Bounding the Locus of the Center of Mass for a Part with Shape Variation

Fatemeh Panahi   A. Frank van der Stappen [*]

Abstract

The shape and center of mass of a part are crucial parameters to algorithms for planning automated manufacturing tasks. As industrial parts are generally manufactured to tolerances, the shape is subject to variations, which, in turn, also cause variations in the location of the center of mass. Planning algorithms should take into account both types of variation to prevent failure when the resulting plans are applied to manufactured incarnations of a model part.

We study the relation between variation in part shape and variation in the location of the center of mass for a convex part with uniform mass distribution. We consider a general model for shape variation that only assumes that every valid instance contains a polygon P_I while it is contained in another polygon P_E. We characterize the worst-case displacement of the center of mass in a given direction in terms of P_I and P_E. The characterization allows us to determine an adequate polygonal approximation of the locus of the center of mass. We also show that the worst-case displacement is small if P_I is fat and the distance between the boundary of P_E and P_I is bounded.

1 Introduction

Many automated part manufacturing tasks involve manipulators that perform physical actions—such as pushing, squeezing [1], or pulling [2]—on the parts. Over the past two decades, researchers in robotics in general and algorithmic automation in particular have thoroughly studied the effect of physical actions as well as their potential role in accomplishing high-level tasks like orienting or sorting. It is evident that shape and—in many cases (see e.g. [1, 3, 4, 5, 6, 7])—location of the center of mass are important parameters in determining the effect of a physical action on a part.

Industrial parts are always manufactured to tolerances as no production process is capable of delivering parts that are perfectly identical. Tolerance models [12, 13] are therefore used to specify the admitted variations with respect to the CAD model. A consequence of

these variations [8, 9] is that actions that are computed on the basis of a CAD model of a part may easily lead to different behavior when executed on a manufactured incarnation of that part, and thus to failure to accomplish the higher-level task. It is important to note that the shape variations not only directly affect the behavior of the part but indirectly as well because they also cause a displacement of the center of mass of the part.

To extend the planning algorithms to imperfect manufactured incarnations, it is important to understand the effects of variations and take them into account during planning. Larger variations in part shape and center-of-mass location inevitably result in a larger range of possible part behaviors, which reduces the likeliness that a manufacturing task can be accomplished. Therefore we will study how variations in part shape influence the location of the center of mass. (Note that variations in shape and center of mass are not the only sources of uncertainty in robotics. Additional uncertainty can result from the inaccuracy of the actuators and manipulators [11] and sensors [10].)

Several geometric approaches have been proposed to overcome the problems occurring in the presence of uncertainty and to smooth the effects of errors. Among the existing approaches are the model of $\epsilon-$geometry [14], tolerance and interval geometry [15, 16] and region-based models [17]. Generally, in all these models an uncertain point is represented by a region in which it may vary. The model of $\epsilon-$geometry assumes that a point can vary within a disk of radius ϵ. Tolerance and interval-geometry take into account coordinate errors which results in an axis-aligned rectangular region in which a point can vary. In general, region-based models represent a point by any convex region. After modeling uncertainty as a point surrounded by a region, it is possible to study worst (and best) cases for a problem under the specific uncertainty model.

As observed before, variation of the shape causes variation of the center of mass of a part. The locus of the centroid of a set of points with approximate weights has been studied by Bern et al. [19]. Akella et al. [18] estimated the locus for a polygon under the $\epsilon-$geometry model [18]. The problem of finding the locus of the center of mass of a part with shape variation and uniformly distributed mass has been mentioned as an open problem [9, 18]. Akella et al. [18] studied rotating a convex polygon whose vertices and the center of mass lie inside predefined circles centered at their nominal loca-

tions. The problem of orienting a part by fence has been studied by Chen et al. [9]. They define disk and square regions for the vertices of a part and proposed a method for computing the maximum allowable uncertainty radius for each vertex. They also discussed in a more general way the key role of the center of mass and the successfulness of part feeding (or orienting) algorithms in a setting of shape variation. Chen el al. [20] presented algorithms for squeezing and pushing problems. Kehoe et al. [21] explored cloud computing in a context of grasping and push-grasping under shape variation.

