Multi-guard covers for polygonal regions*

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Abstract

We study the problem of finding optimal covers of polygonal regions using multiple mobile guards. By our definition, a point is covered if, at some time, it lies within the convex hull of the guards from which it is visible. The definition captures our desire that guards both "see" and "surround" points that they cover. Guards move along continuous time-parameterized curves within a polygonal region P. An optimal m-guard cover of P is a set of m guard paths of minimum total length that cover all points in P.

In this paper, we restrict our attention to the case where P is convex, and m is either two or three. We first address the apparently simpler problem of optimally covering all points on the boundary, ∂P, of P. Although the guard paths are not restricted to ∂P, we prove that in every optimal two-guard boundary cover the guards remain on ∂P. When there are three guards, an optimal boundary cover may require a guard to cross the interior of the polygon. We show, however, that every optimal three-guard boundary cover is simple (i.e., guard paths do not cross one another). We provide complete characterizations of the form of optimal two- and three-guard boundary covers for convex polygons that support polynomial-time algorithms for their construction. Finally, we show that, for convex P, any optimal two- or three-guard cover of ∂P is also a (necessarily optimal) cover of the full polygon P.

1 Introduction

Given a polygon P, we consider the problem of finding mobile guard paths of minimum total length to cover P. We require the guards to both "see" and "surround" a point in order to cover it. In particular, we associate with a guard i a continuous path $G_i(t)$ (parameterized by time) contained in P. A point $p \in P$ is *covered* at time t by a set of guards, if $p \in \mathrm{CH}(\bigcup_{i:\overline{pG_i(t)}\subset P} G_i(t))$ (the convex hull of the guards that see p at time t). Although similar problems have been studied in the past [2, 3, 4, 5] the definitions of coverage used do not require enclosure, i.e. they are based on visibility considerations alone. Our model also differs from previous work in the form of admissible guard paths: we allow paths to start and end anywhere in P, to cross the interior of P, and to even intersect.

We restrict our attention to covering convex polygons. This allows us to focus on the complications that arise from requiring enclosure in a setting in which visibility is not an issue. Thus, a point p in a convex polygon P is *covered* at time t by a set of guards, if $p \in \mathrm{CH}(\bigcup_i G_i(t))$ (the convex hull of all the guards at time t).

Furthermore, we initially consider only covering the boundary, ∂P, of P. In such a cover, we call a maximal connected subset of ∂P that is not visited (i.e., part of a guard path) a *free (boundary) segment*. Having too many free segments in a boundary cover makes path intersections unavoidable, since the endpoints of a free segment are visited by two different guards at the same time. A boundary cover is *simple* if the paths of the guards do not intersect in the interior of P. We show that every optimal two- or three-guard boundary cover of a convex polygon is simple. In the two-guard case, this implies that the guards stay on ∂P, however in the three-guard case, there are convex polygons whose optimal cover requires a guard to cross the interior of P.

We begin by analyzing the problem for two guards to help set the notation and intuition. The same problem for three guards turns out to have a surprisingly involved solution.

2 Two-guard boundary covers of convex polygons

If the two-guard cover is simple then it leaves at most two free segments on ∂P. Thus a shortest boundary cover of this type has total length at most $|\partial P| - |\ell_1| - |\ell_2|$ where ℓ_1 and ℓ_2 are the two longest boundary edges. The following theorem shows that an optimal boundary cover has exactly this form. Note that this implies that we can find an optimal two-guard boundary cover of a convex polygon with n vertices in $O(n)$ time.

Theorem 1 *Every optimal two-guard boundary cover of a convex polygon P is simple.*

Proof. If the boundary cover has fewer than two free segments then it is not optimal. Let s be the earliest covered free segment and t the latest. Since s is the earliest free segment, either a guard starts its path at the endpoint of s or its starting point is hidden, i.e.,

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it is not the endpoint of any free segment. A similar argument can be made about t. Therefore s and t are the only free segments that can be adjacent to path endpoints (start or end of the paths). We call the rest of the free segments (if they exist) *intermediate*.

We show how to untangle intersecting paths and produce a shorter boundary cover. Assume that P is positioned in a way that s and t intersect a horizontal line, which divides ∂P into two series of edges that connect s and t on top and bottom. Without loss of generality we assume that s is on the left and t on the right. The following steps explain how a two-guard boundary cover can be untangled, see Fig. 1.

- Guard 1 starts its path at the top endpoint of s, takes the top-most path at each intersection until it reaches the top endpoint of t.
- Guard 2 starts its path at the bottom endpoint of s, takes the bottom-most path at each intersection until it reaches the bottom endpoint of t.

