Unfolding Face-Neighborhood Convex Patches: Counterexamples and Positive Results

Joseph O'Rourke*

Abstract

We address unsolved problems of unfolding polyhedra in a new context, focusing on special *convex patches*—disk-like polyhedral subsets of the surface of a convex polyhedron. One long-unsolved problem is edge-unfolding *prismatoids*. We show that several natural strategies for unfolding a prismatoid can fail, but obtain a positive result for "petal unfolding" topless prismatoids, which can be viewed as particular convex patches. We also show that the natural extension of an earlier result on face-neighborhood convex patches fails, but we obtain a positive result for nonobtusely triangulated face-neighborhoods.

1 Introduction

Define a *convex patch* as a connected subset of faces of a convex polyhedron $\mathcal{P}$, homeomorphic to a disk. A convex patch is convexly curved in 3D, but its boundary need not be convex: it could be quite "jagged." I propose studying edge-unfolding of convex patches to simple (non-overlapping) polygons in the plane, as presumably easier versions of the many unsolved convex-polyhedron unfolding problems. (Here, *edge-unfolding* cuts only edges of $\mathcal{P}$; we leave that understood until the final discussion.) Toward this end, I study here special convex patches, various *face-neighborhoods*, and obtain several positive and negative results.

Face Neighborhoods. Let F be a face of a convex polyhedron $\mathcal{P}$. There are two natural "face-neighborhoods" of F: the *edge-neighborhood* $N_e(F)$, F together with every face of $\mathcal{P}$ that shares an edge with F, and the *vertex-neighborhood* $N_v(F)$, F together with every face incident to a vertex of F.[1] Clearly, $N_v(F) \supseteq N_e(F)$. A "dome" polyhedron $\mathcal{P}$ is one with a "base face" B such that $N_e(B) = \mathcal{P}$. Domes were earlier proved to unfold without overlap [6, p. 323ff]. Pincu [12] subsequently proved that $N_e(F)$ unfolds without overlap for any F, generalizing the dome result. Both the dome and the edge-neighborhood unfoldings are what I am now calling "petal unfoldings," described next in the context of prismatoids.

Prismatoids and Prismoids. A *prismatoid* is the convex hull of two convex polygons A (above) and B (base), that lie in parallel planes. Despite its simple structure, it remains unknown whether or not every prismatoid has a non-overlapping edge-unfolding, a narrow special case of what has become known as Dürer's Problem: whether every convex polyhedron has a non-overlapping edge-unfolding [6, Prob. 21.1] [11].

If A and B are angularly similar with their edges parallel, then all lateral faces are trapezoids. Such a polyhedron is called a *prismoid*. These special prismatoids are known to edge-unfold without overlap [6, p. 322].

Band and Petal Unfoldings. There are two natural unfoldings of a prismatoid. A *band unfolding* cuts one lateral edge and unfolds all lateral faces connected in *band*, leaving A and B attached each by one uncut edge to opposite sides of the band (see, e.g., [2]). Aloupis showed that the lateral cut-edge can be chosen so that the band alone unfolds [1], but I showed that, nevertheless, there are prismoids such that every band unfolding overlaps [8]. The example, Fig. 1, is repeated here, as it plays a role in the closing discussion (Sec. 4). Note

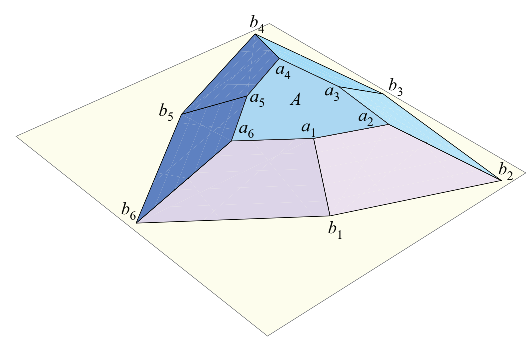

Figure 1: A convex patch with no band unfolding.

that this example also establishes that not every edge-neighborhood patch of a face of $\mathcal{P}$ has a band unfolding: $N_e(A)$ has no band unfolding.