All the previous models for shape variation only allow the vertices to vary. In this paper we use a more general model for shape variation. For given convex shapes P_I and P_E such that $P_I \subseteq P_E$ we consider the family of shapes P satisfying $P_I \subseteq P \subseteq P_E$. In the practical setting of toleranced parts the shapes P_I and P_E will be fairly similar. We will show in Section 3 that the valid instance that yields the largest displacement of the center of mass in a given direction is a shape that combines a part of P_I with a part of P_E. The corresponding displacement is computable in $O(n)$ time where n is the complexity of P_I and P_E; it can be used to obtain a k-vertex outer approximation of the set of all possible loci of the center of mass in $O(kn)$ time.

In Section 4, we will study the size of the set of possible center-of-mass loci. Fatness of the objects under consideration has led to lower combinatorial complexities and more efficient algorithms for various problems, including union complexities [24], motion planning [22], hidden surface removal [25], and range searching [26]. We show that fatness of P_I together with the assumption that no point in P_E has a distance larger than ϵ to P_I leads to a bound on the distance between the centers of mass of any two valid instances of a part which is proportional to ϵ and the fatness of P_I.

2 Preliminaries

In this section, we first present a general model for shape variations, then review the notion of a center of mass, and finally introduce a few notions that allow us to characterize the shapes that maximize the displacement of the center of mass. Let P_M be the model part. The part P_M has a uniform mass distribution.

No production process ever delivers parts that are perfectly identical to the model part P_M and therefore industrial parts are manufactured to tolerances. We use a very general model for permitted shape variations that only requires that any manufactured instance of P_M contains a given convex subshape P_I of P_M while it is contained in a convex supershape P_E of P_M. As a result, the set of acceptable instances of P_M is a family of shapes $V(P_I, P_E) = \{P \subset \mathbb{R}^2 | P_I \subseteq P \subseteq P_E\}$ for given P_I and P_E satisfying $P_I \subseteq P_M \subseteq P_E$. In other
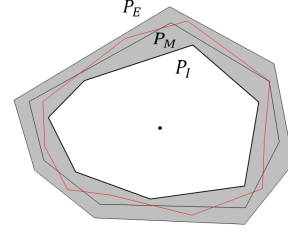

Figure 1: A family of shapes specified by a subshape P_I and a supershape P_E of a model part P_M, along with a valid instance $P \in V(P_I, P_E)$.

words, the boundary ∂P of an instance $P \in V(P_I, P_E)$ should be entirely contained in $Q = P_E - int(P_I)$ where $int(P)$ denotes the interior of the set P. The region Q is referred to as the *tolerance zone*. The objects P_I and P_E are assumed to be closed polygons with a total of n vertices. (Figure 1 shows and example of a model part P_M, shapes P_I and P_E, and a valid instance $P \in V(P_I, P_E)$.) We denote by $COM(P_I, P_E)$ the set of all centers of mass of instances $P \in V(P_I, P_E)$.

We let $X_c(P)$ denote the x-coordinate of the center of mass and $A(P)$ be the area of the object P. The x-coordinate of the center of mass of an object with uniform mass distribution satisfies

$$X_c(P) = \frac{1}{A} \int_A x \, dA,$$

where A is the area of the object. A similar equality holds for the y-coordinate of the center of mass. In the case of uniform mass distribution the center of mass corresponds to the centroid of the object. We will often decompose an object P into sub-objects P_i ($1 \le i \le n$) and then express its center of mass as a function of the centers of mass of its constituents, through the equation

$$X_c(P) = \frac{\sum_{i=1}^n X_c(P_i) A(P_i)}{\sum_{i=1}^n A(P_i)}. \qquad (1)$$

We conclude this section by defining useful objects. Disks play a prominent role in Section 4. We denote by $D_r(p)$ the closed disk with radius r centered at p, and use the abbreviation $D_r = D_r(O)$ where O is the origin.