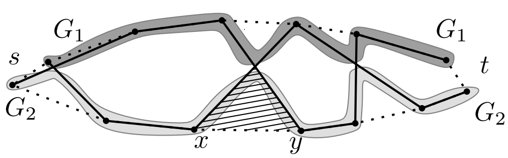

Figure 1: Finding the top-most and bottom-most paths in a two-guard boundary cover.

When reaching an intermediate free segment xy, a guard will choose a path that leaves the boundary at one endpoint of xy, walks around the interior of the *xy-pocket* (made by xy and the interior bottom-most (or top-most) paths around xy) and gets to the other endpoint of xy, see Fig. 1.

Since in the original boundary cover, both guards start at s and end at t, any cut that separates s from t crosses each of the guard paths. This implies that the total length of the top- and bottom-most paths is at most the length of the original boundary cover.

By the triangle inequality, we can shorten the top- and bottom-most paths by replacing the interior path of an xy-pocket with xy. We refer to this replacement of a pocket with its free segment as *flattening* the pocket. By flattening all pockets, we create new paths from s to t for the guards that are limited to ∂P and still form a boundary cover of P. □

3 Three-guard boundary covers of convex polygons

When we increase the number of guards to three, it may appear that, as in the two-guard case, an optimal boundary cover exists in which all guards remain on ∂P. However, for certain convex polygons, crossing the interior is required to make an optimal cover. Figures 2 and 3 show optimal boundary covers where a guard must cross the interior of the polygon one or more times.

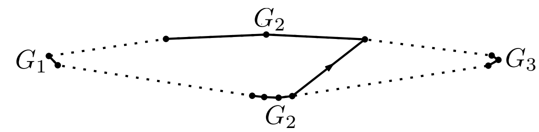

Figure 2: Optimal boundary cover may cross interior.

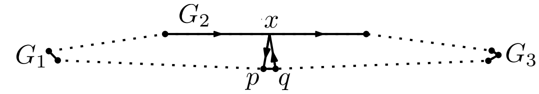

Figure 3: Optimal boundary cover may cross interior twice.

It may even appear possible at this point that for some convex polygon an optimal boundary cover requires guard paths to intersect, i.e., is not simple. We will show that this is not the case, that is every optimal boundary cover is simple.

In most cases, intersecting guard paths can be untangled nicely. For example, in Fig. 4, the paths G_1 and G_2 intersect but can be untangled (see the grey highlights) without lengthening the paths. However,

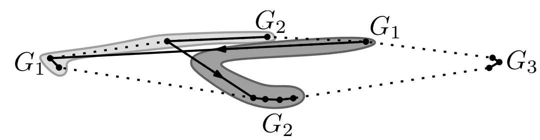

Figure 4: A simple untangling step.

there are some cases that can only be partially untangled since we might encounter pockets that cannot be flattened without possibly increasing the total length of the guard paths. Figures 5 and 6 show boundary covers that have been partially untangled using a method similar to the two-guard method. The regular pockets along the paths have been flattened, but the shaded pocket in Fig. 5 cannot be flattened since the pocket contains a path endpoint and the interior path of the pocket doesn't connect x and y. Thus, although it does not appear so in the figure, the interior path of an xy-pocket could be shorter than xy.

In the following theorem, we argue that all boundary covers that can only be partially untangled result in a set of paths that have one of the two general forms shown in Figures 7 and 8. (For example, the highlighted curves of Fig. 5 are of the form shown in Fig. 7.) We call these two forms the *single* and *double intersection forms*, respectively. In fact, the paths in these forms may not provide a proper boundary cover. However, they do provide a *relaxed* boundary cover in the sense that every point on the boundary is either visited by a guard or is part of a segment connecting two points $G_i(t_1)$ and $G_j(t_2)$ for $i \neq j$, where t_1 and t_2 may differ. We show that for a relaxed boundary cover in these forms there is always a shorter *simple* relaxed boundary cover. This is a consequence of the following two deceptively simple lemmas, whose lengthy proofs constitute a major part of the first author's thesis [1] and have been

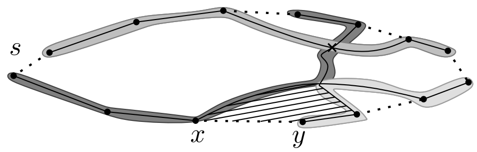

Figure 5: The pocket cannot be flattened and no further untangling is possible.