The second natural unfolding of a prismatoid is a

*Department of Computer Science, Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu. This is a revision of [10]. Omitted proofs are in the full version.

[1] This is my own terminology. $N_e(F)$ is called the "face-neighborhood" in [7].

petal unfolding.[2] The three positive results mentioned above are all via petal unfoldings: the dome unfolding, the prismoid unfolding, and Pincu's edge-neighborhood patch unfolding. Thus Fig. 1 without its base, which is a edge-neighborhood patch, can be petal-unfolded: simply cut each lateral edge $a_i b_i$. We henceforth concentrate on petal unfoldings (until the final Sec. 4).

New Results. Given the collection of partial results and unsolved problems reviewed above, it is natural to explore petal unfoldings of vertex-neighborhood patches. Our results are as follows:

1. Define a *topless prismatoid* as one with A removed; so it is a special (non-jagged) vertex-neighborhood $N_v(B)$. We prove that every topless prismatoid whose lateral faces are triangles has a petal unfolding without overlap (Thm. 7). This shows that, in some sense, placing the top A is an obstruction to unfolding prismatoids.
2. Via a counterexample polyhedron $\mathcal{P}$ (Fig. 8), we show that not every vertex-neighborhood patch $N_v(F)$ has a non-overlapping petal unfolding.
3. However, if $\mathcal{P}$ is non-obtusely triangulated, $N_v(F)$ does have a non-overlapping petal unfolding for every face of $\mathcal{P}$ (Thm. 8).
4. This leads to a non-overlapping unfolding of a restricted class of prismatoids (Cor. 9).

I am hopeful that the main proof technique—obtaining a result for flat patches and then lifting into $z > 0$—will lead to further results.

We conclude in Section 4 with a conjecture that not every edge-neighborhood has a non-overlapping "zipper unfolding."

2 Topless Prismatoid Petal Unfolding

Let $\mathcal{P}$ be a prismatoid, and assume all lateral faces are triangles, the generic and seemingly most difficult case. Let $A = (a_1, a_2, \ldots)$ and $B = (b_1, b_2, \ldots)$. Call a lateral face that shares an edge with B a *base* or B-triangle, and a lateral face that shares an edge with A a *top* or A-triangle. A petal unfolding cuts no edge of B, and unfolds every base triangle by rotating it around its B-edge into the base plane. The collection of A-triangles incident to the same b_i vertex—the *A-fan* AF_i—must be partitioned into two groups, one of which rotates clockwise (cw) to join with the unfolded base triangle to its left, and the other group rotating counterclockwise (ccw) to join with the unfolded base triangle to its right. Either group could be empty. Finally, the top A is attached to one A-triangle. So a petal unfolding has choices for how to arrange the A-triangles, and which A-triangle connects to the top.

[2] Called a "volcano unfolding" in [6, p. 321].

It remains possible that every prismatoid has a petal unfolding: so far I have not been able to find a counterexample. Now we turn to our main result: every topless prismatoid has a petal unfolding. An example of a petal unfolding of a topless prismatoid is shown in Fig. 2.

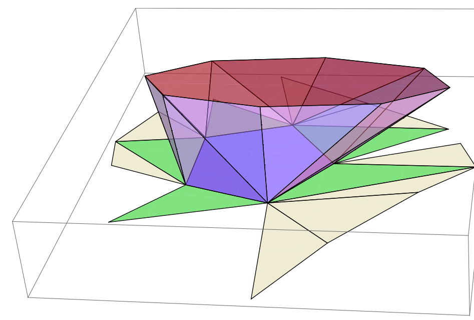

Figure 2: Unfolding of a topless prismatoid. A-fans are lightly shaded.

