Cole's Parametric Search Technique Made Practical

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Abstract

Parametric search has been widely used in geometric algorithms. Cole's improvement provides a way of saving a logarithmic factor in the running time over what is achievable using the standard method. Unfortunately, this improvement comes at the expense of making an already complicated algorithm even more complex; hence, this technique has been mostly of theoretical interest. In this paper, we provide an algorithm engineering framework that allows for the same asymptotic complexity to be achieved probabilistically in a way that is both simple and practical (i.e., suitable for actual implementation). The main idea of our approach is to show that a variant of quicksort, known as *boxsort*, can be used to drive comparisons, instead of using a sorting network, like the complicated AKS network, or an EREW parallel sorting algorithm, like the fairly intricate parallel mergesort algorithm. This results in a randomized optimization algorithm with a running time matching that of using Cole's method, with high probability, while also being practical. We show how this results in practical implementations of some geometric algorithms utilizing parametric searching and provide experimental results that prove practicality of the method.

1 Introduction

Parametric search [24] has proven to be a useful technique in design of efficient algorithms for many geometric and combinatorial optimization problems (e.g., see [2, 3, 28]). Example applications include ray shooting [1], slope selection [13], computing the Fréchet distance between two polygonal curves [6, 8], matching drawings of planar graphs [5], labeling planar maps with rectangles [22], and various other matching and approximation problems (e.g., see [15, 16, 17]).

Although it has been superseded in some applications by Chan's randomized optimization technique [9, 10], for many problems asymptotically best known results still depend on parametric searching.

The technique is applied to a decision problem, B, whose solution depends on a real parameter, λ, in a monotonic way, so that B is true on some interval $(-\infty, \lambda^*)$. The goal is to determine the value of λ^*, the maximum for which B is true. To achieve this goal, the parametric search approach utilizes two algorithms. The first algorithm, $\mathcal{C}$, is a sequential *decision* algorithm for B that can determine if a given λ is less than, equal to, or greater than λ^*. The second algorithm, $\mathcal{A}$, is a *generic* parallel algorithm whose inner workings are driven by "comparisons," which are either independent of λ or depend on the signs of low-degree polynomials in λ. Because $\mathcal{A}$ works in parallel, its comparisons come in batches, so there are several independent such comparisons that occur at the same time. The idea, then, is to run $\mathcal{A}$ on the input that depends on the unknown value λ^*, which will result in actually finding that value as a kind of by-product (even though we do not know λ^*, $\mathcal{C}$ can be used to resolve comparisons that appear during the execution of $\mathcal{A}$). The next step is to simulate an execution of $\mathcal{A}$ sequentially. To resolve comparisons that occur in a single step of this simulation, we can use the algorithm $\mathcal{C}$ to perform binary search among the (ordered) roots of the polynomials in λ for these comparisons, which allows us to determine signs of all these polynomials, hence, allows us to continue the simulation. When the simulation completes, we will have determined the value of λ^*. Moreover, the running time for performing this simulation is $O(P(n)T(n) + C(n)T(n)\log P(n))$, where $C(n)$ is the (sequential) running time of $\mathcal{C}$, $T(n)$ is the (parallel) running time of $\mathcal{A}$, and $P(n)$ is the number of processors used by $\mathcal{A}$.

Cole [11] shows how to improve the asymptotic performance of the parametric search technique when sorting is the problem solved by $\mathcal{A}$. His improvement comes from an observation that performing a separate binary search for each step of the algorithm $\mathcal{A}$ will often "waste" calls to $\mathcal{C}$ to resolve a relatively small number of comparisons. Rather than resolve all the comparisons of a single step of $\mathcal{A}$, he instead assumes that $\mathcal{A}$ is implemented as the AKS sorting network [4] or an optimal EREW parallel sorting algorithm [12, 18], which allows for comparisons on multiple steps of $\mathcal{A}$ to be considered at the same time (so long as their preceding comparisons have been resolved). This improvement results in a running time for the optimization problem that is $O(P(n)T(n) + C(n)(T(n) + \log P(n)))$.

From an algorithm engineering perspective, the "clas-

sical" parametric search technique (utilizing a parallel algorithm) is admittedly difficult to implement, although some implementations do exist [29, 30, 31]. Cole's improvement is even more complex, however, and we are not familiar with any implementations of his parametric search optimization.