For an object P and a value m we define its right portion $P^+[m]$ with respect to m by $P^+[m] = \{(x, y) \in P | x \ge m\}$. Similarly, we define its left portion $P^-[m]$ with respect to m by $P^-[m] = \{(x, y) \in P | x \le m\}$. With these portions we can define *minmax* objects, which allow us to capture the intuition that the largest displacement of the center of mass in a given direction is achieved by the object from $V(P_I, P_E)$ that 'maximizes mass' in that direction and 'minimizes mass' in the opposite direction. The minmax object $P^*[m]$ consists of a left portion of P_I and a right portion of P_E with respect to the same m, so $P^*[m] = P_I^-[m] \cup P_E^+[m]$
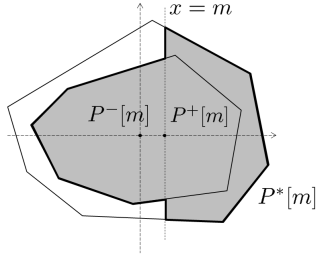
Figure 2: A minmax object

(see Figure 2). Note that an alternative way to describe $P^*[m]$ is by the equation $P^*[m] = P_I \cup Q^+[m]$

3 Displacement of the center of mass

In this section, we find an upper bound on the displacement of the center of mass in a given direction. The resulting bound allows us to determine a good polygonal outer approximation of the set $COM(P_I, P_E)$ of possible loci of the center of mass.

3.1 Bounding the displacement in one direction

Without loss of generality we assume that P_I and P_E are positioned and oriented in such a way that the center of mass of P_I coincides with the origin (so $X_c(P_I) = 0$) and that the direction in which we want to bound the displacement aligns with the positive x-axis. This assumption and the equation $P^*[m] = P_I \cup Q^+[m]$ allow us to simplify Equation 1 to

$$X_c(P^*[m]) = \frac{X_c(Q^+[m])A(Q^+[m])}{A(P_I) + A(Q^+[m])}. \qquad (2)$$

Although we will bound the displacement with respect to the center of mass of P_I we observe that the result also induces a bound with respect to the center of mass of P_M as $P_M \in V(P_I, P_E)$ by definition. We let $X_r = \max_{(x,y) \in P_E} x$.

Our first lemma establishes a connection between the minmax objects $P^*[x]$ for $0 \le x \le X_r$ and the location of their centers of mass.

Lemma 1 *There is exactly one minmax object $P^*[m]$ ($0 \le m \le X_r$) that satisfies $X_c(P^*[m]) = m$. Moreover $x < X_c(P^*[x]) \le m$ for all $0 \le x < m$ and $X_c(P^*[x]) < m$ for all $m < x \le X_r$.*

Proof. From $X_c(P_I) = 0$ and $X_c(Q^+[0]) \ge 0$ and the fact that $P^*[0] = P_I \cup Q^+[0]$ it follows that $X_c(P^*[0]) \ge 0$; moreover, it is clear that $X_c(P^*[X_r]) \le X_r$. As the center of mass of $P^*[x]$ moves continuously as x increases from 0 to X_r there must be at least one x such that $X_c(P^*[x]) = x$. It remains to show that there is also at most one such x. Let m be such that

$X_c(P^*[m]) = m$. We consider a minmax object $P^*[x]$ for $x \ne m$ and distinguish two cases: (i) $0 \le x < m$ and (ii) $m < x \le X_r$.

Consider case (i). Using the notation $Q' = Q^+[m]$ and $Q'' = P^*[x] - P^*[m] = Q^+[x] - Q^+[m]$ we have that $P^*[m] = P_I \cup Q'$ and $P^*[x] = P_I \cup Q' \cup Q''$. Note that $Q'' \subset [x, m] \times \mathbb{R}$ and thus

$$x \le X_c(Q'') \le m.$$

As $x < X_c(P^*[m]) = X_c(P_I \cup Q') = m$ it follows from Equation 1 that

$$x(A(P_I) + A(Q')) < X_c(Q')A(Q') \le m(A(P_I) + A(Q')).$$

After observing that $X_c(P_I) = 0$ we apply Equation 1 to $P^*[x] = P_I \cup Q' \cup Q''$ and use the aforementioned inequalities to obtain

$$
\begin{aligned}
X_c(P^*[x]) &= \frac{X_c(Q')A(Q') + X_c(Q'')A(Q'')}{A(P_I) + A(Q') + A(Q'')} \\
&> \frac{x(A(P_I) + A(Q')) + xA(Q'')}{A(P_I) + A(Q') + A(Q'')} = x
\end{aligned}
$$

and

$$
\begin{aligned}
X_c(P^*[x]) &= \frac{X_c(Q')A(Q') + X_c(Q'')A(Q'')}{A(P_I) + A(Q') + A(Q'')} \\
&\le \frac{m(A(P_I) + A(Q')) + mA(Q'')}{A(P_I) + A(Q') + A(Q'')} = m.
\end{aligned}
$$