Even topless prismatoids present challenges. For example, consider the special case when there is only one A-triangle between every two B-triangles. Then the only choice for placement of the A-triangles is whether to turn each ccw or cw. It is natural to hope that rotating all A-triangles consistently ccw or cw suffices to avoid overlap, but this can fail. A more nuanced approach would turn each A-triangle so that its (at most one) obtuse angle is not joined to a B-triangle, but this can fail also.

The proof that topless prismatoids have petal unfoldings follows this outline:

1. An "altitudes partition" of the plane exterior to the *base unfolding* (petal unfolding of $N_e(B)$) is defined and proved to be a partition.
2. It is shown that both $\mathcal{P}$ and this partition vary in a consistent manner with respect to the separation z between the A- and B-planes.
3. An algorithm is detailed for petal unfolding the A-triangles for the "flat prismatoid" $\mathcal{P}(0)$, the limit of $\mathcal{P}(z)$ as $z \to 0$, such that these A-triangles fit inside the regions of the altitude partition.
4. It is proved that nesting within the partition regions remains true for all z.

2.1 Altitude Partition

We use a_i and b_j to represent the vertices of $\mathcal{P}$, and primes to indicate unfolded images on the base plane.

Let $B_i = \triangle b_i b_{i+1} a'_j$ be the i-th base triangle. Say that $B^{\cup} = B \cup (\bigcup_i B_i)$ is the *base unfolding*, the petal unfolding of $N_e(B)$ without any A-triangles. The altitude partition partitions the plane exterior to $B^\cup$.

Let r_i be the *altitude ray* from a'_j along the altitude of B_i. Finally, define R_i to be the region of the plane incident to b_i, including the edges of the B_{i-1} and B_i triangles incident to b_i, and bounded by r_{i-1} and r_i. See Fig. 3.

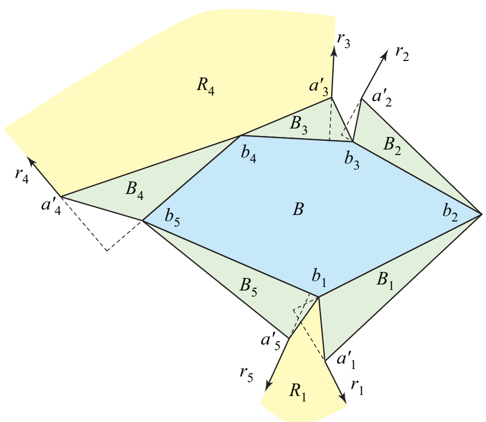

Figure 3: Partition exterior to $B^\cup$ by altitude rays r_i. Here both A and B are pentagons; in general there would not be synchronization between the b_i and a_i indices. The A-triangles are not shown.

Lemma 1 *No pair of altitude rays cross in the base plane, and so they define a partition of that plane exterior to the base unfolding $B^\cup$.*

Our goal is to show that the A-fan AF_i incident to b_i can be partitioned into two groups, one rotated cw, one ccw, so that both fit inside R_i.

2.2 Behavior of $\mathcal{P}(z)$

We will use "(z)" to indicate that a quantity varies with respect to the height z separating the A- and B-planes.

Lemma 2 *Let $\mathcal{P}(z)$ be a prismatoid with height z. Then the combinatorial structure of $\mathcal{P}(z)$ is independent of z, i.e., raising or lowering A above B retains the convex hull structure.*

We will call $\mathcal{P}(0) = \lim_{z\to 0} \mathcal{P}(z)$ a *flat prismatoid*. Each lateral face of $\mathcal{P}(0)$ is either an *up-face* or a *down-face*, and the faces of $\mathcal{P}(z)$ retain this classification in that their outward normals either have a positive or a negative vertical component.

