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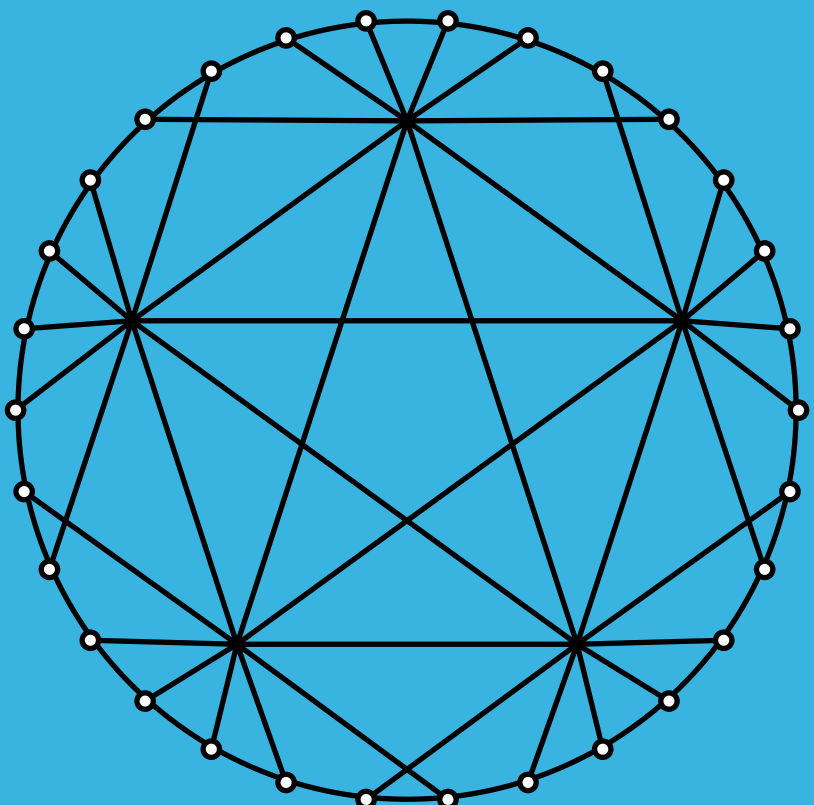

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Distance-k locating-dominating sets in graphs

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Abstract. Let G be a graph with vertex set V, and let k be a positive integer. A set $D \subseteq V$ is a *distance-k dominating set* of G if, for each vertex $u \in V - D$, there exists a vertex $w \in D$ such that $d(u, w) \leq k$, where $d(u, w)$ is the minimum number of edges linking u and w in G. Let $d_k(x, y) = \min\{d(x, y), k+1\}$. A set $R \subseteq V$ is a *distance-k resolving set* of G if, for any pair of distinct $x, y \in V$, there exists a vertex $z \in R$ such that $d_k(x, z) \neq d_k(y, z)$. The *distance-k domination number* $\gamma_k(G)$ (*distance-k dimension* $\dim_k(G)$, respectively) of G is the minimum cardinality of all distance-k dominating sets (distance-k resolving sets, respectively) of G. The *distance-k location-domination number*, $\gamma_L^k(G)$, of G is the minimum cardinality of all sets $S \subseteq V$ such that S is both a distance-k dominating set and a distance-k resolving set of G. Note that $\gamma_L^1(G)$ is the well-known location-domination number introduced by Slater in 1988. For any connected graph G of order $n \geq 2$, we obtain the following sharp bounds: (1) $\gamma_k(G) \leq \dim_k(G) + 1$; (2) $2 \leq \gamma_k(G) + \dim_k(G) \leq n$; (3) $1 \leq \max\{\gamma_k(G), \dim_k(G)\} \leq \gamma_L^k(G) \leq \min\{\dim_k(G)+1, n-1\}$. We characterize G for which $\gamma_L^k(G) \in \{1, |V|-1\}$. We observe that $\frac{\dim_k(G)}{\gamma_k(G)}$ can be arbitrarily large. Moreover, for any tree T of order $n \geq 2$, we show that $\gamma_L^k(T) \leq n - ex(T)$, where $ex(T)$ denotes the number of exterior major vertices of T, and we characterize trees T achieving equality. We also examine the effect of edge deletion on the distance-k location-domination number of graphs.

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1 Introduction

Let G be a finite, simple, undirected, and connected graph with vertex set $V(G)$ and edge set $E(G)$. Let k be a positive integer. For $x, y \in V(G)$, let $d(x,y)$ denote the length of a shortest path between x and y in G, and let $d_k(x,y) = \min\{d(x,y), k+1\}$. The *diameter*, $\text{diam}(G)$, of a graph G is $\max\{d(x,y) : x,y \in V(G)\}$. For $v \in V(G)$ and $S \subseteq V(G)$, let $d(v,S) = \min\{d(v,w) : w \in S\}$. The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. More generally, for $v \in V(G)$, let $N^k[v] = \{u \in V(G) : d(u,v) \leq k\}$. The *degree* of a vertex $v \in V(G)$ is $|N(v)|$. For distinct $x, y \in V(G)$, x and y are called *twin vertices* if $N(x) - \{y\} = N(y) - \{x\}$ in G. A *major vertex* is a vertex of degree at least three, a *leaf* (also called an *end-vertex*) is a vertex of degree one, and a *support vertex* is a vertex that is adjacent to a leaf. A leaf ℓ is called a *terminal vertex* of a major vertex v if $d(\ell, v) < d(\ell, w)$ for every other major vertex w in G. The terminal degree, $ter(v)$, of a major vertex v is the number of terminal vertices of v in G. A major vertex v is an *exterior major vertex* if it has positive terminal degree. We denote the number of exterior major vertices of G by $ex(G)$ and the number of leaves of G by $\sigma(G)$. We denote by $\overline{G}$ the complement of G, i.e., $V(\overline{G}) = V(G)$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$ for any distinct vertices x and y in G. The *join* of two graphs G and H, denoted by $G + H$, is the graph obtained from the disjoint union of G and H by joining an edge between each vertex of G and each vertex of H. Let P_n, C_n, and K_n denote respectively the path, the cycle, and the complete graph on n vertices; let $K_{s,n-s}$ denote the complete bi-partite graph on n vertices with parts of sizes s and $n-s$. Let $\mathbb{Z}^+$ be the set of positive integers and $k \in \mathbb{Z}^+$. For $\alpha \in \mathbb{Z}^+$, let $[\alpha] = \{1, 2, \ldots, \alpha\}$.

A vertex subset $D \subseteq V(G)$ is a *distance-k dominating set* of G if, for each vertex $u \in V(G) - D$, there exists a vertex $w \in D$ such that $d(u,w) \leq k$. The *distance-k domination number*, $\gamma_k(G)$, of G is the minimum cardinality over all distance-k dominating sets of G. The concept of distance-k domination was introduced by Meir and Moon [20]. We note that $\gamma_1(G)$ is the well-known domination number of G, which is often denoted by $\gamma(G)$ in the literature. Applications of domination can be found in resource allocation on a network, determining efficient routes within a network, and designing secure systems for electrical grids, to name a few. It is known that determining the domination number of a general graph is an NP-hard problem (see [9]). For a survey on domination in graphs, see [14].