Consider case (ii). Using the notation $Q' = Q^+[x]$ and $Q'' = P^*[m] - P^*[x] = Q^+[m] - Q^+[x]$ we have that $P^*[x] = P_I \cup Q'$ and $P^*[m] = P_I \cup Q' \cup Q''$. Note that $Q'' \subset [m, x] \times \mathbb{R}$ and thus

$$m \le X_c(Q'') \le x.$$

As $X_c(P^*[m]) = X_c(P_I \cup Q' \cup Q'') = m$ it follows from Equation 1 that

$$X_c(Q')A(Q') = m(A(P_I) + A(Q')) + (m - X_c(Q''))A(Q'').$$

We apply Equation 1 to $P^*[x] = P_I \cup Q'$ and use the aforementioned equations and inequality to obtain

$$
\begin{aligned}
X_c(P^*[x]) &= \frac{X_c(Q')A(Q')}{A(P_I) + A(Q')} \\
&\le \frac{m(A(P_I) + A(Q')) - (X_c(Q'') - m)}{A(P_I) + A(Q')} \\
&\le \frac{m(A(P_I) + A(Q'))}{A(P_I) + A(Q')} = m.
\end{aligned}
$$

Combining both cases we find that there is no $x \ne m$ that satisfies $X_c(P^*[x]) = x$. $\qquad \square$

In addition to the fact that there is only one minmax object $P^*[m]$ that satisfies $X_c(P^*[m]) = m$, Lemma

1 also reveals that $X_c(P^*[x]) > x$ for $x < m$ and $X_c(P^*[x]) < x$ for $x > m$. Moreover, it shows that $X_c(P^*[x]) < m$ for all $x \neq m$ which means that the minmax object $P^*[m]$ with $X_c(P^*[m]) = m$ achieves larger displacement of the center of mass in the direction of the positive x-axis than any other minmax object $P^*[x]$ with $x \neq m$. The following theorem shows that $P^*[m]$ in fact achieves the largest displacement of the center of mass among *all* objects in $V(P_M)$.

Theorem 2 *Let $P^*[m]$ $(0 \leq m \leq X_r)$ be the unique minmax object that satisfies $X_c(P^*[m]) = m$. Then $X_c(P) < X_c(P^*[m])$ for all $P \in V(P_I, P_E), P \neq P^*[m]$.*

Proof. Let $P \in V(P_I, P_E), P \neq P^*[m]$ be the object that yields the largest displacement $m' \geq m$ of the center of mass, so $X_c(P) = m'$. If $P = P^*[m']$ then it follows immediately from Lemma 1 that $m' = m$. Now assume for a contradiction that $P \neq P^*[m'] = P_I^-[m'] \cup P_E^+[m']$ which implies that (i) $P_E^+[m'] - P^+[m'] \neq \emptyset$ or (ii) $P^-[m'] - P_I^-[m'] \neq \emptyset$.

Consider case (i) and let R be a closed connected subset with $A(R) > 0$ of $P_E^+[m'] - P^+[m']$. Observe that $P \cup R \in V(P_I, P_E)$. Note that $R \subset (m', \infty) \times \mathbb{R}$ and thus $X_c(R) > m'$. We get

$$X_c(P \cup R) = \frac{X_c(P)A(P) + X_c(R)A(R)}{A(P) + A(R)}$$
$$> \frac{m'A(P) + m'A(R)}{A(P) + A(R)} = m'$$

which contradicts the assumption that P is the object in $V(P_I, P_E)$ that achieves the largest displacement of the center of mass.