Even without Cole's improvement, a challenge for implementing the parametric search technique is the simulation of a parallel algorithm on a sequential machine. This difficulty has motivated some researchers to abandon the use of parametric searching entirely and instead use other paradigms, such as expander graphs [21], geometric random sampling [23], and ϵ-cuttings [7] (see also [2]).

Interestingly, van Oostrum and Veltkamp [31] show that, for sorting-based parametric search applications, one can use the well-known `quicksort` algorithm to drive comparisons instead of a parallel sorting algorithm. Unfortunately, as van Oostrum and Veltkamp note in their paper, Cole's improvement cannot be applied in this case. The main difficulty is that, when viewed as a kind of parallel algorithm, comparisons to be done at one level of `quicksort` become known only after all the comparisons on the level above have been resolved. Thus, comparisons cannot be pipelined in the way required by Cole's optimization when using this approach. The result, of course, is that this sets up an unfortunate tension between theory and practice, forcing algorithm designers to choose between a practical, but asymptotically inferior, implementation or an impractical algorithm whose running time is asymptotically better by a logarithmic factor.

1.1 Our Results

We show that it is, in fact, possible to implement Cole's parametric search technique in a manner that is efficient and practical (i.e., fast and easy to implement). The main idea is to use a variant of quicksort, known as `boxsort` [26], to drive comparisons (instead of sorting networks, like the complicated AKS network or an EREW parallel sorting algorithm). We apply a potential function to comparisons in the `boxsort` algorithm, which, together with a weighted-median-finding algorithm, allows us to schedule these comparisons in a pipelined fashion and achieve, with high probability, the same asymptotic running time as Cole's method, while also being practical. Moreover, we provide experimental results that give empirical evidence supporting these claims for the "median-of-lines" problem [24] and the geometric optimization problems of matching planar drawings [5] and labeling planar maps with rectangles [22].

2 Parametric Search Explained

In this section, we provide a more in-depth description of the parametric search technique. Recall that B is a problem that we want to solve. Furthermore, we restrict ourselves to the case where the generic algorithm $\mathcal{A}$ is a sorting algorithm. We require of B the following.

1. There is a *decision algorithm*, $\mathcal{C}$, which, for any value λ, resolves a comparison $\lambda < \lambda^*$ in time $C(n)$ without actually knowing λ^* (note that $C(n)$ is a function of the size of input to B). Typically, $C(n)$ is at least $\Omega(n)$, as opposed to $O(1)$ comparison time which is usual for classical sorting algorithms.

2. There is an efficient way of generating values x_i (with each x_i being either a real value or a real-valued function of λ) from an input to problem B. Ideally, it produces $O(n)$ such values.

3. For each $x_i < x_j$ comparison, the answer is determined by the sign of a low-degree polynomial in λ at $\lambda = \lambda^*$ (polynomials for different comparisons may differ).

4. *Critical values* (values λ that, based on combinatorial properties of B, have the potential of being equal to λ^*) form a subset of the set of roots of the polynomials determining answers to every possible comparison $x_i < x_j$.

Then, as a by-product of sorting values x_i, we get (directly or indirectly) the answers to all comparisons $\lambda < \lambda^*$, where λ's are roots of all comparisons $x_i < x_j$. Therefore, we are able to find λ^*.

We can solve B in the following way: generate x_i's, sort them using algorithm $\mathcal{A}$ and recover λ^* from the answer. If $\mathcal{A}$ sorts n items in $T(n)$ comparisons and each comparison is resolved in time $O\big(C(n)\big)$ (it requires determining whether $\lambda < \lambda^*$ for a constant number of roots λ), solving B this way takes time $T(n)C(n)$.

It is important to note that if there are k comparisons $x_i < x_j$, we can avoid calling $\mathcal{C}$ on every single root of their polynomials, and still resolve them all. This is because resolving $\lambda < \lambda^*$ automatically resolves comparisons for values $\lambda' \leq \lambda$ (if the result was `YES`) or $\lambda'' > \lambda^*$ (if the result was `NO`). Therefore, we can solve k comparisons in only $O(\log k)$ calls to $\mathcal{C}$, if in every iteration we use a standard median-finding algorithm (e.g., see [14]) to find the median root λ, and then resolve it by a call to $\mathcal{C}$ (each iteration halves the number of unresolved comparisons).