A vertex subset $R \subseteq V(G)$ is a *resolving set* of G if, for any pair of distinct vertices $x, y \in V(G)$, there exists a vertex $z \in R$ such that $d(x,z) \neq d(y,z)$. The *metric dimension*, $\dim(G)$, of G is the minimum cardinality over all resolving sets of G. The concept of metric dimension was introduced independently by Slater [23] and by Harary and Melter [13]. A vertex subset $S \subseteq V(G)$ is a *distance-k resolving set* (also called a k-truncated resolving set) of G if, for any distinct vertices $x, y \in V(G)$, there exists a vertex $z \in S$ such that $d_k(x,z) \neq d_k(y,z)$. The *distance-k dimension* (also called the k-truncated dimension), $\dim_k(G)$, of G is the minimum cardinality over all distance-k resolving sets of G. The metric dimension of a metric space (V, d_k) is studied in [2]. The distance-k dimension corresponds to the $(1, k+1)$-metric dimension in [5] and [6]. We note that $\dim_1(G)$ is also called the adjacency dimension, introduced in [16], and it is often denoted by $\text{adim}(G)$ in the literature. For detailed results on $\dim_k(G)$, we refer to [8], which is a merger of [12] and [24], along with some additional results. For an ordered set $S = \{u_1, u_2, \ldots, u_\alpha\} \subseteq V(G)$ of distinct vertices, the distance-k metric code of $v \in V(G)$ with respect to S, denoted by $\text{code}_{S,k}(v)$, is the α-vector $(d_k(v,u_1), d_k(v,u_2), \ldots, d_k(v,u_\alpha))$. We denote by $(\mathbf{k+1})_\alpha$ the α-vector with $k+1$ on each entry. Applications of metric dimension can be found in robot navigation, network discovery and verification, and combinatorial optimization, to name a few. It is known that determining the metric dimension and the adjacency dimension of a general graph are NP-hard problems (see [19] and [7]). For a discussion on computational complexity of the distance-k dimension of graphs, see [6].

Slater [22] introduced the notion of locating-dominating set and location-domination number. A set $A \subseteq V(G)$ is a *locating-dominating set* of G if A is a dominating set of G and $N(x) \cap A \neq N(y) \cap A$ for distinct vertices $x, y \in V(G) - A$. The *location-domination number*, $\gamma_L(G)$, of G is the minimum cardinality over all locating-dominating sets of G. The notion of location-domination by Slater is a natural marriage of its two constituent notions, where a subset of vertices functions both to locate (via d_1 metric) each node of a network and to dominate (supply or support) the entire network. Viewed in this light, the following is but a natural extension of the notion of Slater. For $(s,t) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, let $S \subseteq V(G)$ be a distance-s resolving set of G and a distance-t dominating set of G, which we call an (s,t)-locating-dominating set of G. Then the (s,t)-location-domination number of G, denoted by $\gamma_L^{(s,t)}(G)$, is defined to be the minimum cardinality of S as S varies over all (s,t)-locating-dominating sets of G. When $s = k = t$, we will abbreviate and simply speak of distance-k locating-dominating set and distance-k location-domination number, and we will simplify $\gamma_L^{(k,k)}(G)$ to $\gamma_L^k(G)$.

In this paper, we study the distance-k location-domination number of graphs. We examine the relationship among $\gamma_k(G)$, $\dim_k(G)$ and $\gamma_L^k(G)$. Let G be a connected graph of order $n \geq 2$, and let $k \in \mathbb{Z}^+$. In Section 2, we show that $\gamma_k(G) \leq \dim_k(G) + 1$ and that $\dim_k(G) - \gamma_k(G)$ can be arbitrarily large. We also show that $2 \leq \gamma_k(G) + \dim_k(G) \leq n$, and we characterize G satisfying $\gamma_k(G) + \dim_k(G) = 2$. In Section 3, we show that $1 \leq \max\{\gamma_k(G), \dim_k(G)\} \leq \gamma_L^k(G) \leq \min\{\dim_k(G) + 1, n - 1\}$, where the bounds are sharp. We also characterize G satisfying $\gamma_L^k(G)$ equals 1 and $n - 1$, respectively. Moreover, for a non-trivial tree T, we show that $\gamma_L^k(T) \leq n - ex(T)$ and we characterize trees T achieving equality. In Section 4, we determine $\gamma_L^k(G)$ when G is the Petersen graph, a complete multipartite graph, a cycle or a path. In Section 5, we examine the effect of edge deletion on the distance-k location-domination number of graphs.

2 Relations between $\gamma_k(G)$ and $\dim_k(G)$

In this section, we examine the sum and difference between $\gamma_k(G)$ and $\dim_k(G)$. Let G be a non-trivial connected graph, and let $k \in \mathbb{Z}^+$. We show that $\gamma_k(G) \leq \dim_k(G) + 1$, where the bound is sharp, and we observe that $\dim_k(G) - \gamma_k(G)$ can be arbitrarily large. We also show that $2 \leq \gamma_k(G) + \dim_k(G) \leq |V(G)|$, and we characterize G satisfying $\gamma_k(G) + \dim_k(G) = 2$. We begin with the following observation.

Observation 2.1. *Let G be any connected graph, and let $s, s', t, t', k, k' \in \mathbb{Z}^+$. Then*

(a) for $k > k'$, $\gamma_k(G) \leq \gamma_{k'}(G) \leq \gamma_1(G)$;

(b) [2, 5, 6] *for $k > k'$, $\dim(G) \leq \dim_k(G) \leq \dim_{k'}(G) \leq \dim_1(G)$;*

(c) more generally, we have $\gamma_L^{(s,t)}(G) \geq \gamma_L^{(s',t')}(G)$ for $s \leq s'$ and $t \leq t'$, since an (s,t)-locating-dominating set of G is an (s', t')-locating-dominating set of G.

For any minimum distance-k resolving set S of a connected graph G, we show that there is a vertex $v \in V(G) - S$ such that $S \cup \{v\}$ is a distance-k dominating set of G.

Proposition 2.2. *For any non-trivial connected graph G and for any $k \in \mathbb{Z}^+$,*

$$\gamma_k(G) \leq \dim_k(G) + 1.$$

Proof. Let S be any minimum distance-k resolving set of G. Then there exists at most one vertex, say w, in $V(G)-S$ such that $d(w,S)>k$; notice that $\text{code}_{S,k}(w)=(\mathbf{k+1})_{|S|}$. If $d(u,S)\le k$ for each $u\in V(G)$, then S is a distance-k dominating set of G, and hence $\gamma_k(G)\le |S|=\dim_k(G)$. If there exists a vertex $v\in V(G)$ such that $d(v,S)>k$, then $S\cup\{v\}$ forms a distance-k dominating set of G, and thus $\gamma_k(G)\le |S|+1=\dim_k(G)+1$. □

Next, we show the sharpness of the bound in Proposition 2.2.