Lemma 3 *Let $\mathcal{P}(z)$ be a prismatoid with height z, and $B^\cup(z)$ its base unfolding. Then the apex $a'_j(z)$ of each $B'_i(z)$ triangle $\triangle b_i b_{i+1} a'_j(z)$ in $B^\cup(z)$ lies on the fixed line containing the altitude of $B'_i(z)$.*

Thus the vertices $a'_j(z)$ of the base unfolding "ride out" along the altitude rays r_i as z increases (see ahead to Fig. 6 for an illustration). Therefore the combinatorial structure of the altitude partition is fixed, and R_i only changes geometrically by the lengthening of the edges $b_i a'_j$ and $b_{i+1} a'_j$ and the change in the angle gap $\kappa_{b_i}(z)$ at b_i.

2.3 Structure of A-fans

Henceforth we concentrate on one A-fan, which we always take to be incident to b_2, and so between $B_1 = \triangle b_1 b_2 a_1$ and $B_2 = \triangle b_2 b_3 a_k$. The *a-chain* is the chain of vertices $a_1, \ldots, a_k$. Note that the plane in $\mathbb{R}^3$ containing face B_1 of $\mathcal{P}$ supports A at a_1, and the plane containing B_2 supports A at a_k. Let $\beta = \beta_2$ be the base angle at b_2: $\beta = \angle b_1 b_2 b_3$. We state here a few facts true of every A-fan.

1. An a-chain spans at most "half" of A, i.e., a portion between parallel supporting lines to A (because $\beta > 0$).
2. If an A-fan is unfolded as a unit to the base plane, the a-chain consists of convex, reflex, and convex portions, any of which may be empty. So, excluding the first and last vertices, the interior vertices of the chain have convex angles, then reflex, then convex.
3. Correspondingly, an A-fan consists of down-faces followed by up-faces followed by down-faces, where again any (or all) of these three portions could be empty.
4. All four possible combinations of up/down are possible for the B_1 and B_2 triangles.

The second fact above is not so easy to see. The intuition is that there is a limited amount of variation possible in an a-chain. It is the third fact that we will use essentially; it will become clear shortly.

2.4 Flat Prismatoid Case Analysis

How the A-fan is proved to sit inside its altitude region R for $\mathcal{P}(0)$ depends primarily on where b_2 sits with respect to A, and secondarily on the three B-vertices (b_1, b_2, b_3). Fig. 4 illustrates one of the easiest cases, when b_2 is in C, the convex region bounded by the a-chain and extensions of its extreme edges. Then all the A-faces are down-faces, the a-chain is convex, one of the two B-faces is a down-face (B_2 in the illustration), and we simply leave the A-fan attached to that B down-face.

A second case occurs when b_2 is on the reflex side of A. An instance when both B-triangles are down-faces is illustrated in Fig. 5. Now the A-fan consists of down-faces and up-faces, the up-faces incident to the reflex side of the a-chain. These up-faces must be flipped in

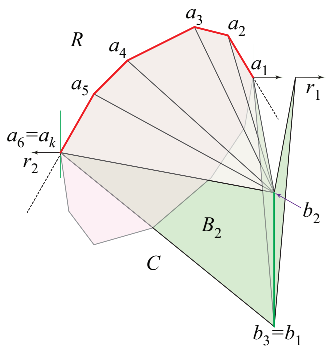

Figure 4: Case 1b. Here we have illustrated $b_1 = b_3$ to allow for the maximum a-chain extent.

the unfolding, reflected across one of the two tangents from b_2 to A. A key point is that not always will both flips be "safe" in the sense that they stay inside the altitude region. Fortunately, one of the two flips is always safe:

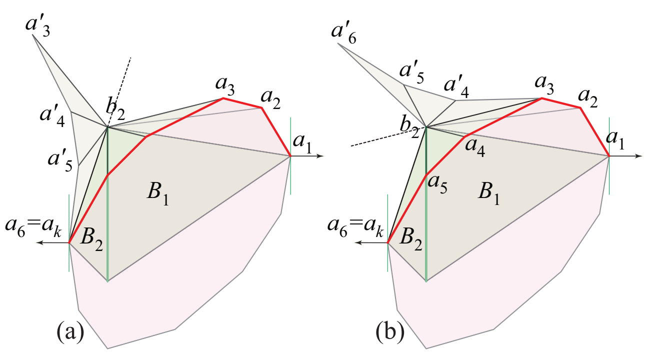

Figure 5: Case 2a. The A-triangles between the tangents b_2 to a_3 and b_2 to a_6 are up-faces. (a) shows the up-faces flipped over the left tangent b_2a_6, and (b) when flipped over the right tangent b_2a_3.