The above observation lies at the heart of the original parametric search, as introduced by Megiddo [24]. Note that we can group the comparisons in such a way only if they are *independent* of each other. To assure this, one chooses $\mathcal{A}$ to be a *parallel* sorting algorithm, running in

time $T(n)$ on $P(n)$ processors. At every step of $\mathcal{A}$, there are $O(P(n))$ independent comparisons, and they can be resolved in time $O\big(P(n) + \log(P(n)) \cdot C(n)\big)$ according to the previous observation. Resolving comparisons at all $T(n)$ steps of $\mathcal{A}$ takes time $O\big(T(n) \cdot P(n) + T(n) \cdot \log(P(n)) \cdot C(n)\big)$. Simulating $\mathcal{A}$ on a sequential machine takes time $O\big(T(n)P(n)\big)$. Therefore, parametric search, as originally introduced, helps solve B in time $O\big(T(n) \cdot P(n) + T(n) \cdot \log(P(n)) \cdot C(n)\big)$.

2.1 Cole's Improvement

Cole [11] was able to improve on Megiddo's result by using a sorting network or an EREW parallel sorting algorithm as $\mathcal{A}$, and changing the order of comparison resolution by assigning weights to comparisons and resolving the *median weighted comparison* at each step.

In the case of a sorting network, a straightforward notion of *active* comparisons and *active* wires was introduced. Initially, all input wires (and no others) are *active*. A comparison is said to be *active* if it is not resolved and both its input wires are *active*. When active comparison gets resolved, its output wires now become *active*, possibly activating subsequent comparisons. Informally, *active* comparisons have not been resolved yet, but both of their inputs are already determined.

Weight is assigned to every comparison, being equal to 4^{-j} for a comparison at depth j. The *active weight* is defined as the weight of all *active* comparisons. The weighted median comparison can be found in $O(n)$ time [27], and resolving it automatically resolves a weighted half of the comparisons.

It is shown that for a sorting network of width $P(n)$ and depth $T(n)$, or an EREW sorting algorithm with $P(n)$ processors and time $T(n)$, the method of resolving weighted median comparison requires only $O(T(n) + \log(P(n)))$ direct calls to $\mathcal{C}$. Including simulation overhead, we solve B in time $O(P(n) \cdot T(n) + \big(T(n) + \log(P(n))\big) \cdot C(n))$.

This is completely impractical, however, as the bounds for the AKS network have huge constant factors. In a subsequent work [12], Cole shows that one can substitute an EREW parallel sorting algorithm for the AKS network, which makes using his optimization more implementable, but arguably still not practical, since the existing optimal EREW parallel sorting algorithms [12, 18] are still fairly intricate.

2.2 Applying `quicksort` to Parametric Search

Van Oostrum and Veltkamp [31] have shown that the `quicksort` algorithm [20] can be used as $\mathcal{A}$. Recall that in the randomized version of this algorithm we sort a set of elements by picking one of them (called the *pivot*) at random, and recursively sorting elements smaller than the pivot and greater than the pivot. A key observation here is that all the comparisons with the pivot(s) at a given level of recursion are independent of each other. It leads to a practical algorithm, running in $O(n \log n + \log^2 n \cdot C(n))$ expected-time, for solving B (it becomes $O(n \log n + \log n \cdot C(n))$ under additional assumption about distribution of the roots of polynomials). Comparisons are resolved by resolving the median comparison among unresolved comparisons at the current level. As `quicksort` is expected to have $O(\log n)$ levels of recursion, and $O(n)$ comparisons at each level can be resolved in time $O(n + \log n \cdot C(n))$, time bound follows.

Cole's improvement cannot be applied in this case, because all comparisons at one level have to be resolved before we even know what comparisons have to be done at the next level (that is, we don't know the splits around pivots until the very last comparison is resolved).

3 Our Practical Version of Cole's Technique

In this section, we describe our algorithm engineering framework for making Cole's parametric search technique practical. Our approach results in a randomized parametric search algorithm with a running time of $O(n \log n + \log n \cdot C(n))$, with high probability, which makes no assumptions about the input. Our framework involves resolving median-weight comparison, according to a potential function based on Cole-style weights assigned to comparisons of a fairly obscure sorting algorithm, which we review next.

3.1 The `boxsort` Algorithm

We use the `boxsort` algorithm due to Reischuk [26] (see also [25]) as $\mathcal{A}$. This algorithm is based on an extension of the main idea behind randomized `quicksort`, namely splitting elements around pivots and recursing into subproblems. While `quicksort` randomly selects a single pivot and recurses into two subproblems, `boxsort` randomly selects $\sqrt{n}$ pivots and recurses into $\sqrt{n}+1$ subproblems in a single stage. We think of it as a parallel algorithm, in the sense that the recursive calls on the same level are independent of each other. The pseudocode is shown in Algorithm 1.