Observation 2.3. *Let G be a non-trivial connected graph.*

- *(a) If there exists a vertex $v\in V(G)$ such that $N^k[v]=V(G)$, then $\{v\}$ is a distance-k dominating set of G and $\gamma_k(G)=1$.*
- *(b) Suppose $\cup_{i=1}^{x}\{v_i\}\subseteq V(G)$ satisfies $N^k[v_i]\cap N^k[v_j]=\emptyset$ for $i\neq j$. Then any distance-k dominating set of G must contain a vertex of $N^k[v_i]$ for each $i\in[x]$. Thus $\gamma_k(G)\ge x$.*

Remark 2.4. *For each $k\in\mathbb{Z}^+$, there is a connected graph G with $\gamma_k(G)=\dim_k(G)+1$.*

Proof. Let G be a tree with $ex(G)=x\ge 1$ such that $v_1,v_2,\ldots,v_x$ are the exterior major vertices of G with $ter(v_i)=\alpha\ge 3$ for each $i\in[x]$, and let $v_1,v_2,\ldots,v_x$ form an induced path of order x in G. For each $i\in[x]$, let $\{\ell_{i,1},\ell_{i,2},\ldots,\ell_{i,\alpha}\}$ be the set of the terminal vertices of v_i in G such that $d(v_i,\ell_{i,j})=k+1=1+d(v_i,\ell_{i,\alpha})$ for each $j\in[\alpha-1]$. For each $i\in[x]$ and for each $j\in[\alpha-1]$, let $s_{i,j}$ be the neighbor of v_i lying on the $v_i-\ell_{i,j}$ path in G. See Fig. 1 when $k=3$.

First, note that $\gamma_k(G)=x\alpha$: (i) $\gamma_k(G)\le x\alpha$ since $D=\cup_{i=1}^{x}\{\ell_{i,1},\ldots,\ell_{i,\alpha}\}$ forms a distance-k dominating set of G with $|D|=x\alpha$; (ii) $\gamma_k(G)\ge x\alpha$ by Observation 2.3(b) and the fact that $N^k[\ell_{i,j}]\cap N^k[\ell_{s,t}]=\emptyset$ for $(i,j)\neq(s,t)$. Second, note that $\dim_k(G)=x\alpha-1$: (i) $\dim_k(G)\le x\alpha-1$ since $R=(\cup_{i=1}^{x}\{s_{i,1},s_{i,2},\ldots,s_{i,\alpha-1}\})\cup(\cup_{i=1}^{x-1}\{\ell_{i,\alpha}\})$ forms a distance-k resolving set of G with $|R|=x\alpha-1$; (ii) $\dim_k(G)\ge\gamma_k(G)-1=x\alpha-1$ by Proposition 2.2. Therefore, $\gamma_k(G)=x\alpha=\dim_k(G)+1$. □

Based on Proposition 2.2 and Remark 2.4, we have the following

Question 2.5. *Can we characterize graphs G satisfying*

$$\gamma_k(G)=\dim_k(G)+1\ ?$$

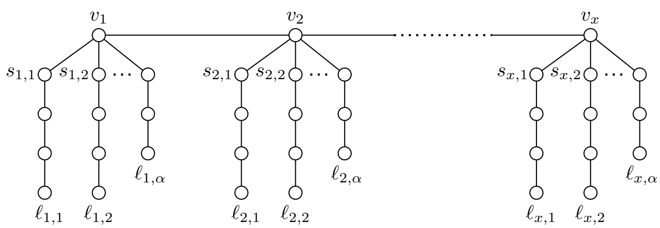

Figure 1: Graphs G with $\gamma_3(G) = \dim_3(G) + 1$.

Question 2.6. *Can we characterize graphs G satisfying*

$$\gamma_k(G) = \dim_k(G) \ ?$$

Next, we show that $\frac{\dim_k(G)}{\gamma_k(G)}$ can be arbitrarily large; thus, $\dim_k(G) - \gamma_k(G)$ can be arbitrarily large. We recall the connected graphs G of order n for which $\dim_k(G) \in \{1, n-2, n-1\}$; here, we note that Theorem 2.7(a),(d) for the case $k = 1$ is obtained in [16]. See Theorem 3.9 in [11] for a characterization of all graphs G having $\dim_1(G) = m$ for each $m \in \mathbb{Z}^+$.

Theorem 2.7. *Let G be a connected graph of order $n \geq 2$, and let $k \in \mathbb{Z}^+$. Then $1 \leq \dim_k(G) \leq n-1$, and we have the following:*

- *(a)* [5, 24] $\dim_k(G) = 1$ *if and only if* $G \in \cup_{i=2}^{k+2}\{P_i\}$*;*
- *(b)* [8, 12, 24] *for* $n \geq 4$*,* $\dim_1(G) = n-2$ *if and only if* $G = P_4$*,* $G = K_{s,t}$ *($s, t \geq 1$),* $G = K_s + \overline{K}_t$ *($s \geq 1, t \geq 2$), or* $G = K_s + (K_1 \cup K_t)$ *($s, t \geq 1$);*
- *(c)* [8, 12, 24] *for* $k \geq 2$ *and for* $n \geq 4$*,* $\dim_k(G) = n-2$ *if and only if* $G = K_{s,t}$ *($s, t \geq 1$),* $G = K_s + \overline{K}_t$ *($s \geq 1, t \geq 2$), or* $G = K_s + (K_1 \cup K_t)$ *($s, t \geq 1$);*
- *(d)* [8, 12, 24] $\dim_k(G) = n-1$ *if and only if* $G = K_n$*.*

Proposition 2.8. *For a connected graph G and for $k \in \mathbb{Z}^+$, $\frac{\dim_k(G)}{\gamma_k(G)}$ can be arbitrarily large.*

Proof. Let G be a connected graph of order $n \geq 4$. First, note that $\dim_k(K_n) = n-1$ by Theorem 2.7(d) and $\gamma_k(K_n) = 1$ by Observation 2.3(a); thus $\frac{\dim_k(K_n)}{\gamma_k(K_n)} = n - 1 \to \infty$ as $n \to \infty$.

For another example, let G be the graph obtained from $K_{1,\alpha}$, where $\alpha \geq 3$, by subdividing each edge of $K_{1,\alpha}$ exactly $k-1$ times; let v be the central vertex of degree α in G and let $\ell_1, \ell_2, \dots, \ell_\alpha$ be the leaves of G such that $d(v, \ell_i) = k$ for each $i \in [\alpha]$. Let $N(v) = \{s_1, s_2, \dots, s_\alpha\}$ such that s_i lies on the $v - \ell_i$ path in G, and let P^i denote the $s_i - \ell_i$ path, where $i \in [\alpha]$. Then $\gamma_k(G) = 1$ since $\{v\}$ is a minimum distance-k dominating set of G by Observation 2.3(a). Note that $\dim_k(G) = \alpha - 1$: (i) $\dim_k(G) \le \alpha - 1$ since $N(v) - \{s_1\}$ forms a distance-k resolving set of G; (ii) $\dim_k(G) \ge \alpha - 1$ since $S \cap (V(P^i) \cup V(P^j)) \neq \emptyset$ for any distance-k resolving set S of G and for distinct $i, j \in [\alpha]$, as $S \cap (V(P^i) \cup V(P^j)) = \emptyset$ implies $\text{code}_{S,k}(s_i) = \text{code}_{S,k}(s_j)$. So, $\frac{\dim_k(G)}{\gamma_k(G)} = \alpha - 1 \to \infty$ as $\alpha \to \infty$. $\square$

Next, for any connected graph G of order $n \ge 2$ and for any $k \in \mathbb{Z}^+$, we show that $2 \le \gamma_k(G) + \dim_k(G) \le n$ and we characterize G with $\gamma_k(G) + \dim_k(G) = 2$. We recall the following results.

