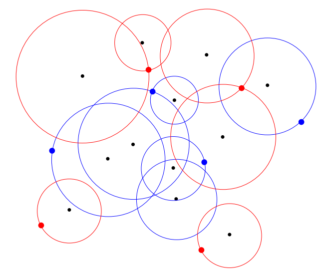

Figure 1: Arrangement of the set of circles $\mathcal{C}_{FS}$

Consider the set of circles $\mathcal{C}_{FS}$ where each circle $C \in \mathcal{C}_{FS}$ is centered at some user u_i and passing through the facility closest to u_i among the set of facilities $F \cup S$ (see Figure 1 where facilities placed by P1 are shown in red and the facilities placed by P2 shown in blue). Denote the arrangement of the set of circles $\mathcal{C}_{FS}$ by $\mathcal{A}(\mathcal{C}_{FS})$. We also include the lines joining any pair of users into $\mathcal{C}_{FS}$.

For any placement of facility x by P1 and for any user u_i, let $C_i(x)$ be the circle centered on u_i and passing though the facility closest to u_i from the set of facilities $F \cup S \cup \{x\}$. Consider all such circles $\mathcal{C}(x)$.

Let x and y be two points that belong to the same cell of $\mathcal{A}(\mathcal{C}_{FS})$ but $\delta(x) \neq \delta(y)$. Now as x and y belong to the same cell of $\mathcal{A}(\mathcal{C}_{FS})$ therefore $U_{S\setminus x} = U_{S\setminus y}$. That means for the placement of facilities x and y, $\max_{s \in \mathbb{R}^2}(|U_x(s)| + |U_{F\setminus x}(s)|) \neq \max_{s \in \mathbb{R}^2}(|U_y(s)| + |U_{F\setminus y}(s)|)$.

Now observe that for any placement of facility x, $\max_{s \in \mathbb{R}^2}(|U_x(s)| + |U_{F\setminus x}(s)|)$ denotes the maximum number of circles among the set of circles $\mathcal{C}(x)$ that can be pierced by a single point. Hence for each cell $\lambda \in \mathcal{A}(\mathcal{C}_{FS})$ if we can subdivide λ further such that in each sub-cell of λ for all points x maximum number of circles among the set of circles $\mathcal{C}(x)$ that can be pierced by a single point remains fixed, then we are done.

Lemma 2 *If x, y belong to some cell of $\mathcal{A}(\mathcal{C}_{FS})$ with $\delta(x) \neq \delta(y)$, then there exist three users $u_i, u_j, u_k \in U_{FS}(x) \cup U_{F\setminus x}$ such that $C_i(x) \cap C_j(x) \cap C_k(x) \neq \emptyset$ and $C_i(y) \cap C_j(y) \cap C_k(y) = \emptyset$ or vice versa.*

Proof. Without loss of generality assume $\delta(x) > \delta(y)$. For any placement of facility f by P1, U_f be the maximum cardinality set of users served by P2. As x and y belong to the same cell of $\mathcal{A}(\mathcal{C}_{FS})$, $U_x \nsubseteq U_{FS}(x)$. Also we can assume that the cardinality of U_x and U_y is at least three. We shall prove this result by contradiction. Suppose that for every three users $u_i, u_j, u_k \in$

$U_{FS}(x) \cup U_{F\backslash x}$, such that $C_i(x) \cap C_j(x) \cap C_k(x) \neq \emptyset$ we also have $C_i(y) \cap C_j(y) \cap C_k(y) \neq \emptyset$.

Therefore, for any three users $u_i, u_j, u_k \in U_x$, $C_i(x) \cap C_j(x) \cap C_k(x) \neq \emptyset$. By assumption, this implies $C_i(y) \cap C_j(y) \cap C_k(y) \neq \emptyset$. Therefore, by Helly's theorem, $\bigcap_{u_i \in U_x} C_i(y) \neq \emptyset$, which means that the number of users that can be served by P2 by placing one facility from the set of users $U_{FS}(y) \cup U_{F\backslash y}$ is at least $|U_x|$. Therefore, $\delta(y) \geq |U_x| + |U_{S\backslash y}| = |U_x| + |U_{S\backslash x}| = \delta(x)$, which is a contradiction and the result holds. □

In light of lemma 2 we define, for each triplet of users $u_i, u_j, u_k \in U$, and any placement of facility $x \in \mathbb{R}^2$ by P1, the indicator variable,

$$\beta_{ijk}(x) = \begin{cases} 1 & \text{if } C_i(x) \cap C_j(x) \cap C_k(x) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Let $\beta(x)$ be the 3-dimensional array with cardinality $|U| \times |U| \times |U|$ where each cell $\beta_{ijk}(x)$ is defined as above. From Lemma 2 and the above definition, the following observation is immediate.