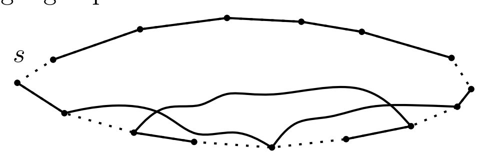

Figure 6: Another example that cannot be untangled.

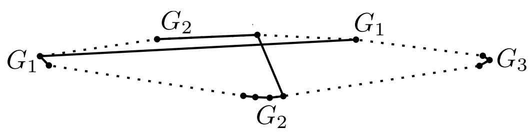

Figure 7: Single intersection form.

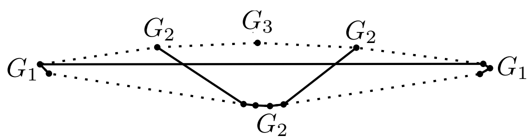

Figure 8: Double intersection form.

omitted from this abstract.

Lemma 2 *[1, Theorem 1] If a relaxed boundary cover* $\{G_1, G_2, G_3\}$ *has the single intersection form (Fig. 7) with interior segments* S_1 *and* S_2 *then there exist two free segments* M_1 *and* M_2 *such that (i)* $|M_1| + |M_2| < |S_1| + |S_2|$ *and (ii)* $((G_1 \cup G_2 \cup G_3) \backslash (S_1 \cup S_2)) \cup M_1 \cup M_2$ *can be partitioned into a simple relaxed boundary cover.*

Lemma 3 *[1, Theorem 2] If a relaxed boundary cover* $\{G_1, G_2, G_3\}$ *has the double intersection form (Fig. 8) with interior segments* S_1*,* S_2 *and* S_3 *then there exist three free segments* M_1*,* M_2 *and* M_3 *such that (i)* $|M_1| + |M_2| + |M_3| < |S_1| + |S_2| + |S_3|$ *and (ii)* $((G_1 \cup G_2 \cup G_3) \backslash (S_1 \cup S_2 \cup S_3)) \cup M_1 \cup M_2 \cup M_3$ *can be partitioned into a simple relaxed boundary cover.*

The proof of the following theorem reduces the general problem of untangling a non-simple boundary cover to the two special cases captured in the preceding lemmas.

Theorem 4 *Every optimal three-guard boundary cover of a convex polygon* P *is simple.*

Sketch of Proof. Untangling a non-simple three-guard boundary cover is similar to the two-guard case, but some additional complications need to be addressed. One critical issue is that the paths that arise in the untangling process may not admit a parameterization that makes them a proper boundary cover. We start by ignoring parameterization and show that any non-simple three-guard boundary cover can be transformed into a shorter simple relaxed boundary cover.

The general idea is to create new paths for the guards by finding the top-most and bottom-most paths as in the two-guard case. If a top-most (or bottom-most) path starting at s reaches a path endpoint before reaching an endpoint of t, we backtrack to the last visited intersection and follow the next top-most (or bottom-most) path. See, for example, the top-most path in Fig. 5. The path of the third guard connects the two remaining path endpoints in the original boundary cover. It is then possible to show that the new set of paths (after further untangling) are either (i) simple, (ii) form a single intersection (if the third guard's path connects one path endpoint on the top to another at the bottom of P, see Fig. 5), or (iii) form a double intersection (if the third guard's path connects two path endpoints on the same side, see Fig. 6).

However, finding the top-most or bottom-most paths may not be possible, for example, if the path of a guard in the original boundary cover that starts at s (or ends at t) does not intersect any other path. We show that all boundary covers of this type that are not simple can be untangled to either a simple relaxed boundary cover or a set of paths that form a single intersection, see Fig. 9 (the highlighted curves show the untangled paths). For further details on this case see [1, Chapter 4].

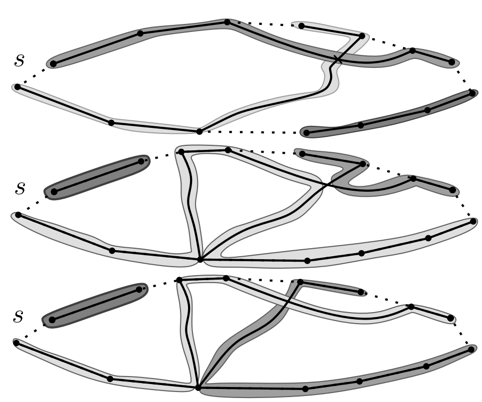

Figure 9: Finding the top- or bottom-most path is not possible.