Lemma 2.9. [1] *Let G be a connected graph. Then there exists a minimum dominating set for G which does not have any pair of twin vertices.*

Theorem 2.10. [1] *Let G be a connected graph of order $n \ge 2$. Then $\gamma(G) + \dim(G) \le n$, and equality holds if and only if $G \in \{K_n, K_{s,n-s}\}$ for $2 \le s \le n-2$.*

Proposition 2.11. *Let G be any connected graph of order $n \ge 2$, and let $k \in \mathbb{Z}^+$. Then $2 \le \gamma_k(G) + \dim_k(G) \le n$, and $\gamma_k(G) + \dim_k(G) = 2$ if and only $G \in \cup_{i=2}^{k+2}\{P_i\}$.*

Proof. Let G be a connected graph of order $n \ge 2$, and let $k \in \mathbb{Z}^+$. Since $\gamma_k(G) \ge 1$ and $\dim_k(G) \ge 1$, we have $\gamma_k(G) + \dim_k(G) \ge 2$. Note that $\gamma_k(G) + \dim_k(G) = 2$ if and only if $\gamma_k(G) = 1 = \dim_k(G)$ if and only if $G \in \cup_{i=2}^{k+2}\{P_i\}$ by Observation 2.3(a) and Theorem 2.7(a).

To prove $\gamma_k(G) + \dim_k(G) \le n$, it suffices to show that $\gamma_1(G) + \dim_1(G) \le n$ by Observation 2.1. The proof given for Theorem 2.10 in [1] actually shows $\gamma_1(G) + \dim_1(G) \le n$. To see this, we can take a minimum dominating set D of G that contains no twin vertices by Lemma 2.9. Suppose $x, y \in D$ have the same neighbors in $V(G) - D$; this implies that neither x nor y has a neighbor in D, because if, say, y has a neighbor in D, then $D - \{y\}$ remains a dominating set, and thus x and y have the same neighbors in $V(G)$, contradicting the choice of D. Since no two vertices of D have the same neighborhood in $S = V(G) - D$, S is a distance-1 resolving set of G, and we have $\gamma_1(G) + \dim_1(G) \le |D| + |S| = n$. $\square$

In contrast to Theorem 2.10, we note that if $G \in \{P_4, K_n, K_{s,n-s}\}$ with $2 \le s \le n-2$, then $\gamma_1(G) + \dim_1(G) = |V(G)|$. So, we have the following

Question 2.12. *Can we characterize graphs G satisfying*

$$\gamma_k(G) + \dim_k(G) = |V(G)|\ ?$$

3 Bounds on $\gamma_L^k(G)$

In this section, for any connected graph G of order $n \ge 2$ and for any $k \in \mathbb{Z}^+$, we show that $1 \le \max\{\gamma_k(G), \dim_k(G)\} \le \gamma_L^k(G) \le \min\{\dim_k(G) + 1, n-1\}$; we characterize G satisfying $\gamma_L^k(G) = 1$ and $\gamma_L^k(G) = n-1$, respectively. For any non-trivial tree T, we show that $\gamma_L^k(T) \le |V(T)| - ex(T)$ and we characterize trees T achieving equality.

Theorem 3.1. *For any connected graph G of order $n \ge 2$ and for any $k \in \mathbb{Z}^+$,*

$$\max\{\gamma_k(G), \dim_k(G)\} \le \gamma_L^k(G) \le \min\{1 + \dim_k(G), n-1\}.$$

Proof. Let G be a connected graph of order $n \ge 2$, and let $k \in \mathbb{Z}^+$. Since a minimum distance-k locating-dominating set of G is both a distance-k dominating set of G and a distance-k resolving set of G, we have $\gamma_L^k(G) \ge \max\{\gamma_k(G), \dim_k(G)\}$.

Next, we show that $\gamma_L^k(G) \le \min\{1 + \dim_k(G), n-1\}$. Suppose S is a minimum distance-k resolving set of G; then at most one vertex in G has the distance-k metric code $(\mathbf{k{+}1})_{|S|}$ with respect to S. If $\text{code}_{S,k}(u) \ne (\mathbf{k{+}1})_{|S|}$ for each $u \in V(G)$, then S is a distance-k locating-dominating set of G. If $\text{code}_{S,k}(w) = (\mathbf{k{+}1})_{|S|}$ for some $w \in V(G)$, then $S \cup \{w\}$ forms a distance-k locating-dominating set of G. So, $\gamma_L^k(G) \le |S| + 1 = \dim_k(G) + 1$. Now, $\gamma_L^k(G) \le n-1$ follows from the fact that any vertex subset $S' \subseteq V(G)$ with $|S'| = n-1$ is a distance-k locating-dominating set of G. $\square$

Theorems 2.7(d) and 3.1 imply that $\max\{\gamma_k(K_n), \dim_k(K_n)\} = \gamma_L^k(K_n) = \min\{1 + \dim_k(K_n), n-1\}$ for $n \ge 2$ and for $k \ge 1$. Since $\gamma_k(G) \ge 1$ and $\dim_k(G) \ge 1$, Theorem 3.1 implies the following.

Corollary 3.2. *For any connected graph G of order $n \ge 2$ and for any $k \in \mathbb{Z}^+$, $1 \le \gamma_L^k(G) \le n-1$.*

Next, we characterize connected graphs G of order n satisfying $\gamma_L^k(G) = 1$ and $\gamma_L^k(G) = n-1$, respectively, for all $k \in \mathbb{Z}^+$. We recall the following observation.

Observation 3.3. [8] *Let x and y be distinct twin vertices of G, and let $k \in \mathbb{Z}^+$. Then, for any distance-k resolving set S_k of G, $S_k \cap \{x,y\} \neq \emptyset$.*

Theorem 3.4. *Let G be a connected graph of order $n \geq 2$, and let $k \in \mathbb{Z}^+$. Then*

(a) $\gamma_L^k(G) = 1$ if and only if $G \in \cup_{i=2}^{k+1}\{P_i\}$;

(b) $\gamma_L^1(G) = n-1$ if and only if $G \in \{K_n, K_{1,n-1}\}$;

(c) for $k \geq 2$, $\gamma_L^k(G) = n-1$ if and only if $G = K_n$.

Proof. Let G be a connected graph of order $n \geq 2$, and let $k \in \mathbb{Z}^+$.

(a) If $G \in \cup_{i=2}^{k+1}\{P_i\}$, then a leaf of G forms a distance-k locating-dominating set of G; thus, $\gamma_L^k(G) = 1$. Now, suppose $\gamma_L^k(G) = 1$; then $\gamma_k(G) = 1 = \dim_k(G)$. By Theorem 2.7(a), $\dim_k(G) = 1$ implies $G \in \cup_{i=2}^{k+2}\{P_i\}$, where any minimum distance-k resolving set consists of a leaf whereas a leaf of P_{k+2} fails to form a distance-k dominating set of P_{k+2} since $\text{diam}(P_{k+2}) = k+1$. So, $\gamma_L^k(G) = 1$ implies $G \in \cup_{i=2}^{k+1}\{P_i\}$.