Sorting in lines 3 and 6 is done in a brute-force manner, by comparing all pairs of items, in time $O(n^2)$ in line 3, and $O(n)$ in line 6 (note that since all these comparisons are independent, they can all be realized in a single parallel step).

Once the *marked* items are sorted in line 6, splitting in line 7 is simply $n - \sqrt{n}$ independent binary searches through the *marked* items (to determine, for each unmarked element, the subproblem where it lands). It takes $O(n \log \sqrt{n})$ time (when realized in a sequential way). Equivalently, we think of the sorted set of *marked*

```
    // N – original number of items
proc boxsort(A[i...j])
 1: n ← (j − i + 1)
 2: if n < log N then        // base case
 3:    sort A[i...j]
 4: else
 5:    randomly mark √n items
 6:    sort the marked items
 7:    use the marked items to split A[i...j] into subproblems A_1, A_2, ..., A_{√n+1}
 8:    for all i ← 1...√n + 1 do
 9:       boxsort(A_i)
10:    end for
11: end if
```

Algorithm 1: `boxsort`

items as forming a perfectly balanced binary search tree. Locating a destination subproblem for an item is then done by *routing* the item through this tree. The tree has $\log \sqrt{n}$ levels, and all routing comparisons are independent between different unmarked items. Therefore, *routing* can be realized in $\log \sqrt{n}$ parallel steps.

3.2 Weighting Scheme

Motivated by Cole's approach, we assign weight to every *active* comparison, and resolve the weighted median comparison in a single step. For simplicity, we identify each comparison $x_i < x_j$ with a single comparison against the optimum value, i.e., $\lambda_{ij} < \lambda^*$ for real λ_{ij} (in essence, we assume that comparison polynomials have degree 1). It is straightforward to extend the scheme for the case of higher degrees of comparison polynomials.

It makes sense here to think of `boxsort` in a network-like fashion, in order to understand how the weights are assigned to comparisons. Here, nodes represent comparisons, and directed edges represent dependence on previous comparisons. Furthermore, we imagine the network with edges directed downward, and refer to edge sources as *parents*, and destinations as *children*. Comparison becomes *active* as soon as all its dependencies become resolved (and stops when it gets resolved).

Our "network" also contains nodes for *virtual comparisons*. These are not real comparisons, and don't appear during actual execution of the algorithm. Their sole purpose is to make it easy to assign weights to *real* comparisons once they become *active* (we will later see that, in fact, they are not necessary even for that; but they make it easy to understand how the weights are computed). When a *virtual* comparison becomes *active*, it is automatically resolved (reflecting the fact that there is no *real* work assigned to a virtual comparison).

Contrary to Cole's weighting scheme for sorting networks, our scheme does not rely only on comparison's depth when assigning weights. In fact, different comparisons at the same level of the network may have different weights. Weights are assigned to comparisons (virtual or not) according to the following *weight rule*:

> When comparison C of weight w gets resolved and causes m comparisons $C_1, \ldots, C_m$ to become *active*, each of these comparisons gets weight $w/2m$.

Informally, resolved comparison distributes half of its weight among its newly activated children. Each comparison gets its weight only once, from its last resolved parent (the scheme guarantees that all parents of a comparison have equal weight).

3.3 The Algorithm

Simulating a single recursive call of `boxsort` (including the *virtual* parts) consists of the following steps.

1. Randomly *mark* $\sqrt{n}$ items.
2. Create $\sqrt{n} \cdot (\sqrt{n} - 1)/2 = O(n)$ comparisons for sorting *marked* items.
3. Construct a complete binary tree of virtual comparisons (comparisons from Step 2 are leaves).
4. Create *routing* trees from section 3.1 for routing unmarked elements; make the root of each such tree depend on the root of the tree from Step 3.
5. Route items through the tree of *marked* items;
6. Construct a binary tree of virtual comparisons (leaves are last comparisons from *routing* trees).
7. Split items into boxes
8. Assign weights for comparisons in the next level of recursion (after the items are split into boxes) by making them children of the root from Step 6.
9. Recurse into subproblems (simultaneously).