Consider case (ii) and let R be a closed connected subset with $A(R) > 0$ of $P^-[m'] - P_I^-[m']$. Observe that $P - R \in V(P_M)$. Note that $R \subset (-\infty, m') \times \mathbb{R}$ and thus $X_c(R) < m'$. We get

$$X_c(P - R) = \frac{X_c(P)A(P) - X_c(R)A(R)}{A(P) - A(R)}$$
$$> \frac{m'A(P) - m'A(R)}{A(P) - A(R)} = m$$

which again contradicts the assumption that P is the object in $V(P_I, P_E)$ that achieves the largest displacement of the center of mass. As a result we find that $P^*[m]$ with $X_c(P^*[m]) = m$ is the unique object in $V(P_I, P_E)$ that achieves the largest displacement of the center of mass. $\square$

The theorem shows that the set $COM(P_I, P_E)$ does not extend beyond (i.e., to the right of) the line $x = m$ where m is such that $X_c(P^*[m]) = m$. The bound is tight because $P^*[m] \in V(P_I, P_E)$. In fact, the theorem shows that $P^*[m]$ is the only instance in $V(P_I, P_E)$ that has its center of mass on that line. Since the result holds in any direction, this implies that $COM(P_I, P_E)$ is convex.
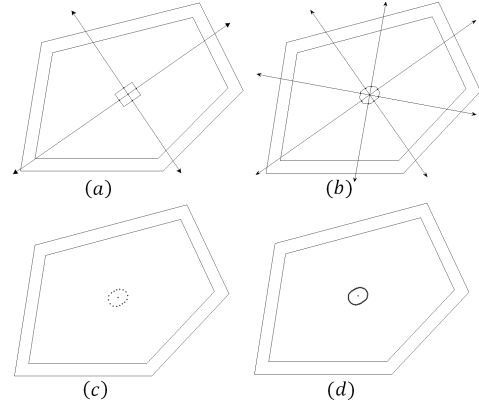

Figure 3: Outer approximations of $COM(P_I, P_E)$ with (a) 4, (b) 8, (c) 16, and (d) 64 vertices.

3.2 A k-vertex approximation for $COM(P_I, P_E)$

The results in the previous subsection suggest an easy approach to determine an outer approximation of the set $COM(P_I, P_E)$ of possible centers of mass of instances in $V(P_I, P_E)$. If we select k different directions that positively span the plane and apply Theorem 2 in each of these directions then we obtain a bounded polygon with k edges enclosing $COM(P_I, P_E)$. To find the largest displacement in the positive x-direction, we sweep a vertical line from $x = 0$ to $x = X_r$ while maintaining $X_c(P^*[m])$ using Equation 2. The mathematical descriptions of $A(Q^+[m])$ and $X_c(Q^+[m])$ only change when the line hits a vertex of P_I or P_E, as the boundary of these two shapes determine the boundary of Q. The corresponding update of these mathematical descriptions at such a vertex can be accomplished in constant time. Moreover, the check to decide whether the equation $X_c(P^*[m]) = m$ has a solution between the current vertex and the next vertex hit by the line also takes constant time as it requires (for polygonal P_I and P_E) computing the roots of a polynomial function of degree four. Since P_I and P_E have n vertices we obtain the following theorem.

Theorem 3 *A polygonal k-vertex outer approximation of $COM(P_I, P_E)$ can be computed in $O(kn)$ time.*

Lemma 1 suggests that we can use binary search to identify the m satisfying $X_c(P^*[m]) = m$. Unfortunately it seems impossible to evaluate $A(Q^+[m])$ and $X_c(Q^+[m])$ for an arbitrary m in $o(n)$ time, which would be crucial to improve the time bound reported in Theorem 3.

Figure 3 shows 4-, 8-, 16-, and 64-vertex outer approximations of $COM(P_I, P_E)$ for a given P_I and P_E. Recall that every edge of the polygonal approximation contains one point of the convex set $COM(P_I, P_E)$, so $COM(P_I, P_E)$ strongly resembles its approximation.

