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SPHERICAL FUNCTIONS ON ORDERED SYMMETRIC SPACES

by J. FARAUT, J. HILGERT and G. ÓLAFSSON

To Sigurdur Helgason on his sixtyfifth birthday

0. Introduction.

A semisimple symmetric space $\mathcal{M} = G/H$ is said to be ordered if it carries a partial order which is invariant under the action of G and infinitesimally generated, i.e. determined by the tangent cone of the set of positive elements at the origin. If $\mathcal{M}$ is irreducible and ordered then it is never Riemannian, i.e., H is non compact. Examples are one sheeted hyperboloids.

A spherical function φ on the ordered symmetric space $\mathcal{M}$ is a function defined on the positive part $\{x \in \mathcal{M} \mid x > \mathbf{1}H\}$ of $\mathcal{M}$ which satisfies the functional equation

$$\int_H \varphi(xhy)dh = \varphi(x)\varphi(y),$$

where φ is viewed as an H-bi-invariant function on G. Such functions were first studied for special cases in [FV86], where they were used to diagonalize certain integral equations with symmetry and causality conditions.

In this paper we present a construction of spherical functions for general semisimple ordered symmetric spaces, but in order to keep the necessary background on the structure theory of those spaces at a minimum we do not hesitate to restrict ourselves to certain representatives of locally

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isomorphic spaces even though more general results can be obtained. In the first four sections we describe the geometry of ordered symmetric spaces to the extend we need in the sequel. In Section 5 we construct a family of spherical functions parametrized by an open subset $\mathcal{E} + i\mathfrak{a}^* \subset \mathfrak{a}_{\mathbb{C}}^*$, where $\mathfrak{a}$ is a certain abelian subalgebra in $\mathfrak{q}$, the tangent space of $\mathcal{M}$ at $\mathbf{1}H$. The formula is similar to that for Riemannian symmetric spaces :

$$\varphi_\lambda(x) = \int_H e^{\langle \rho - \lambda, A(hx) \rangle} dh.$$

As H is non-compact in this case, one needs to restrict both λ and x. In particular we have to assume that $x > \mathbf{1}H$. Section 6 is devoted to the study of the asymptotic behavior of φ_λ. We introduce the c-function associated to the ordered symmetric space $\mathcal{M}$. This function is a product of two c-functions, one of them being the Harish-Chandra c-function associated to a Riemannian sub-symmetric space contained in $\mathcal{M}$ and the other a function constructed by an integral over a bounded real symmetric domain. In Section 7 we relate the spherical functions to H-spherical distributions associated with principal series representations of G.

In Section 8 we introduce the spherical Laplace transform of invariant causal kernels on $\mathcal{M}$, and, what is the same, H-invariant functions on the positive part. The Laplace transform is defined by

$$\mathcal{L}(f)(\lambda) = \int_{\mathcal{M}} f(x) e^{\langle \rho - \lambda, A(x) \rangle} dx.$$

This integral does not converge for all λ. If f has compact support modulo H, then the integral converges for $\lambda \in \mathcal{E} + i\mathfrak{a}^*$. We also introduce the Abel transform of an invariant causal kernel, and show that the spherical Laplace and Abel transforms are related by a classical Laplace transform. In the last two sections we present special cases for which we can actually invert the spherical Laplace transform. For $\mathcal{M}$ of the form $G_{\mathbb{C}}/G_{\mathbb{R}}$, by using a formula of Delorme, we are able to invert both Laplace and Abel transforms. His formula also shows, that $c(\lambda)^{-1}\varphi_\lambda$ has an analytic extension to $i\mathfrak{a}^*$. Let S be the semigroup

$$S = \{g \in G \mid gH \geqslant \mathbf{1}H\}.$$

We prove

THEOREM 9.7. — *Let $\mathcal{M}$ be an ordered symmetric space of type $G_{\mathbb{C}}/G_{\mathbb{R}}$ and let f be an H-invariant smooth function on $S^o \cdot x_o$ such that*