Lemma 4 *Let b_2 have tangents touching a_s and a_t of A. Then either reflecting the enclosed up-faces across the left tangent, or across the right tangent, is "safe" in the sense that no points of a flipped triangle falls outside the rays r_1 or r_k.*

The remaining cases are minor variations on those illustrated, and will not be further detailed.

2.5 Nesting in $\mathcal{P}(z)$ regions

The most difficult part of the proof is showing that the nesting established above for $\mathcal{P}(0)$ holds for $\mathcal{P}(z)$. A key technical lemma is this:

Lemma 5 *Let $\triangle b, a_1(z), a_2(z)$ be an A-triangle, with angles $\alpha_1(z)$ and $\alpha_2(z)$ at $a_1(z)$ and $a_2(z)$ respectively. Then $\alpha_1(z)$ and $\alpha_2(z)$ are monotonic from their $z = 0$ values toward $\pi/2$ as $z \to \infty$.*

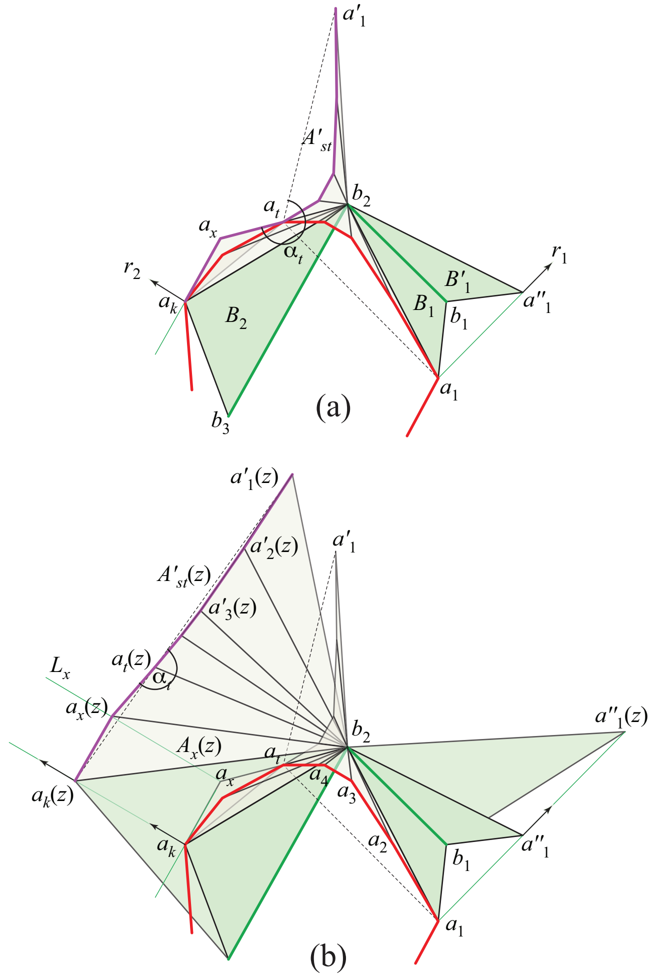

Figure 6: (a) $z = 0$. $\triangle a_t a_x a_k$ encloses the convex section, and $\triangle a_1 b_2 a_t$ encloses the reflex section. (b) $z > 0$. Reflex angle $\alpha_t(z)$ decreases as z increases.