4 Bounding the size of $COM(P_I, P_E)$

The admitted shape variation for a manufactured part is usually small compared to the dimensions of the part itself. As a result, the enclosed shape P_I and enclosing shape P_E do not deviate much from the model shape P_M, and therefore also not from each other. To capture this similarity we will assume that $P_I \subseteq P_E \subseteq P_I \oplus D_\epsilon$, where D_ϵ is the disk of radius ϵ centered at the origin and $\oplus$ denotes the Minkowski sum. Note that this means that every point in P_E is within a distance of at most ϵ from some point in P_I. In Subsection 4.1 we will see that this distance constraint alone is not enough to obtain a bound on the diameter of $COM(P_I, P_E)$ that is independent of the size of P_I and P_E. In Subsection 4.2 we show that the additional assumption that P_I is fat leads to a bound on the diameter of $COM(P_I, P_E)$ that depends on ϵ and the fatness.

4.1 A thin part

When P_I is a sufficiently long and narrow box the set $V(P_I, P_E)$ contains shapes whose centers of mass are a distance proportional to the diameter of P_I apart. Let $L \gg \epsilon$ and pick δ such that $0 < \delta < \epsilon^2/(2L - \epsilon)$. We define $P_I = [-L/2, L/2] \times [-\delta/2, \delta/2]$ and $P_E = [-(L+\epsilon)/2, (L+\epsilon)/2] \times [-(\delta+\epsilon)/2, (\delta+\epsilon)/2]$, and note that $P_E \subseteq P_I \oplus D_\epsilon$. Now consider the object $P^*[L/2] = P_I^-[L/2] \cup P_E^+[L/2] = P_I \cup P_E^+[L/2]$. We observe that $A(P_I) = \delta L$, $X_c(P_I) = 0$, $A(P_E^+[L/2]) = \epsilon(\epsilon + \delta)/2$, and $X_c(P_E^+[L/2]) = L/2 + \epsilon/4 > L/2$. The upper bound on δ implies that $A(P_E^+[L/2]) > A(P_I)$. From Equation 1 it follows that $X_c(P^*[L/2]) > L/4$ showing that the diameter of $COM(P_I, P_E)$ is *not* proportional to ϵ in this case.

4.2 Fat parts

In this subsection we add the assumption that P_I is fat to deduce a bound on the diameter of $COM(P_I, P_E)$ that depends on ϵ and the fatness. There are many different definitions of fatness and we will use the one by De Berg et al. [23], which is based on a similar definition presented in the thesis of van der Stappen [22].

Definition 1 *Let $P \subseteq \mathbb{R}^2$ be an object and let β be a constant with $0 < \beta \leq 1$. Define $U(P)$ as the set of all disks centered inside P whose boundary intersects P. We say that the object P is β-fat if for all disks $D \in U(P)$ we have $A(P \cap D) \geq \beta \cdot A(D)$. The fatness of P is defined as the maximal β for which P is β-fat.*

For bounded objects the value of β is at most $1/4$; larger values only occur for unbounded objects [22].

In the remainder of this section we assume that P_I is β-fat $(0 < \beta \leq 1)$. The main implication of this assumption is that it provides us with a lower bound on the area of P_I in terms of its diameter.

The following lemma is not strictly necessary yet it leads to a better bound in our main theorem. We omit the proof and immediately apply it in the theorem.

Lemma 4 *Let P be a convex polygon with diameter d. Then no point in P has distance larger than $\frac{2d}{3}$ to the center of mass of P.*

Theorem 5 *Let P_I be a bounded convex β-fat object $(0 < \beta \leq 1)$ and let P_E be a bounded object satisfying $P_I \subseteq P_E \subseteq P_I \oplus D_\epsilon$. Then the diameter of $COM(P_I, P_E)$ is bounded by $\frac{5}{2}\beta^{-1}\epsilon$.*

Proof. We use d to denote the diameter of P_I and once again assume without loss of generality that the center of mass of P_I coincides with the origin. Theorem 2 shows that it suffices to consider objects $P^*[m]$ $(0 \leq m \leq X_r)$ to bound the size of $COM(P_I, P_E)$. Lemma 4 says that P_I lies completely inside the disk $D(2d/3)$. As a consequence, the object $P^*[m]$ must lie entirely inside $D(2d/3 + \epsilon)$, which implies that $X_c(P^*[m]), X_c(Q^+[m]) \leq 2d/3 + \epsilon$. We distinguish two cases based on the ratio of ϵ and d.