$f|_{S^o\cap A}$ has compact support. Then there exists a constant $c > 0$ only depending on normalization of measures such that, for $a \in S^o \cap A$,

$$f(a) = c\int_{\mathfrak{a}^*} \mathcal{L}(f)(i\lambda)\varphi_{-i\lambda}(a)\frac{d\lambda}{c(i\lambda)c(-i\lambda)}.$$

In Section 10 we consider for $n \geqslant 2$ the symmetric space $\mathcal{M} = \mathrm{SO}_o(1,n)/\mathrm{SO}_o(1,n-1)$. Here we first invert the Abel transform by using the Riemann-Liouville transform and then by using that we invert the spherical Laplace transform.

The paper is organized as follows :

1. Causal structures
2. Causal symmetric spaces
3. Symmetric spaces of Olshanskii type
4. Ordered symmetric spaces
5. Spherical functions
6. Convergence of integrals and asymptotics
7. Spherical functions and H-spherical distributions
8. Invariant causal kernels and the spherical Laplace transform
9. Inversion formula for spaces of Olshanskii type
10. Inversion formulas for spaces of rank 1

1. Causal structures.

Let $\mathcal{M}$ be a differentiable manifold of dimension n. A *causal structure* on $\mathcal{M}$ is a field of cones $\mathcal{M} \ni x \mapsto C_x \subset T_x\mathcal{M}$. The cone C_x is assumed to be closed, convex, proper ($C_x \cap -C_x = 0$), and with non-empty interior (i.e., generating $C_x - C_x = T_x\mathcal{M}$). Furthermore the cone C_x depends smoothly on x. More precisely, for a family of open subsets U covering $\mathcal{M}$, there exist smooth maps

$$\phi_U : U \times \mathbb{R}^n \to T(\mathcal{M})$$

with $\phi_U(x,\xi) \in T_x(\mathcal{M})$, and there is a cone C in $\mathbb{R}^n$ such that

$$C_x = \phi_U(x, C)$$

for $x \in U$. A piecewise C^1-curve $\gamma: [\alpha, \beta] \to \mathcal{M}$ is said to be *causal*, if, for all t, the derivative $\dot{\gamma}(t)$ belongs to the cone $C_{\gamma(t)}$ (the right derivative if $\dot{\gamma}(t)$ has a discontinuity at t). The causal structure is said to be *global* if there exists no non trivial closed causal curve. In that case one defines a partial ordering on $\mathcal{M}$ in the following way : one writes $x \leqslant y$ if there exists a causal curve from x to y. For x, y in $\mathcal{M}$, we define the interval $D(x, y)$,

$$D(x, y) = \{z \in \mathcal{M} \mid x \leqslant z \leqslant y\}.$$

In general these order intervals are not closed, but in the case we will consider they are even compact. The causal manifold is said to be *globally hyperbolic* if the intervals are all compact.

Assume that $\mathcal{M}$ is a homogeneous manifold, i.e., $\mathcal{M} = G/H$, where G is a Lie group and H a closed subgroup. For g in G we denote by ℓ_g the map

$$\ell_g: aH \mapsto gaH.$$

The causal structure is said to be *G-invariant* if, for all $g \in G$, and $x \in \mathcal{M}$

$$C_{\ell_g(x)} = d\ell_g(x)\big(C_x\big).$$

Let $x_o = \mathbf{1}H$, where $\mathbf{1}$ is the unit in G. Then a G-invariant causal structure is determined by the cone C_{x_o} in $T_{x_o}(\mathcal{M})$, which is invariant under H, i.e., under the linear transformations $d\ell_h(x_o)$, $h \in H$.

To a global invariant causal structure on $\mathcal{M}$, one associates the semigroup

$$S = \{g \in G \mid gx_o \geqslant x_o\}.$$

One can easily see, that $S \cap S^{-1} = H$. For more information on causal structures on homogeneous spaces we refer to [La89].