Observation 1 *If x, y belong to the same cell of $\mathcal{A}(\mathcal{C}_{FS})$ and the two arrays $\beta(x)$ and $\beta(y)$ are equal in every coordinate, then $\delta(x) = \delta(y)$.*

Our next goal is to tessellate $\mathcal{A}(\mathcal{C}_{FS})$ into a finer set of cells such that for any two points x and y on the same cell $\beta(x) = \beta(y)$. This implies that for any three users $u_i, u_j, u_k \in U$, either $C_i(x) \cap C_j(x) \cap C_k(x) \neq \emptyset$ or $C_i(x) \cap C_j(x) \cap C_k(x) = \emptyset$, for all points x in a fixed finer cell of the tessellation. Observation 1 would then imply that for all points in a cell the effective depth remains constant. Hence, by checking each cell once we can find the point with minimum effective depth.

Therefore, for each cell of $\mathcal{A}(\mathcal{C}_{FS})$, we want a further subdivision such that for every point x in the fixed subdivided cell $C_i(x) \cap C_j(x) \cap C_k(x) \neq \emptyset$, for every triplet of users $u_i, u_j, u_k \in U$. Note that for any placement x by P1 and a user $u \in U \backslash U_{S\backslash x}$, either u is served by x or by some existing facility in F. The following definition distinguishes these two cases:

Definition 2.1 *Given any placement x by P1 and a user $u \in U \backslash U_{S\backslash x}$, the circle $C(x)$ is called an* old circle *if it is centered at u and passes through some facility $f_j \in F$, where f_j be the facility closest to u among the set of facilities $F \cup \{x\}$, that is, $u \in U_{F\backslash x}$. The circle $C(x)$ is called a* new circle *if it is centered at u and passes through x, that is $u \in U_{FS}(x)$.*

For every three fixed users $u_i, u_j, u_k \in U$ and a fixed point $x \in \mathbb{R}^2$, denote by $N_{ijk}(x)$ the subset of the circles in $\{C_i(x), C_j(x), C_k(x)\}$, which are new. For $S \subseteq \{u_i, u_j, u_k\}$, define the following sets:

$$\Gamma_{ijk}(S) = \{x \in \mathbb{R}^2 : C_i(x) \cap C_j(x) \cap C_k(x) = \emptyset \text{ and } C_a(x) \in N_{ijk}(x) \text{ if } u_a \in S\}$$

Moreover, for $z \in \{0, 1, 2, 3\}$, let $\Gamma^z_{ijk} = \bigcup_{S:|S|=z} \Gamma_{ijk}(S)$, where the union is taken over all sets $S \subseteq \{u_i, u_j, u_k\}$ such that $|S| = z$.

Lemma 3 *Let D_a be the circle centered at u_a passing through the facility in $F \cup S$ closest to u_a, for $a \in 1, 2, \ldots n$. Then for three fixed users $u_i, u_j, u_k \in U$, we have*

(a) $\Gamma_{ijk}(\emptyset) = (D_i \cup D_j \cup D_k)^c$.

(b) For $S = \{u_i, u_j, u_k\}$, $\Gamma_{ijk}(S)$ is the interior of the triangle formed by u_i, u_j and u_k.

(c) For $S = \{u_k\}$, $\Gamma_{ijk}(S)$ is the interior of the circle centered at u_k and passing through the point in $D_i \cap D_j$ closest to u_k.

Proof. It is easy to show (a) and (b) from the definitions.

For proving (c), let p_c be the point in $D_i \cap D_j$ which is closer to u_k. Hence, for all points p in the open disk D, centered at u_k and passing through p_c, $C_i(x) \cap C_j(x) \cap D = \emptyset$, and for all points p outside the open disk D, $p_c \in C_i(x) \cap C_j(x) \cap D$. Therefore, if $S = \{u_k\}$, then $\Gamma_{ijk}(S) = D$. □

As it turns out, when S consists of 2 elements then the sets $\Gamma_{ijk}(S)$ has a complicated geometric structure. The following lemma provides a complete characterization of the set $\Gamma_{ijk}(S)$ where $|S| = 2$.