If $\epsilon \geq d/6$ then $X_c(P^*[m]) \leq 2d/3 + \epsilon \leq 5\epsilon$. Since $P^*[m]$ is bounded we know that $\beta \leq 1/4$ and thus $X_c(P^*[m]) \leq 5\epsilon \leq 5\beta^{-1}\epsilon/4$.

If $\epsilon \leq d/6$ we use Equation 2 to obtain an upper bound $X_c(P^*[m])$ by combining the upper bound $X_c(Q^+[m]) \leq 2d/3 + \epsilon$ with a lower bound on $A(P_I)$ and upper and lower bounds on $A(Q^+[m])$. The lower bound on $A(P_I)$ follows from the fatness of P_I. As d is the diameter of P_I there must be two points $p_1, p_2 \in P_I$ that are d apart. The boundary of the disk $D_d(p_1)$ contains p_2 and thus belongs to the set $U(P_I)$. The β-fatness of P_I implies that $A(P_I) \geq \beta \cdot A(D_d(p_1)) = \beta\pi d^2$.

It remains to bound $A(Q^+[m])$. Recall that $Q = P_E - int(P_I)$. Due to our assumption $P_E \subseteq P_I \oplus D_\epsilon$ we obtain that $Q \subseteq Q' = (P_I \oplus D_\epsilon) - int(P_I)$, from which it follows that $A(Q^+[m]) \leq A(Q) \leq A(Q')$. We observe that the convexity of P_I implies that $A(Q') = l\epsilon + \pi\epsilon^2$, where l denotes the perimeter of P_I. As P_I is contained in $D(2d/3)$ the perimeter l of P_I is bounded by $4\pi d/3$. Combining these observations with a trivial lower on $A(Q^+[m])$ we get $0 \leq A(Q^+[m]) \leq 4\pi\epsilon d/3 + \pi\epsilon^2$.

Plugging all the inequalities into Equation 2 and using $\epsilon/d \leq 1/6$ yields

$$
\begin{aligned}
X_c(P^*[m]) &= \frac{X_c(Q^+[m])A(Q^+[m])}{A(P_I) + A(Q^+[m])} \\
&\leq \frac{(\frac{2}{3}d + \epsilon)(\frac{4}{3}\pi\epsilon d + \pi\epsilon^2)}{\beta\pi d^2} \\
&= \beta^{-1}\epsilon(\frac{8}{9} + 2(\frac{\epsilon}{d}) + (\frac{\epsilon}{d})^2) \\
&\leq \frac{5}{4}\beta^{-1}\epsilon.
\end{aligned}
$$

which shows $COM(P_I, P_E) \subseteq D(\frac{5}{4}\beta^{-1}\epsilon)$. $\qquad\square$

Theorem 5 confirms the intuition that the variation of the center of mass grows if the admitted shape variation increases or the fatness decreases.

5 Conclusion

We have considered a very general model for admitted shape variations of a model part, based on enclosed convex shape P_I and an enclosing convex shape P_E. We have identified the valid instance that maximizes the displacement of the center of mass in a given direction, and used this result to find a k-vertex polygonal outer approximation of the set of all possible center-of-mass loci in $O(kn)$ time, where n is the number of vertices of P_I and P_E. If P_I is β-fat and every point of P_E is within a distance ϵ of P_I then the diameter of the set of all center-of-mass loci can be shown to be $O(\beta^{-1}\epsilon)$.

We expect that our results will generalize to three-dimensional objects. It is interesting to see under which circumstances the results can be extended to non-convex shapes P_I (and P_E), and to parts with non-uniform mass distribution.

References

[1] K. Goldberg, Orienting polygonal parts without sensors. *Algorithmica*, 10(2), pp. 201–225, 1993.

[2] R.-P. Berretty, M.H. Overmars, A.F. van der Stappen, Orienting polyhedral parts by pushing. *Computational Geometry: Theory and Applications*, 21, pp. 21–38, 2002.

[3] S. Akella, M.T. Mason, Posing polygonal objects in the plane by pushing. *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 2255–2262, 1992.