2. Causal symmetric spaces.

The results of this section are taken from [Óla90]. The earliest reference to the objects studied is [Ol82]. Let (G, H) be a symmetric pair, i.e., G is a connected Lie group, H is a closed subgroup, and there exists an involutive automorphism τ of G such that

$$(G^\tau)_o \subseteq H \subseteq G^\tau,$$

where $G^\tau = \{g \in G \mid \tau(g) = g\}$, and $(G^\tau)_o$ is the identity component in G^τ. As in the introduction we let $\mathcal{M} = G/H$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H and denote the differential of τ also by the same letter. Then

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \tau(X) = X\}.$$

Let

$$\mathfrak{q} = \{X \in \mathfrak{g} \mid \tau(X) = -X\}.$$

The tangent space at x_o of $\mathcal{M}$ can be identified with $\mathfrak{q}$. In this identification $d\ell_h(x_o)$, $h \in H$ corresponds to $\mathrm{Ad}(h)$. Therefore an invariant causal structure on $\mathcal{M}$ is determined by a cone C in $\mathfrak{q}$ with the properties

- C is closed, convex, proper, generating
- C is $\mathrm{Ad}(H)$-invariant.

To see that the cone field $gH \mapsto (d\ell_g)(x_o)(C)$ is smooth, we choose a zero neighborhood $U \subset \mathfrak{q}$ such that $\mathrm{Exp} : X \mapsto \exp X \cdot x_o$ is a diffeomorphism of U onto $V := \mathrm{Exp}(U)$. Let $g \in G$. Then ϕ_{gV} is defined by

$$\phi_{gV}(g\mathrm{Exp}X, Y) = (d\ell_{g\exp(X)})(x_o)(Y),\ X \in U, Y \in \mathfrak{q}.$$

Assume that G is semisimple with finite center. Let θ be a Cartan involution of G commuting with τ. Let K be the corresponding maximal compact subgroup of G. Its Lie algebra is given by

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}.$$

Define

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}.$$

Let $\mathfrak{q}_0$ be the space of $\mathrm{Ad}(H \cap K)$-invariant vectors in $\mathfrak{q}$. There exists in $\mathfrak{q}$ a cone C with the properties

- C is closed, convex, proper
- C is $\mathrm{Ad}(H)$-invariant

if and only if $\mathfrak{q}_0 \neq \{0\}$.

Assume further that $(\mathfrak{g}, \mathfrak{h})$ is *irreducible*, i.e., there is no non-trivial ideal in $\mathfrak{g}$ which is invariant under τ. If $\mathfrak{q}_0 \neq \{0\}$, then one of the following cases occurs :

Case (1) $\dim \mathfrak{q}_0 = 1,\ \mathfrak{q}_0 \subseteq \mathfrak{q} \cap \mathfrak{k}$.
Case (2) $\dim \mathfrak{q}_0 = 1,\ \mathfrak{q}_0 \subseteq \mathfrak{q} \cap \mathfrak{p}$.
Case (3) $\dim \mathfrak{q}_0 = 2$.

In Case (1) let C be a cone in $\mathfrak{q}$ which is closed, convex, proper, and $\mathrm{Ad}(H)$-invariant, then C is generating and $C^o \cap (\mathfrak{q} \cap \mathfrak{k}) \neq \varnothing$, where C^o denotes the interior of C, or equivalently $C \cap \mathfrak{k} \neq \{0\}$. This defines on $\mathcal{M}$ an invariant causal structure which is *not* global. In fact, if X_o is a non-zero element of $\mathfrak{q}_0$, then the curve $t \mapsto \mathrm{Exp}(tX_o)$ is causal and closed.

In Case (2) let C be a cone in $\mathfrak{q}$ which is closed, convex, proper, and $\mathrm{Ad}(H)$-invariant, then C is generating, and $C^o \cap (\mathfrak{q} \cap \mathfrak{p}) \neq \varnothing$. We will see below that in this case the associated causal structure is global and globally hyperbolic.