(b) First, suppose $G \in \{K_n, K_{1,n-1}\}$. Note that $\gamma_L^1(K_n) = n-1$ by Theorems 2.7(d) and 3.1. For $n \geq 3$, if v is the central vertex of $K_{1,n-1}$ and $N(v) = \{s_1, s_2, \ldots, s_{n-1}\}$, then $\dim_1(K_{1,n-1}) = n-2$ by Theorem 2.7(b) and $|S \cap N(v)| = n-2$ for any minimum distance-1 resolving set S of $K_{1,n-1}$ by Observation 3.3; without loss of generality, let $S' = \{s_1, s_2, \ldots, s_{n-2}\}$ be a minimum distance-1 resolving set of $K_{1,n-1}$. Since $d(s_{n-1}, S') = 2$, S' fails to be a distance-1 locating-dominating set of $K_{1,n-1}$; thus, $\gamma_L^1(K_{1,n-1}) \geq n-1$. By Theorem 3.1, $\gamma_L^1(K_{1,n-1}) = n-1$.

Second, suppose $\gamma_L^1(G) = n-1$. By Theorem 3.1, $\dim_1(G) \in \{n-2, n-1\}$. To see this, if $\dim_1(G) \leq n-3$, then $\gamma_L^1(G) \leq \dim_1(G)+1 \leq n-2$ by Theorem 3.1. If $\dim_1(G) = n-1$, then $G = K_n$ by Theorem 2.7(d). If $\dim_1(G) = n-2$, then $G = P_4$, $G = K_{s,t}$ with $s,t \geq 1$, $G = K_s + \overline{K}_t$ with $s \geq 1, t \geq 2$, or $G = K_s + (K_1 \cup K_t)$ with $s,t \geq 1$ by Theorem 2.7(b). We note the following: (i) $\gamma_L^1(P_4) = 2$ since the two leaves of P_4 form a minimum distance-1 locating-dominating set of P_4; (ii) $\gamma_L^1(K_{1,t}) = \gamma_L^1(K_1 + \overline{K}_t) = t$ as shown above; (iii) for $s,t \geq 2$, $\gamma_L^1(K_{s,t}) = s+t-2 = \gamma_L^1(K_s + \overline{K}_t)$ since all but one vertex from each of the two partite sets form a minimum distance-1 locating-dominating set of $K_{s,t}$; (iv) $K_1 + (K_1 \cup K_1) = K_{1,2}$ and

$\gamma_L^1(K_{1,2}) = 2$ as shown above; (v) for $t \geq 2$, $\gamma_L^1(K_1 + (K_1 \cup K_t)) = t$ since all but one vertex of the K_t and the leaf of $K_1 + (K_1 \cup K_t)$ form a minimum distance-1 locating-dominating set of $K_1 + (K_1 \cup K_t)$; (vi) for $s \geq 2$ and $t \geq 1$, $\gamma_L^1(K_s + (K_1 \cup K_t)) = s + t - 1$ since all but one vertex of the K_s and all vertices of the K_t form a minimum distance-1 locating-dominating set of $K_s + (K_1 \cup K_t)$. So, $\gamma_L^1(G) = n - 1$ implies $G = K_n$ or $G = K_{1,n-1}$.

(c) Let $k \geq 2$. Note that $\gamma_L^k(K_n) = n - 1$ by Theorems 2.7(d) and 3.1. So, suppose $\gamma_L^k(G) = n - 1$. Then $\dim_k(G) \in \{n-1, n-2\}$ by Theorem 3.1. If $\dim_k(G) = n - 1$, then $G = K_n$ by Theorem 2.7(d). If $\dim_k(G) = n - 2$ for $n \geq 4$, then, by Theorem 2.7(c), $G = K_{s,t}$ with $s, t \geq 1$, $G = K_s + \overline{K}_t$ with $s \geq 1, t \geq 2$, or $G = K_s + (K_1 \cup K_t)$ with $s, t \geq 1$; then $\operatorname{diam}(G) = 2$ and any minimum distance-k resolving set of G is also a distance-k dominating set of G. So, $\dim_k(G) = n - 2$ implies $\gamma_L^k(G) = n - 2$ for $k \geq 2$. $\square$

Question 3.5. *Can we characterize graphs G of order n such that*

$$\gamma_L^k(G) = \beta,$$

where $\beta \in \{2, 3, \ldots, n-2\}$?

Next, we examine the relation between $\gamma_L^k(G)$ and other parameters in Theorem 3.1.

Proposition 3.6. *Let G be a non-trivial connected graph, and let $k \in \mathbb{Z}^+$. Then*

(a) $\gamma_L^k(G) - \dim_k(G) \in \{0, 1\}$;

(b) $\gamma_L^k(G) - \gamma_k(G)$ *can be arbitrarily large;*

(c) $(|V(G)| - 1) - \gamma_L^k(G)$ *can be arbitrarily large.*

Proof. Let $k \in \mathbb{Z}^+$. For (a), $0 \leq \gamma_L^k(G) - \dim_k(G) \leq 1$ by Theorem 3.1.

For (b) and (c), let G be a tree obtained from the path $v_1, v_2, \ldots, v_x$ ($x \geq 2$) by adding leaves $\ell_{i,1}, \ell_{i,2}, \ldots, \ell_{i,\alpha}$ ($\alpha \geq 3$) to each vertex v_i, where $i \in [x]$; notice that $|V(G)| = x(\alpha + 1)$. Since $\cup_{i=1}^{x}\{v_i\}$ is a distance-k dominating set of G, $\gamma_k(G) \leq x$. Note that $\gamma_L^k(G) \geq x(\alpha - 1)$ by Observation 3.3 since any distinct vertices in $\{\ell_{i,1}, \ell_{i,2}, \ldots, \ell_{i,\alpha}\}$ are twin vertices in G. Also, note that $\gamma_L^k(G) \leq x\alpha$ since $V(G) - \cup_{i=1}^{x}\{\ell_{i,\alpha}\}$ is a distance-k locating-dominating set of G. So, $\gamma_L^k(G) - \gamma_k(G) \geq x(\alpha - 1) - x = x(\alpha - 2) \to \infty$ as $x \to \infty$ or $\alpha \to \infty$, and $|V(G)| - 1 - \gamma_L^k(G) \geq x(\alpha + 1) - 1 - x\alpha = x - 1 \to \infty$ as $x \to \infty$. $\square$

In view of Theorem 3.1 and Proposition 3.6(b), we have the following

Question 3.7. *Can we characterize graphs G such that $\gamma_L^k(G)=\gamma_k(G)$?*

Next, for a graph G with $\gamma_L^k(G)=\beta$, we determine the upper bound of $|V(G)|$.

Theorem 3.8. [8, 12] *If $\dim_k(G)=\beta$, then $|V(G)|\le(\lfloor\frac{2(k+1)}{3}\rfloor+1)^\beta+\beta\sum_{i=1}^{\lceil\frac{k+1}{3}\rceil}(2i-1)^{\beta-1}$ and the bound is sharp.*

By Theorem 3.1, $\gamma_L^k(G)=\beta$ implies $\dim_k(G)\le\beta$. Theorem 3.8 is sharp, and a graph G attaining the maximum order must contain a vertex $\omega\in V(G)$ with $\text{code}_{S,k}(\omega)=(\mathbf{k+1})_{|S|}$ for any minimum distance-k resolving set S of G. The deletion of ω from G leaves intact distance relations and code vectors; thus, we have the following sharp bound.