Blue steps (3, 6) deal with trees of virtual comparisons, while *red* steps (4, 8) represent relationships that make *real* comparisons depend on *virtual* ones. The idea behind *blue* steps is to ensure synchronization (that is, guarantee that all *real* comparisons on the levels above have been resolved), and *red* steps are there to ensure proper assignment of weights. For simplicity, we present heights/weights as if there were exactly n (instead of $\sqrt{n} \cdot (\sqrt{n} - 1)/2$) comparisons between *marked* items, and exactly n (instead of $n - \sqrt{n}$) unmarked items to be routed. This assumption also applies to the following.

Steps 1 and 7 do not involve any comparisons, and they do not affect weights. Comparisons from Step 2 start with weight w. The tree from Step 3 has height

$\log n$, so its root, according to the *weight rule* gets weight $w/(2^{\log n}) = w/n$. Dependencies introduced in Step 4 between that root and roots of the *routing trees* cause their weight to be $w/2n^2$ (weight w/n divided among n comparisons). *Routing trees* have height $\log \sqrt{n}$, so the comparisons at their bottom have weight $w/2n^{2.5}$ ($w/2n^2$ divided by $2^{\log \sqrt{n}}$, because, as the routing progresses, the *routing trees* get whittled down to paths, and resolving a routing comparison *activates* at most one new routing comparison. Step 6 is essentially the same as Step 3, so the root of the second *virtual* tree gets weight $w/2n^{3.5}$. All initial comparisons in the subsequent recursive calls (sorting of new *marked* items and/or sorting in the base case) depend on this root (Step 8), and they are given weight $w/4n^{4.5}$ (much like in Step 4). The height of the dependence network is $O(\log n)$, and at any given moment the number of currently *active* comparisons does not exceed n.

From now on, comparisons are independent across different subproblems. For subsequent subproblems, n from the above discussion gets substituted by $\hat{n}$, the size of the subproblem. Since subproblem sizes may differ, comparisons on the same level of the network (general level, for the entire algorithm) are no longer guaranteed to have same weights (weights of comparisons belonging to the same subproblem are however equal).

The above discussion shows that, as advertised, we don't really need *virtual* comparisons in order to assign weights to *real* comparisons, as these depend only on n, the size of the subproblem. Therefore, the actual algorithm only consists of steps 1, 2, 5, 7, and 9 and is the following.

1. Randomly *mark* $\sqrt{n}$ items
2. Sort *marked* items by comparing every pair in $O(n)$ comparisons, each of weight w.
3. When the last comparison finishes, *activate* comparisons for routing through the tree of *marked* items, each of weight $w/2n^2$.
4. Route items through the trees, following the *weight rule* when a comparison gets resolved.
5. When the destination for the last item is determined, split items into boxes (no additional comparisons resolved here).
6. Assign weight $w/4n^{4.5}$ to initial comparisons in new subproblems.
7. Recurse into subproblems (simultaneously).

3.4 Analysis

Assume that initially all comparisons at the highest level were given weight 1. Here, we also include *virtual* comparisons. Motivatad by Cole's analysis [11], we get the following (for details, refer to the full version [19]).

Lemma 1 *$O(f(n) + \log n)$ rounds of resolving the median-weight comparison suffice to resolve every comparison, where $f(n)$ is the height of* `boxsort`*'s network.*

We also have the following fact about `boxsort`.

Lemma 2 *(Theorem 12.2 of [25]) There is a constant $b > 0$ such that* `boxsort` *terminates in $O(\log n)$ parallel steps with probability at least $1 - \exp(-\log^b n)$.*

Originally, `boxsort` requires $O(\log n)$ parallel steps to execute a single recursive call for a problem of size n. We noted that the dependence network for a single recursive call in our simulation has height $O(\log n)$ for a problem of size n as well. This means that Lemma 2 applies here and proves that, with high probability, the dependence network for the entire simulation has height $O(\log n)$.

Combining that with Lemma 1 and the observation that any level in the dependence network contains $O(n)$ comparisons, we get the following.

Theorem 3 *With high probability, the presented algorithm requires $O(\log n)$ calls to $\mathcal{C}$, yielding an $O(n \log n + \log n \cdot C(n))$ time parametric search solution to problem B.*

4 Conclusion

We have introduced a practical version of Cole's optimization of the parametric search technique. Our method results in a randomized algorithm whose running time matches that of using Cole's technique, with high probability, while being easily implementable. We have implemented it and, based on experimentation performed on some geometric problems (details in the full paper [19]), showed that our approach is competitive with the previous practical parametric search technique of van Oostrum and Veltkamp [31], while having superior asymptotic performance guarantees.

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