In Case (3) we have $\mathfrak{q}_0 = \mathfrak{q}_0 \cap \mathfrak{k} + \mathfrak{q}_0 \cap \mathfrak{p}$ and each of these subspaces of $\mathfrak{q}_0$ has dimension 1. There are four invariant causal structures on $\mathcal{M}$, two of which are global and globally hyperbolic whereas the others are not global.

We consider a few examples, for a complete classification see [Óla90].

Case (1)

(i) Let G be a connected simple group, then G can be considered as a symmetric space, the group $G \times G$ acting on G by $(a, b) \cdot x = axb^{-1}$. The corresponding involution is $\tau(a, b) = (b, a)$. There exists a biinvariant causal structure on G if and only if the Lie algebra $\mathfrak{g}$ of G is Hermitean, i.e., $\mathfrak{k}$ has a non-trivial center. If G has finite center, this causal structure is not global.

(ii) The symmetric spaces $\mathrm{SU}(p, q)/\mathrm{SO}_o(p, q)$ for $p \geqslant 1, q \geqslant 2$. Let as usual

$$\mathrm{SU}(p, q) = \{a \in \mathrm{SL}(n, \mathbb{C}) \mid a^\top \mathbf{1}_{p,q} \bar{a} = \mathbf{1}_{p,q}\},$$

with $n = p + q$, and where

$$\mathbf{1}_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}.$$

Let $\mathrm{su}(p, q)$ be the Lie algebra of $\mathrm{SU}(p, q)$. Define τ by $\tau(a) = \bar{a} = \mathbf{1}_{p,q}(a^{-1})^\top \mathbf{1}_{p,q}$, $a \in \mathrm{SU}(p, q)$. Then $H = G_o^\tau = \mathrm{SO}_o(p, q)$, $\mathfrak{h} = \mathrm{so}(p, q)$

and

$$\mathfrak{q} = \{X \in \mathrm{su}(p,q) \mid \overline{X} = -X\}.$$

Let $\theta(X) = -X^*$ be the usual Cartan involution on $\mathrm{su}(p,q)$. Then

$$\mathfrak{q} \cap \mathfrak{k} = \left\{ i \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A^\top = A,\ D^\top = D,\ \mathrm{Tr}A + \mathrm{Tr}D = 0,\ A, D \text{ real} \right\}.$$

Let C_k be the cone in $\mathfrak{q} \cap \mathfrak{k}$ such that $A \geqslant 0$ and $D \leqslant 0$. Then the closure of the convex hull of $\mathrm{Ad}(H)C_k$ is a closed, convex, proper and generating H-invariant cone in $\mathfrak{q}$ such that $C^o \cap \mathfrak{k} = C_k^o \neq 0$.

(iii) The symmetric space $\mathrm{SO}_o(2, n-1)/\mathrm{SO}_o(1, n-1)$ for $n \geqslant 3$. Define the involution τ as conjugation by $\mathbf{1}_{1,n}$. Then $H = G_o^\tau \simeq \mathrm{SO}_o(1, n-1)$. The bilinear form $-B|\mathfrak{q}$, B the Killing form on $\mathrm{so}(2, n-1)$, defines a Lorentzian form on $\mathfrak{q}$. Each component of the light cone $\{X \in \mathfrak{q} \mid -B(X,X) \geqslant 0\}$ defines an $\mathrm{SO}_o(2, n-1)$-invariant causal structure on the Lorentzian manifold $\mathrm{SO}_o(2, n-1)/\mathrm{SO}_o(1, n-1)$ which is not global.