Even if the top- and bottom-most paths can be found, they may not be shorter than the original boundary cover, since it is possible that they use some part of the original boundary cover more than once, as shown in Fig. 10 (segment a is used by both). If this is the case, then there is a cut (a curve) that separates P into P_s (containing s) and P_t (containing t) that intersects only one path (at a) of the original boundary cover. Such a boundary cover can be untangled to a single intersection form [1], see the highlighted curves in Fig. 10.

Thus any three-guard boundary cover can be untan-

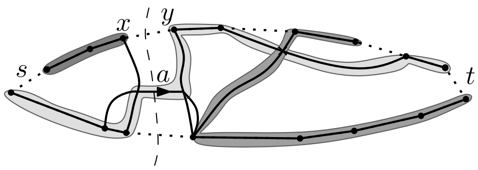

Figure 10: The cut intersects only one path.

gled to a simple relaxed boundary cover or a single or double intersection form. Since Lemmas 2 and 3 provide shorter simple relaxed boundary covers for the single or double intersection form, the outcome is a simple relaxed boundary cover that is shorter than the original.

To complete the proof, it suffices to show that any optimal simple relaxed three-guard boundary cover is, under suitable parameterization, also an optimal simple proper boundary cover.

We begin by proving that a simple relaxed boundary cover has at most four free segments. Suppose there are more than four free segments. It follows that at least one guard must visit an endpoint of two non-consecutive (around P) free segments. Thus its path must cross the interior of P and cut P into two polygons P_L and P_R each of which contains at least one free segment. Since no guard can cross this path and since each free segment must be visited by two different guards, one guard is constrained to P_L and another to P_R. Thus the number of free segments in P_L and P_R is at most two, which contradicts our assumption.

If there are at most three free segments then none of the guards need to leave ∂P, thus choosing the three longest edges of P as free segments will lead to the shortest boundary cover of this form. If there are four free segments then an optimal simple relaxed boundary cover has only one guard crossing the interior of P (see [1]) and Figures 2 and 3 show the possible forms of such a boundary cover. As in Figures 2 and 3, we can easily parameterize the guard paths to provide a proper boundary cover. □

If P has n vertices, finding the shortest boundary cover with three free segments requires finding the three longest edges of P, which can be accomplished in $O(n)$ time. Finding the shortest simple boundary cover with four free segments, requires the algorithm to choose the four free edges so that the sum of the other $n-4$ edges and the interior paths are minimized. There are $\binom{n}{4}$ possible choices of four free edges. For each choice, we calculate the shortest corresponding simple boundary cover of each of the two possible forms. In a boundary cover as in Fig. 3, the point x where G_2 leaves ∂P is either at a vertex or a point inside an edge e. The latter will be chosen if the point that is distance $ad/(a+b)$ from the projection of p on the line ℓ through e is inside e, where a, b, and d are the lengths indicated in Fig. 11. Such a point minimizes the interior part of G_2 since $|\hat{p}x| + |xq|$ is minimized when $\hat{p}q$ is a straight line, where

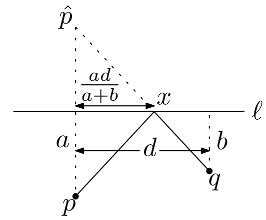

Figure 11: $|px| + |xq|$ is minimized when $\hat{p}q$ is a straight line.

$\hat{p}$ is the reflection of p across ℓ. Finding all possible points where G_2 may leave ∂P takes $O(n)$ time. Thus the algorithm finds the shortest simple boundary cover in $O(n^5)$ time. We expect that it will be possible to improve this algorithm by analyzing the shape of P.

4 Polygon Coverage

The shortest two- and three-guard boundary covers of a convex polygon P are also shortest polygon covers of P. For the two-guard case, this is straight-forward.

For the three-guard case, if the guards remain on the boundary of P then there are three free segments covered at times $t_1 \leq t_2 \leq t_3$. Let G_1 and G_2 cover the free segment at time t_2. The triangle with vertices $G_1(t_2), G_2(t_2), G_3(t_2)$ covers some part of P and partitions the remainder of P into two parts. One of these parts is covered by G_1 and G_3 and the other by G_2 and G_3. If a guard crosses the interior of P (as in Figures 2 and 3) its path partitions P into two or three parts. Each of these parts is covered by a pair of guards.

As a result, the shortest two- and three-guard boundary covers of an n-vertex polygon P that we find in $O(n)$ and $O(n^5)$ time, respectively, are also polygon covers.

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