I should note that it is not true, as one might hope, that the apex angle at b of that A-triangle, $\angle a_1(z), b, a_2(z)$, shrinks monotonically with increasing z, even though its limit as $z \to \infty$ is zero. Nor is the angle gap $\kappa_b(z)$ necessarily monotonic. These nonmonotonic angle variations complicate the proof.

Another important observation is that the sorting of ba_i edges by length in $\mathcal{P}(0)$ remains the same for all $\mathcal{P}(z)$, $z > 0$. More precisely, let $|ba_i| > |ba_j|$ for two lateral edges connecting vertex $b \in B$ to vertices $a_i, a_j \in A$ in $\mathcal{P}(0)$. Then $|ba_i(z)| > |ba_j(z)|$ remains true for all $\mathcal{P}(z)$, $z > 0$.

For the nesting proof, I will rely on a high-level description, and one difficult instance. At a high level, each of the convex or reflex sections of the a-chain are enclosed in a triangle, which continues to enclose that portion of the a-chain for any $z > 0$ (by Fact 1, Sec. **??**). The reflex enclosure is determined by the tangents from b_2 to A: $\triangle a_s b_2 a_t$. So then the task is to prove these (at most three) triangles remain within $R(z)$. Fig. 6 shows a case where there is both a convex and a reflex section. Were there an additional convex section, it would remain attached to $B_1(z)$ and would not increase the challenge.

Lemma 6 *If the a-chain consists of a convex and a reflex section, and the safe flip (by Lemma 4) is to a side with a down-face (B_2 in the figure), then $AF'(z) \subset R(z)$: the A-fan unfolds within the altitude region for all z.*

I have been unsuccessful in unifying the cases in the analysis, despite their similarity. Nevertheless, the conclusion is this theorem:

Theorem 7 *Every triangulated topless prismatoid has a petal unfolding.*

It is natural to hope that further analysis will lead to a safe placement of the top A (which, alas, might not fit into any altitude-ray region).

3 Unfolding Vertex-Neighborhoods

We now return to arbitrary face-neighborhoods. As mentioned previously, Pincu proved that the petal unfolding of $N_e(B)$ avoids overlap for any face B of a convex polyhedron. Here we show that the vertex-neighborhood $N_v(B)$ does not always have a non-overlapping petal unfolding, even when all faces in the neighborhood are triangles.

A portion of the a 9-vertex example $\mathcal{P}$ that establishes this negative result is shown in Fig. 7. The b_1b_3 edge of B lies on the horizontal xy-plane. The vertices $\{b_2, a_1, a_2, c_1, c_2\}$ all lie on a parallel plane at height z, with b_2 directly above the origin: $b_2 = (0, 0, z)$.

All of $N_v(B)$ is shown in Fig. 8. The structure in Fig. 7 is surrounded by more faces designed to minimize curvatures at the vertices b_i of B. Finally, $\mathcal{P}$ is the convex hull of the illustrated vertices, which just adds a quadrilateral "back" face (p_1, c_1, c_2, p_3) (not shown).

The design is such that there is so little rotation possible in the cw and ccw options for the triangles incident to vertex b_2 of B, that overlap is forced: see Figs. 9, 10, and 11. The thin $\triangle b_2a_1a_2$ overlaps in the vicinity of a_1 if rotated ccw, and in the vicinity of a_2 is cw (illustrated).

One can identify two features of the polyhedron just described that lead to overlap: low curvature vertices (to restrict freedom) and obtuse face angles (at a_1 and a_2) (to create "overhang"). Both seem necessary ingredients. Here I pursue excluding obtuse angles:

Theorem 8 *If $\mathcal{P}$ is nonobtusely triangulated, then for every face B, $N_v(B)$ has a petal unfolding.*

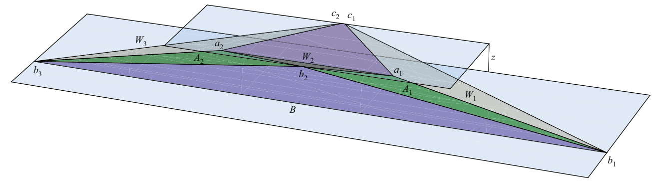

Figure 7: Faces of $\mathcal{P}$ in the immediate vicinity of B.