Lemma 4 *For $S = \{u_i, u_j\}$, $\Gamma_{ijk}(S)$ is an open set bounded by $O(1)$ circular arcs and line segments. As a consequence, the boundary of $\Gamma_{ijk}(S)$ can be computed in constant time.*

We prove this lemma in the next section. Then using Lemma 3 and Lemma 4 we show how the proof of Theorem 1 can be completed.

3 Proof of Lemma 4 and Theorem 1

In this section we prove Lemma 4. The proof is rather technical, and requires careful analysis of the geometry of the points. Using this lemma, we then complete the proof of Theorem 1.

3.1 Proof of Lemma 4

In this section we will characterize $\Gamma_{ijk}(S)$ for $S = \{u_i, u_j\}$ and will complete the proof of Lemma 4. Hence given three users, say u_i, u_j and u_k we want to characterize the set of points $\Gamma_{ijk}(S)$ for $S = \{u_i, u_j\}$ such that for any point x in $\Gamma_{ijk}(S)$ if P1 places a facility at x, two circles $C_i(x)$ and $C_j(x)$ will pass through x and $C_k(x)$ will pass through some other facility and

$C_i(x) \cap C_j(x) \cap C_k(x) = \emptyset$. For notational simplicity, throughout this section we shall denote the region $\Gamma_{ijk}(S)$ as X and $C_k(x)$ as C. It can be shown that X is bounded and open. In the following lemma, we characterize the boundary of X. The proof is involves standard geometric arguments, and can be found in the full version of the paper.

Lemma 5 *Any point p belongs to the boundary of X, $\partial(X)$ if and only if $C_i(p) \cap C_j(p) \cap C$ is a singleton set.* □

From Lemma 5 we know that $\partial(X)$ is the set of points p such that $C_i(p) \cap C_j(p) \cap C$ is a singleton set. Next we will find all such points. Observe that there can be two cases.

Case **1:** Suppose the intersection of any two of the three circles, $C_i(p), C_j(p)$ and C is single point, say x, and the third circle contains x. Now suppose we want to find the set of points p such that $C_i(p) \cap C$ is a single point and $C_j(p)$ contains that point. Let the point closest to u_i in C be p_i. Now the set of points for which $C_i(p) \cap C$ is a single point is the circle $C_i(p_i)$ and for all point $p \in C_i(p_i)$, $C_i(p) \cap C = \{p_i\}$. But the circle $C_j(p)$ must contain p_i. Hence the set of points p such that $C_i(p) \cap C$ is a single point and $C_j(p)$ contains that point, is $C_i(p_i) \setminus C_j(p_i)$ (see Figure 2). Similarly we can find the points p such that $C_j(p) \cap C$ is a single point and $C_i(p)$ contains that point. Set of points p for which $C_i(p) \cap C_j(p)$ is a single point is the set of points on the line segment joining u_i and u_j, $[u_i, u_j]$. But C must contain the point p. Hence the set of points for which $C_i(p) \cap C_j(p)$ is a single point and C contains that point, is equal to $[u_i, u_j] \cap C$ (see Figure 3).