Corollary 3.9. *If $\gamma_L^k(G)=\beta$, then $|V(G)|\le(\lfloor\frac{2(k+1)}{3}\rfloor+1)^\beta-1+\beta\sum_{i=1}^{\lceil\frac{k+1}{3}\rceil}(2i-1)^{\beta-1}$.*

Remark 3.10. *The proof for Theorem 3.8 in [8, 12] uses a method similar to the one in [15]. For a construction of graphs G with $\dim_1(G)=\beta$ of maximum order $\beta+2^\beta$, we refer to [11]. For a construction of graphs G with $\dim_2(G)=\beta$ and of order $\beta+3^\beta$, we refer to [8, 12]; this construction is similar to the one provided in [10].*

Next, for any non-trivial tree T and for $k\in\mathbb{Z}^+$, we show that $\gamma_L^k(T)\le n-ex(T)$ and we characterize trees T achieving equality.

Proposition 3.11. *For any tree T of order $n\ge2$ and for any $k\in\mathbb{Z}^+$, $\gamma_L^k(T)\le n-ex(T)$.*

Proof. Let T be a tree of order $n\ge2$ and let $k\in\mathbb{Z}^+$. If $ex(T)\in\{0,1\}$, then $\gamma_L^k(T)\le n-1\le n-ex(T)$ by Theorem 3.1. So, suppose $ex(T)=x\ge2$; let $v_1,v_2,\ldots,v_x$ be the exterior major vertices of T. For each $i\in[x]$, let $\{\ell_{i,1},\ell_{i,2},\ldots,\ell_{i,\sigma_i}\}$ be the set of terminal vertices of v_i in T with $ter(v_i)=\sigma_i\ge1$. Since $S=V(T)-\cup_{i=1}^{x}\{\ell_{i,1}\}$ is a distance-k locating-dominating set of T with $|S|=n-x=n-ex(T)$, $\gamma_L^k(T)\le n-ex(T)$. $\square$

Next, we characterize non-trivial trees T satisfying $\gamma_L^k(T) = |V(T)| - ex(T)$. We recall some terminology. An *exterior degree-two vertex* is a vertex of degree two that lies on a path from a terminal vertex to its major vertex, and an *interior degree-two vertex* is a vertex of degree two such that the shortest path to any terminal vertex includes a major vertex.

Theorem 3.12. *Let T be any tree of order $n \geq 2$ and let $k \in \mathbb{Z}^+$. Then $\gamma_L^k(T) = n - ex(T)$ if and only if $k = 1$, $ex(T) \geq 1$, and $ex(T) + \sigma(T) = n$.*

Proof. Let T be a tree of order $n \geq 2$ and let $k \in \mathbb{Z}^+$. If $ex(T) = x \geq 1$, let $v_1, v_2, \ldots, v_x$ be the exterior major vertices of T, and let $\{\ell_{i,1}, \ell_{i,2}, \ldots, \ell_{i,\sigma_i}\}$ be the set of terminal vertices of v_i with $ter(v_i) = \sigma_i \geq 1$ in T for each $i \in [x]$.

($\Leftarrow$) Let $k = 1$, $ex(T) = x \geq 1$, and $ex(T) + \sigma(T) = n$; notice that T is a caterpillar. Let S be an arbitrary minimum distance-1 locating-dominating set of T. By Observation 3.3, $|S \cap \{\ell_{i,1}, \ell_{i,2}, \ldots, \ell_{i,\sigma_i}\}| \geq \sigma_i - 1$. Thus, up to a relabeling of vertices of T, we may assume that $S \supseteq V(T) - \cup_{i=1}^{x}\{v_i, \ell_{i,1}\}$. Since $N[\ell_{i,1}] \cap N[\ell_{j,1}] = \emptyset$ for $i \neq j$, a vertex in $\{v_i, \ell_{i,1}\}$ (for each $i \in [x]$) must also belong to S by Observation 2.3(b). So, $\gamma_L^1(T) \geq n - ex(T)$. Since $\gamma_L^1(T) \leq n - ex(T)$ by Proposition 3.11, $\gamma_L^1(T) = n - ex(T)$.

($\Rightarrow$) Let $\gamma_L^k(T) = n - ex(T)$. If $ex(T) = 0$, then $\gamma_L^k(T) < n - ex(T)$ by Theorem 3.1. So, let $ex(T) = x \geq 1$. We will show that T has no major vertex of terminal degree zero and no degree-two vertex; i.e., each vertex in T is either an exterior major vertex or a leaf.

If T contains either an interior degree-two vertex w or a major vertex w' with $ter(w') = 0$, then $A = V(T) - (\{u\} \cup (\cup_{i=1}^{x}\{\ell_{i,1}\}))$, where $u \in \{w, w'\}$, forms a distance-k locating-dominating set of T; thus $\gamma_L^k(T) \leq n - (x+1) < n - ex(T)$. Now, suppose T contains an exterior degree-two vertex, say z. By relabeling the vertices of T if necessary, we may assume that z lies on the $v_i - \ell_{i,1}$ path in T for some $i \in [x]$. If $ter(v_i) \geq 2$, then $B = V(T) - (\{z\} \cup (\cup_{j=1}^{x}\{\ell_{j,\sigma_j}\}))$ forms a distance-k locating-dominating set of T. If $ter(v_i) = 1$, then $C = V(T) - (\{v_i\} \cup (\cup_{j=1}^{x}\{\ell_{j,1}\}))$ forms a distance-k locating-dominating set of T. (It is easy to see that the sets A, B, and C are distance-1 locating-dominating; then apply Observation 2.1(c) for $k \geq 1$.) In each case, $\gamma_L^k(T) \leq n - (x+1) < n - ex(T)$.

So, each vertex in T is either an exterior major vertex or a leaf; thus $ex(T) + \sigma(T) = n$. Now, if $k \geq 2$, then $R = V(T) - (\{v_1\} \cup (\cup_{i=1}^{x}\{\ell_{i,1}\}))$ forms a distance-k locating-dominating set of T, and hence $\gamma_L^k(T) \leq |R| = n - ex(T) - 1 < n - ex(T)$. Thus, $k = 1$. □

4 $\gamma_L^k(G)$ of some classes of graphs

In this section, for any $k \in \mathbb{Z}^+$, we determine $\gamma_L^k(G)$ when G is the Petersen graph, a complete multipartite graph, a cycle or a path. We begin with the following observations.

Observation 4.1. [5, 6, 8, 12] *Let G be a connected graph with $\text{diam}(G) = d \geq 2$, and let $k \in \mathbb{Z}^+$. If $k \geq d-1$, then $\dim_k(G) = \dim(G)$.*

Observation 4.2. *Let G be any connected graph, and let $k, k' \in \mathbb{Z}^+$. Then*

(a) for $k > k'$, $\gamma_L^k(G) \leq \gamma_L^{k'}(G) \leq \gamma_L^1(G)$;

(b) if $k \geq \text{diam}(G)$, then $\gamma_L^k(G) = \dim_k(G)$.

Next, we determine $\gamma_L^k(\mathcal{P})$ for the Petersen graph $\mathcal{P}$.