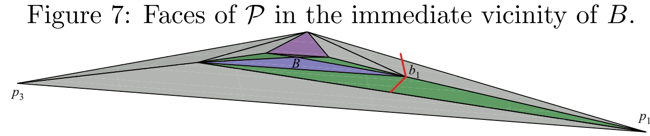

Figure 8: All faces incident to $N_v(B)$, and one more, the purple quadrilateral (a_1, c_1, c_2, a_2). The red vectors are normal to B and to $\triangle b_1p_1c_1$.

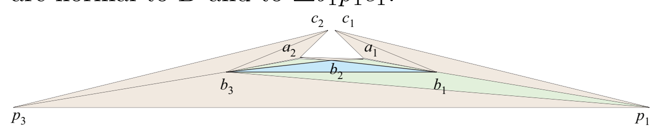

Figure 9: Complete unfolding of all faces incident to B.

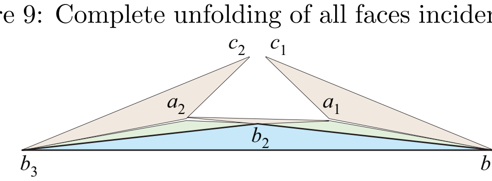

Figure 10: Zoom of Fig. 9.

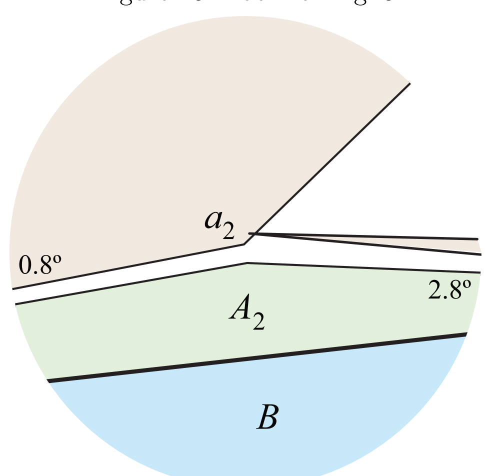

Figure 11: Zoom of Fig. 10 in vicinity of a_2 overlap. The angle gap at b_3 is 0.8°, and the gap at b_2 is 2.8°.

A *nonobtuse triangle* is one whose angles are each $\leq \pi/2$. It is known that any polygon of n vertices has a nonobtuse triangulation by $O(n)$ triangles, which can be found in $O(n \log^2 n)$ time [3]. Open Problem 22.6 [6, p. 332] asked whether every nonobtusely triangulated convex polyhedron has an edge-unfolding. One can view Theorem 8 as a (very small) advance on this problem.[3]

A little more analysis leads to a petal unfolding of a (very special) class of prismatoids (including their tops):

Corollary 9 *Let $\mathcal{P}$ be a triangular prismatoid all of whose faces, except possibly the base B, are nonobtuse triangles, and the base is a (possibly obtuse) triangle. Then every petal unfolding of $\mathcal{P}$ avoids overlap.*

It seems quite possible that this corollary still holds with B an arbitrary convex polygon, but the proof would need significant extension.

4 Discussion

I believe that unfolding convex patches may be a fruitful line of investigation. For example, notice that the edges cut in a petal unfolding of a vertex-neighborhood of a face form a disconnected spanning forest rather than a single spanning tree. One might ask: Does every convex patch have an edge-unfolding via a single spanning cut tree? The answer is NO, already provided by the banded hexagon example in Fig. 1. For such a tree can only touch the boundary at one vertex (otherwise it would lead to more than one piece), and then it is easy to run through the few possible spanning trees and show they all overlap.