[4] M.A. Erdmann, M.T. Mason, An exploration of sensorless manipulation. *IEEE Journal of Robotics and Automation*, 4, pp. 367–379, 1988.

[5] K.M. Lynch, Locally controllable manipulation by stable pushing. *IEEE Transactions on Robotics and Automation*, 15, pp. 314–323, 1999.

[6] R.-P. Berretty, K. Goldberg, M.H. Overmars, A.F. van der Stappen, Computing fence designs for orienting parts. *Computational Geometry: Theory and Applications*, 10(4), pp. 249–262, 1998.

[7] R.C. Brost, Automatic grasp planning in the presence of uncertainty. *International Journal of Robotics Research*, 7(1), pp. 3–17 , 1988.

[8] B.R. Donald, Error Detection and Recovery in Robotics. *Springer-Verlag,* 1987.

[9] J. Chen, K. Goldberg, M.H. Overmars, D. Halperin, K.-F. Böhringer, Y. Zhuang, Shape tolerance in feeding and fixturing. *Robotics: the algorithmic perspective*, pp. 297–311. A.K. Peters, 1998.

[10] S.M. Lavalle, Planning Algorithms, chapter 12: Planning Under Sensing Uncertainty. *Cambridge University press*, 2006.

[11] M. Dogar, S.S. Srinivasa, A Framework for Push-grasping in Clutter. *Robotics: Science and Systems* 2, 2011.

[12] A.A.G. Requicha, Toward a theory of geometric tolerancing. *International Journal of Robotics Research*, 2, 1983.

[13] H. Voelker. A current perspective on tolerancing and metrology. *Manufacturing Review*, 6(4), 1993.

[14] L. Guibas, D. Salesin, J. Stolfi, Epsilon geometry: Building robust algorithms for imprecise computations. *Proc. of the 5th Annual ACM Symp. on Computational Geometry*, pp. 208–217, 1989.

[15] U. Roy, C. Liu, T. Woo. Review of dimensioning and tolerancing. *Computer-Aided Design*, 23(7), 1991.

[16] Y. Ostrovsky-Berman, L. Joskowicz, Tolerance envelopes of planar mechanical parts with parametric tolerances. *Computer-Aided Design*, pp. 531–534, 2005.

[17] M. Löffler, Data Imprecision in Computational Geometry. *PhD Thesis, Utrecht University*, 2009.

[18] S. Akella, M.T. Mason, Orienting Toleranced Polygonal Parts. *International Journal of Robotics Research*, 19(12), pp. 1147–1170, 2000.

[19] M. Bern, D. Eppstein, L.J. Guibas, J.E. Hershberger, S. Suri, J. Wolter, The centroid of points with approximate weights. *Proc. 3rd Eur. Symp. Algorithms*, LNCS 979, pp. 460–472, 1995.

[20] Y.-B. Chen, D.J. Ierardi, The complexity of oblivious plans for orienting and distinguishing polygonal parts. *Algorithmica*, 14, pp. 367–397, 1995.

[21] B. Kehoe, D. Berenson, K. Goldberg, Toward Cloud-Based Grasping with Uncertainty in Shape: Estimating Lower Bounds on Achieving Force Closure with Zero-Slip Push Grasps. *Proc. IEEE Int. Conf. on Robotics and Automation*, 2012.

[22] A.F. van der Stappen, Motion Planning amidst Fat Obstacles. *Ph.D. Thesis, Utrecht University*, 1994.

[23] M. de Berg, M.J. Katz, A.F. van der Stappen, J. Vleugels, Realistic input models for geometric algorithms, *Algorithmica*, 34, pp. 81–97, 2002.

[24] B. Aronov and M. de Berg, Unions of fat convex polytopes have short skeletons. *Discrete and Computational Geometry*, 48(1): 53-64 ,2012.

[25] M.J. Katz, M.H. Overmars, M. Sharir, Efficient hidden surface removal for objects with small union size, *Computational Geometry: Theory and Applications*, 2, pp. 223-234, 1992.

[26] M.H. Overmars, A.F. van der Stappen, Range searching and point location among fat objects, *Journal of Algorithms*, pp. 626-656, 1996.