Case (2)

(iv) Let $\mathfrak{h}$ be a simple Hermitean Lie algebra, G a connected complex Lie group with Lie algebra $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ and assume that the conjugation τ of $\mathfrak{g}$ with respect to $\mathfrak{h}$ can be lifted to G. Then $\mathfrak{q} = i\mathfrak{h}$ and, if $H = (G^\tau)_o$, then $H \cap K$ is a maximal compact subgroup of H. We have $\mathfrak{q}_0 = i\mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{k}_0 = \mathfrak{h} \cap \mathfrak{k}$. These symmetric spaces $\mathcal{M} = G/H$ were first studied by Olshanskii, and we will call them symmetric spaces of Olshanskii type.

(v) The symmetric spaces $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}_o(p,q)$ for $p + q = n, p \geqslant 1, q \geqslant 2$. Here the involution τ is given as for $\mathrm{SU}(p,q)$ by

$$\tau(a) = \mathbf{1}_{p,q}(a^{-1})^\top \mathbf{1}_{p,q}$$

and the invariant cone in $\mathfrak{q}$ is given by $\overline{\mathrm{Ad}(\mathrm{SO}_o(p,q))C_p}$ where C_p is the cone in $\mathfrak{q} \cap \mathfrak{p}$ consisting of matrices of the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, A and D symmetric, $A \geqslant 0$, $D \leqslant 0$ and $\mathrm{Tr}A + \mathrm{Tr}D = 0$.

(vi) The Lorentzian space $\mathrm{SO}_o(1, n)/\mathrm{SO}_o(1, n-1)$ for $n \geqslant 3$. This space is treated in Section 9 of this paper.

Case (3)

(vii) The symmetric space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1) \cong \mathrm{SO}_o(1,2)/\mathrm{SO}_o(1,1)$. Here $\mathcal{M}$ is a hyperboloid with one sheet in $\mathbb{R}^3$.

(viii) The symmetric spaces $\mathrm{Sp}(n,\mathbb{R})/\mathrm{GL}(n,\mathbb{R})$.

(ix) The symmetric spaces $\mathrm{U}(n,n)/\mathrm{GL}(n,\mathbb{C})$.

A semisimple symmetric space equipped with a global causal structure will be said to be *ordered*.

Assume that there exists in $\mathfrak{p}\cap\mathfrak{q}$ a non-zero vector X_o which is invariant under $\mathrm{Ad}(H\cap K)$ and such that the projection on every irreducible component is non-zero. Then it can be shown, see [Óla90], that the centralizer of X_o in $\mathfrak{g}$ equals $\mathfrak{k}\cap\mathfrak{h}\oplus\mathfrak{p}\cap\mathfrak{q}$. Thus if $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{q}$, then $X_o\in\mathfrak{a}$ and $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. Furthermore we may assume that the eigenvalues of $\mathrm{ad}(X_o)$ are $1,0,-1$. Let

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_0 + \mathfrak{g}_{-1}$$

be the corresponding eigenspace decomposition. Then $\mathfrak{g}_0 = \mathfrak{k}\cap\mathfrak{h}\oplus\mathfrak{p}\cap\mathfrak{q}$. As $[\mathfrak{g}_i,\mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$, $i,j=0,1,-1$, it follows, that $\mathfrak{g}_{\pm1}$ are abelian algebras and $[\mathfrak{g}_0,\mathfrak{g}_{\pm1}]\subset\mathfrak{g}_{\pm1}$.

Let $\mathfrak{b}$ be an abelian Lie algebra and $\mathbf{V}$ a finite dimensional semisimple $\mathfrak{b}$-module. We use the notation

$$\begin{aligned}
\mathbf{V}_\alpha &:= \{v\in\mathbf{V} \mid \forall X\in\mathfrak{b} : X\cdot v = \alpha(X)v\}, \quad \alpha\in\mathfrak{b}^*,\\
\Delta(\mathbf{V},\mathfrak{b}) &:= \{\alpha\in\mathfrak{b}^* \mid \alpha\neq 0, \mathbf{V}_\alpha\neq 0\},\\
\mathbf{V}(\Gamma) &:= \bigoplus_{\alpha\in\Gamma}\mathbf{V}_\alpha, \quad \Gamma\subset\Delta(\mathbf{V},\mathfrak{b}), \quad \mathbf{V}^{\mathfrak{b}} := \mathbf{V}_0,\\
\rho(\Gamma) &:= \frac{1}{2}\sum_{\alpha\in\Gamma}(\dim\mathbf{V}_\alpha)\alpha.
\end{aligned}$$