Example 4.3. *Let $\mathcal{P}$ be the Petersen graph with the the following presentation: two disjoint copies of C_5 are given by $u_1, u_2, u_3, u_4, u_5, u_1$ and $w_1, w_3, w_5, w_2, w_4, w_1$, respectively, and the remaining edges are $u_i w_i$ for each $i \in [5]$. Then, for $k \in \mathbb{Z}^+$,*

$$\gamma_L^k(\mathcal{P}) = \begin{cases} \dim_k(\mathcal{P}) + 1 = 4 & \text{if } k = 1, \\ \dim_k(\mathcal{P}) = 3 & \text{if } k \geq 2. \end{cases}$$

To see this, note that $\dim(\mathcal{P}) = 3$ (see [17]) and $\text{diam}(\mathcal{P}) = 2$. For any $k \geq 2$, $\gamma_L^k(\mathcal{P}) = \dim_k(\mathcal{P}) = \dim(\mathcal{P}) = 3$ by Observations 4.1 and 4.2(b). Next, we show that $\gamma_L^1(\mathcal{P}) = 4$. For any minimum distance-1 resolving set S of $\mathcal{P}$, we may assume $u_1 \in S$ since $\mathcal{P}$ is vertex-transitive. It was shown in [18] that there are six such S containing u_1 (i.e., $\{u_1, w_2, w_3\}$, $\{u_1, u_4, w_2\}$, $\{u_1, w_4, w_5\}$, $\{u_1, u_3, w_5\}$, $\{u_1, u_4, w_3\}$ and $\{u_1, u_3, w_4\}$). Since none of those six sets S containing u_1 form a distance-1 dominating set of $\mathcal{P}$, $\gamma_L^1(\mathcal{P}) \geq \dim_1(\mathcal{P}) + 1 = 4$. Since $\{u_1, u_4, w_2, w_3\}$ is a distance-1 locating-dominating set of $\mathcal{P}$, $\gamma_L^1(\mathcal{P}) \leq 4$; thus, $\gamma_L^1(\mathcal{P}) = \dim_1(\mathcal{P}) + 1 = 4$.

Next, we determine $\gamma_L^k(G)$ when G is a complete multipartite graph.

Proposition 4.4. [21] *For $m \geq 2$, let $G = K_{a_1, a_2, \ldots, a_m}$ be a complete m-partite graph of order $n = \sum_{i=1}^m a_i \geq 3$. Let s be the number of partite sets of G consisting of exactly one element. Then*

$$\dim(G) = \begin{cases} n - m & \text{if } s = 0, \\ n - m + s - 1 & \text{if } s \neq 0. \end{cases}$$

Proposition 4.5. *For $m \ge 2$, let $G = K_{a_1,a_2,\ldots,a_m}$ be a complete m-partite graph of order $n = \sum_{i=1}^{m} a_i \ge 3$. For $k \in \mathbb{Z}^+$,*

$$\gamma_L^k(G) = \begin{cases} \dim_k(G) + 1 = n - 1 & \text{if } k = 1 \text{ and } G = K_{1,n-1}, \\ \dim_k(G) & \text{otherwise.} \end{cases}$$

Proof. Let $G = K_{a_1,a_2,\ldots,a_m}$ be a complete m-partite graph of order $n = \sum_{i=1}^{m} a_i \ge 3$, where $m \ge 2$, and let $k \in \mathbb{Z}^+$. Note that $\mathrm{diam}(G) \in \{1, 2\}$, where $\mathrm{diam}(G) = 1$ if and only if $G = K_n$ and $\gamma_L^k(K_n) = \dim_k(K_n) = n-1$, for any $k \ge 1$, by Theorems 2.7(d) and 3.1. If $\mathrm{diam}(G) = 2$ and $k \ge 2$, then $\gamma_L^k(G) = \dim_k(G) = \dim(G)$ by Observations 4.1 and 4.2(b). So, suppose $\mathrm{diam}(G) = 2$ and $k = 1$. Let s be the number of partite sets of G consisting of exactly one element. If $s = 0$, then any minimum distance-1 resolving set of G is also a distance-1 dominating set of G; thus, $\gamma_L^1(G) = \dim_1(G)$. If $s = 1$ with $m = 2$, then $G = K_{1,n-1}$ and $\gamma_L^1(K_{1,n-1}) = n-1 = \dim_1(K_{1,n-1})+1$ by Theorems 2.7(b) and 3.4(b). If either $s = 1$ with $m \ge 3$ or $s \ge 2$, then any minimum distance-1 resolving set of G is also a distance-1 dominating set of G, and hence $\gamma_L^1(G) = \dim_1(G)$. □

Next, we determine $\gamma_L^k(G)$ when G is a cycle or a path.

Theorem 4.6. [8, 12] *Let $k \in \mathbb{Z}^+$. Then*

(a) $\dim_k(P_n) = 1$ *for* $2 \le n \le k + 2$*;*

(b) $\dim_k(C_n) = 2$ *for* $3 \le n \le 3k + 3$*, and* $\dim_k(P_n) = 2$ *for* $k + 3 \le n \le 3k + 3$*;*

(c) for $n \ge 3k + 4$, the formula for $\dim_k(C_n) = \dim_k(P_n)$ is as follows:

$$\begin{cases} \lfloor \frac{2n+3k-1}{3k+2} \rfloor & \text{if } n \equiv 0, 1, \ldots, k+2 \pmod{(3k+2)}, \\ \lfloor \frac{2n+4k-1}{3k+2} \rfloor & \text{if } n \equiv k+3, \ldots, \lceil \frac{3k+5}{2} \rceil - 1 \pmod{(3k+2)}, \\ \lfloor \frac{2n+3k-1}{3k+2} \rfloor & \text{if } n \equiv \lceil \frac{3k+5}{2} \rceil, \ldots, 3k+1 \pmod{(3k+2)}. \end{cases}$$

Proposition 4.7. *Let $G = P_n$ for $n \ge 2$ or $G = C_n$ for $n \ge 3$. For any $k \in \mathbb{Z}^+$, the formula for $\gamma_L^k(G)$ is as follows:*

$$\begin{cases} \dim_k(G) + 1 & \text{if } G \in \{P_n, C_n\} \text{ and } n \equiv 1 \pmod{(3k+2)}, \\ & \text{or } G = P_n \text{ and } n \equiv k+2 \pmod{(3k+2)}, \\ & \text{or } G = C_n, n \ge 3k+4 \text{ and } n \equiv k+2 \pmod{(3k+2)}, \\ \dim_k(G) & \text{otherwise.} \end{cases}$$

Proof. Let $G = P_n$ for $n \geq 2$ or $G = C_n$ for $n \geq 3$. Let $k \in \mathbb{Z}^+$.