The term *zipper unfolding* was introduced in [5] for a non-overlapping unfolding of a convex polyhedron achieved via Hamiltonian cut path. They studied zipper edge-paths, following edges of the polyhedron, but raised the interesting question of whether or not every convex polyhedron has a zipper path, not constrained to follow edges, that leads to a non-overlapping unfolding. This is a special case of Open Problem 22.3 in [6, p. 321] and still seems difficult to resolve.

Given the focus of this work, it is natural to specialize this question further, to ask if every convex patch has a zipper unfolding, using arbitrary cuts (not restricted to edges). I believe the answer is negative: a version of the banded hexagon shown in Fig. 12, a bottomless prismoid, has no zipper unfolding. My argument for this is long and seems difficult to formalize, so I leave the claim as a conjecture. It would constitute an interesting contrast to the recent result that all "nested" prismoids have a zipper edge-unfolding [4].

[3]It can also be used to slightly improve Pincu's "fewest nets" result for this class of polyhedra.

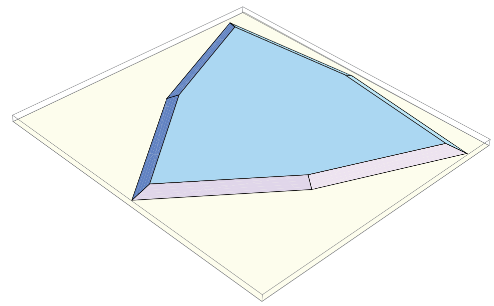

Figure 12: The banded hexagon with a thin band.

References

[1] G. Aloupis. *Reconfigurations of Polygonal Structures.* PhD thesis, McGill Univ., Sch. Comput. Sci., 2005.

[2] G. Aloupis, E. D. Demaine, S. Langermann, P. Morin, J. O'Rourke, I. Streinu, and G. Toussaint. Edge-unfolding nested polyhedral bands. *Comput. Geom. Theory Appl.*, 39(1):30–42, 2007.

[3] M. W. Bern, S. Mitchell, and J. Ruppert. Linear-size nonobtuse triangulation of polygons. *Discrete Comput. Geom.*, 14:411–428, 1995.

[4] E. Demaine, M. Demain, and R. Uehara. Zipper unfoldability of domes and prismoids. In *Proc. 25th Canad. Conf. Comput. Geom.*, Aug. 2013.

[5] E. Demaine, M. Demaine, A. Lubiw, A. Shallit, and J. Shallit. Zipper unfoldings of polyhedral complexes. In *Proc. 22nd Canad. Conf. Comput. Geom.*, pages 219–222, Aug. 2010.

[6] E. D. Demaine and J. O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra.* Cambridge University Press, July 2007. `http://www.gfalop.org`.

[7] H. Guo, A. Maheshwari, D. Nussbaum, and J.-R. Sack. Shortest path queries between geometric objects on surfaces. In *Comput. Sci. Appl.–ICCSA 2007*, pages 82–95. Springer, 2007.

[8] J. O'Rourke. Band unfoldings and prismatoids: A counterexample. Technical Report 087, Smith College, Oct. 2007. arXiv:0710.0811v2 [cs.CG]; `http://arxiv.org/abs/0710.0811`.

[9] J. O'Rourke. Tetrahedron angles sum to π: Bisector plane. MathOverflow: `http://mathoverflow.net/questions/94586/`, Apr. 2012.

[10] J. O'Rourke. Unfolding prismatoids as convex patches: Counterexamples and positive results. arXiv:1205.2048 [cs.CG]. http://arxiv.org/abs/1205.2048, 2012.

[11] J. O'Rourke. Dürer's problem. In M. Senechal, editor, *Shaping Space: Exploring Polyhedra in Nature, Art, and the Geometrical Imagination*, pages 77–86. Springer, 2013.

[12] V. Pincu. On the fewest nets problem for convex polyhedra. In *Proc. 19th Canad. Conf. Comput. Geom.*, pages 21–24, 2007.