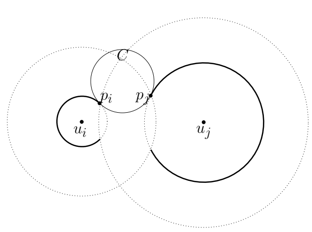

Figure 2: $\partial(X)$ when $C_i(p) \cap C$ is a single point

Case **2:** The intersection of any two of the three circles is not a singleton, but $C_i(p) \cap C_j(p) \cap C$ is a singleton set. For any point p, define p_r to be the reflection of p on the line joining u_i and u_j. Observe that for any point p, the circle $C_i(p)$ and $C_j(p)$ intersects at points p and p_r. Hence, if for any point p, $C_i(p) \cap C_j(p) \cap C$ is a singleton, then $C_i(p_r) \cap C_j(p_r) \cap C$ is also a singleton. Now, we want to find the set of points p such that none of the pairwise intersections of the three circles $C_i(p), C_j(p)$ and C are singletons, but $C_i(p) \cap C_j(p) \cap C$ is a singleton set. Observe that p or p_r must belong to the boundary of C. Without loss of generality we will find the set of points p on the boundary of C such that $C_i(p) \cap C_j(p) \cap C$ is a singleton set. Now consider the line ℓ_i, joining u_i and the center of C (see Figure 4). Observe that for any point p on the boundary of C, C and $C_i(p)$ will intersect at p and p^{-1} where p^{-1} is the reflection of p with respect to the line ℓ_i. Suppose that among p and p^{-1}, p^{-1} is closer to u_j. In that case $C_i(p) \cap C_j(p) \cap C$ is not a singleton because p^{-1} belongs to $C_i(p) \cap C_j(p) \cap C$. Hence all the points on one side of ℓ_i are not in $\partial(X)$. Similarly consider the line ℓ_j, joining u_j and the center of C. All the points in one side of ℓ_j is also not in $\partial(X)$. Remaining points are shown in bold in Figure 4. It can be also shown that any point p on on the region shown in Figure 4 is in $\partial(X)$.

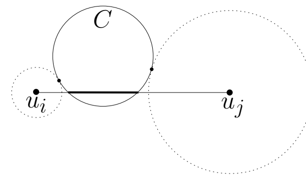

Figure 3: $\partial(X)$ when $C_i(p) \cap C_j(p)$ is a single point and C contains that point

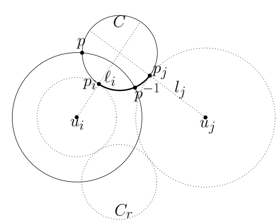

Figure 4: $\partial(X)$ when the intersections of none of the two circles are singletons, but $C_i(p) \cap C_j(p) \cap C$ is a singleton set.

This completes the proof of Lemma 4. The structure of X in the cases where the line joining u_i and u_j does not intersects C, and intersects C, are shown in Figure 5 and Figure 6, respectively.

3.2 *Proof of Theorem 1*

In this section using Lemma 3 and Lemma 4 we complete the proof of Theorem 1. Recall, from Section 2, the definition of Γ^z_{ijk}, for $S \subseteq \{u_i, u_j, u_k\}$, and $z \in \{0, 1, 2, 3\}$. We now define $\Gamma^z = \{\Gamma^z_{ijk} : u_i, u_j, u_k \in U\}$. Consider the tessellation of the plane induced by the

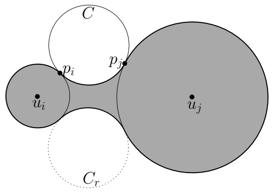

Figure 5: X when the line joining u_i and u_j does not intersect C.

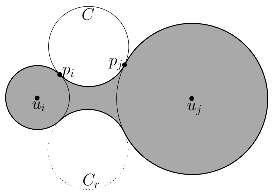

Figure 6: X when the line joining u_i and u_j intersects C.

collection of the sets Γ^z, for $z \in \{0, 1, 2, 3\}$ and $\mathcal{C}_{FS}$. From Lemma 3 we know that $\mathcal{C}_{FS}$ and Γ^0 consists of $O(n)$ circles, and Γ^3 consists $O(n^2)$ lines (set of lines passing through each pair of users in U). Lemma 3 also shows that Γ^1 consists of $O(n^3)$ circles. From Lemma 4 we know for $S \subseteq \{u_i, u_j, u_k\}$, with $|S| = 2$, $\Gamma_{ijk}(S)$ is an open set bounded by $O(1)$ circular arcs and line segments. Therefore, Γ^2 also consists of $O(n^3)$ circles and line segments. Hence, the arrangement generated by Γ^z, for $z \in \{0, 1, 2, 3\}$ and $\mathcal{C}_{FS}$ consists of $O(n^3)$ circles and line segments. The effective depth of any cell in this tessellation is constant. Moreover, the effective depth of a cell is the maximum depth in an arrangement of a set of $O(n)$ circles, which can be computed in $O(n^2)$ time [2]. Hence, by checking the effective depth of all the $O(n^6)$ cells, the minimum effective depth can be obtained in $O(n^8)$ time. Thus, the optimal strategy of P1 in the $G_n(F, S)$ game can be found in $O(n^8)$ time.

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