If $2 \leq n \leq k+1$, then $\gamma_L^k(P_n) = \dim_k(P_n) = 1$ by Theorems 3.4(a) and 4.6(a). If $n = k+2$, then $\gamma_L^k(P_{k+2}) = \dim_k(P_{k+2}) + 1 = 2$ by Theorems 3.1, 3.4(a) and 4.6(a). If $k+3 \leq n \leq 3k+2$ and P_n is obtained from C_n, given by $u_0, u_1, \ldots, u_{n-1}, u_0$, by deleting the edge $u_k u_{k+1}$, then $\{u_0, u_\alpha\}$, where $\alpha = \min\{2k+1, n-1\}$, forms a distance-$k$ locating-dominating set of P_n, and thus $\gamma_L^k(P_n) = \dim_k(P_n) = 2$ by Theorems 3.1 and 4.6(b). If $3 \leq n \leq 3k+2$ and C_n is given by $u_0, u_1, \ldots, u_{n-1}, u_0$, then $\{u_0, u_\alpha\}$, where $\alpha = \min\{2k+1, n-1\}$, forms a distance-$k$ locating-dominating set of C_n, and thus $\gamma_L^k(C_n) = \dim_k(C_n) = 2$ Theorems 3.1 and 4.6(b). If $n = 3k+3$, then, for any minimum distance-k resolving set R of $G \in \{P_{3k+3}, C_{3k+3}\}$, there is a vertex w in G with $\text{code}_{R,k}(w) = (k+1, k+1)$; thus, $\gamma_L^k(G) = \dim_k(G) + 1 = 3$ by Theorem 3.1.

Now, suppose $n \geq 3k+4$, and let $G \in \{P_n, C_n\}$; then $\dim_k(G) \geq 3$. Let S be any minimum distance-k resolving set of G. First, suppose that $|S|$ is odd. If $n \not\equiv k+2 \pmod{3k+2}$, then there exists a minimum distance-k resolving set S_0 of G such that S_0 is also a distance-k dominating set of G (see [8, 12]); thus, $\gamma_L^k(G) = \dim_k(G)$. If $n \equiv k+2 \pmod{3k+2}$, then there exists a vertex w in G with $\text{code}_{R,k}(w) = (\mathbf{k+1})_{|R|}$ for any minimum distance-k resolving set R of G (see [8, 12]); thus, $\gamma_L^k(G) = \dim_k(G) + 1$. Second, suppose $|S|$ is even. If $n \not\equiv 1 \pmod{3k+2}$, then there exists a minimum distance-k resolving set S_1 of G such that S_1 is also a distance-k dominating set of G (see [8, 12]); thus, $\gamma_L^k(G) = \dim_k(G)$. If $n \equiv 1 \pmod{3k+2}$, then there exists a vertex w in G with $\text{code}_{S,k}(w) = (\mathbf{k+1})_{|S|}$ for any minimum distance-k resolving set S of G (see [8, 12]); thus, $\gamma_L^k(G) = \dim_k(G)+1$. $\square$

Based on the proof of Theorem 3.1, we note that $\gamma_L^k(G) = \dim_k(G) + 1$ if and only if, for every minimum distance-k resolving set S of G, there exists a vertex $w \in V(G) - S$ with $d(w, S) > k$. In other words, if there exists a minimum distance-k resolving set S' of G such that $d(v, S') \leq k$ for each $v \in V(G)$, then $\gamma_L^k(G) = \dim_k(G)$.

Question 4.8. *Since* $\dim_k(G) \leq \gamma_L^k(G) \leq \dim_k(G)+1$, *can we characterize* G *for which each of the two (end) inequalities is an equality?*

5 The effect of edge deletion on $\gamma_L^k(G)$

In this section, we examine the effect of edge deletion on the distance-k location-domination number of graphs. Throughout the section, let both G and $G-e$, where $e \in E(G)$, be connected graphs. For the effect of edge deletion on the metric dimension of graphs, we refer to [4]. We recall how the distance-k dimension of a graph changes upon deletion of an edge.

Theorem 5.1. *Let G be a connected graph with $e \in E(G)$, and let $k \in \mathbb{Z}^+$. Then*

(a) [11, 3] $\dim_1(G) - 1 \le \dim_1(G-e) \le \dim_1(G) + 1$*;*

(b) [8, 12] $\dim_2(G-e) \le \dim_2(G) + 1$*;*

(c) [8, 12] *for $k \ge 3$, $\dim_k(G-e) \le \dim_k(G) + 2$;*

(d) [8, 12] *for $k \ge 2$, $\dim_k(G) - \dim_k(G-e)$ can be arbitrarily large.*

Theorem 5.2. *Let G be a connected graph with $e \in E(G)$, and let $k \in \mathbb{Z}^+$. Then*

(a) $\gamma_L^1(G) - 2 \le \gamma_L^1(G-e) \le \gamma_L^1(G) + 2$;

(b) $\gamma_L^2(G-e) \le \gamma_L^2(G) + 2$;

(c) for $k \ge 3$, $\gamma_L^k(G-e) \le \gamma_L^k(G) + 3$.

Proof. Let $k \in \mathbb{Z}^+$. By Theorem 3.1, we have $\dim_k(G) \le \gamma_L^k(G) \le \dim_k(G) + 1$ and $\dim_k(G-e) \le \gamma_L^k(G-e) \le \dim_k(G-e) + 1$.

For (a), note that $\gamma_L^1(G-e) - \gamma_L^1(G) \ge \dim_1(G-e) - (\dim_1(G)+1) \ge -2$ and $\gamma_L^1(G) - \gamma_L^1(G-e) \ge \dim_1(G) - (\dim_1(G-e)+1) \ge -2$ by Theorem 5.1(a); thus, $\gamma_L^1(G) - 2 \le \gamma_L^1(G-e) \le \gamma_L^1(G) + 2$.

For (b), note that $\gamma_L^2(G) - \gamma_L^2(G-e) \ge \dim_2(G) - (\dim_2(G-e)+1) \ge -2$ by Theorem 5.1(b); thus $\gamma_L^2(G-e) \le \gamma_L^2(G) + 2$.

For (c), for any $k \ge 3$, we have $\gamma_L^k(G) - \gamma_L^k(G-e) \ge \dim_k(G) - (\dim_k(G-e)+1) \ge -3$ by Theorem 5.1(c); thus $\gamma_L^k(G-e) \le \gamma_L^k(G) + 3$. □

Theorem 5.3. *For any integer $k \ge 2$, $\gamma_L^k(G) - \gamma_L^k(G-e)$ can be arbitrarily large.*

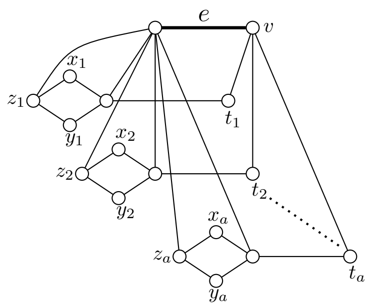

Figure 2: [8] Graphs G such that $\dim_k(G)-\dim_k(G-e)$ can be arbitrarily large, where $k \geq 2$ and $a \geq 3$.

Proof. Let G be the graph in Fig. 2 with $a \geq 3$. It was shown in [8, 12] that, for any $k \geq 2$, $\dim_k(G) = 2a$ and $\dim_k(G-e) = a+1$. For $k \geq 2$, $\gamma_L^k(G) \geq \dim_k(G) = 2a$ and $\gamma_L^k(G-e) \leq \dim_k(G-e)+1 = a+2$ by Theorem 3.1; thus, $\gamma_L^k(G) - \gamma_L^k(G-e) \geq 2a-(a+2) = a-2 \to \infty$ as $a \to \infty$. $\square$

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