Let $\Delta=\Delta(\mathfrak{g},\mathfrak{a})$ and $\Delta_0=\Delta(\mathfrak{g}_0,\mathfrak{a})$. Let

$$\Delta_{\pm1} = \{\alpha\in\Delta(\mathfrak{g},\mathfrak{a}) \mid \mathfrak{g}_\alpha\subseteq\mathfrak{g}_{\pm1}\}.$$

In the root system Δ we choose a positive system Δ^+ such that

$$\Delta^+ = \Delta_0^+\cup\Delta_1$$

with Δ_0^+ a set of positive roots in Δ_0. One obtains in $\mathfrak{a}$ the two cones

$$c_{\max} = \{X \in \mathfrak{a} \mid \forall \alpha \in \Delta_1 \ : \alpha(X) \leqslant 0\},$$
$$c_{\min} = c_{\max}^*,$$

and in $\mathfrak{q}$ the closed convex H-invariant cones $C_{\max}$ and $C_{\min}$ such that

$$C_{\max} \cap \mathfrak{a} = c_{\max},$$
$$C_{\min} \cap \mathfrak{a} = c_{\min}.$$

Set $\mathfrak{n} = \mathfrak{g}(\Delta^+)$, $\mathfrak{n}_{\pm 1} = \mathfrak{g}(\Delta_{\pm 1}) = \mathfrak{g}_{\pm 1}$, $\mathfrak{n}_0 = \mathfrak{g}(\Delta_0^+)$, $\rho = \rho(\Delta^+)$, $\rho_1 = \rho(\Delta_1)$ and $\rho_0 = \rho(\Delta_0^+)$. Moreover we let $N_0 = \exp(\mathfrak{n}_0)$ and $N_{\pm 1} = \exp(\mathfrak{n}_{\pm 1})$. Finally we set $G_0 = K_0 \exp(\mathfrak{p} \cap \mathfrak{q})$ with $K_0 = K \cap H$. Then G_0 is a group, $Z_G(X_o)_o \subset G_0 \subset Z_G(X_o)$ and the Lie algebra of G_0 is $\mathfrak{g}_0$. Moreover, $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1$ is a semidirect product of Lie algebras with $\mathfrak{n}_1$ an ideal and $N = N_0 N_1$ a semidirect product of groups with N_1 abelian and normal. We note that $\tau(N) = \theta(N) = \overline{N}$ and $\tau(N_1) = \theta(N_1) = N_{-1} = \overline{N}_1$ since $\mathfrak{a} \subseteq \mathfrak{p} \cap \mathfrak{q}$.

3. Symmetric spaces of Olshanskii type.

In this section we specify our notation from the last section to the symmetric spaces of Olshanskii type. The notation is the same as in Example (iv).

Let $\mathfrak{h}$ be a simple Hermitean Lie algebra and $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ its complexification. Let G be a connected complex Lie group with Lie algebra $\mathfrak{g}$ such that the conjugation τ of $\mathfrak{g}$ with respect to $\mathfrak{h}$ can be lifted to G, and let H be a subgroup of G with Lie algebra $\mathfrak{h}$,

$$(G^\tau)_o \subseteq H \subseteq G^\tau.$$

Let C be a cone in $i\mathfrak{h}$ with the properties

- C is closed, convex, proper, generating,
- C is $\mathrm{Ad}(H)$-invariant.

The cone C defines an invariant causal structure on $\mathcal{M} = G/H$ which is global. The associated semigroup S is given by

$$S = \exp(C)H$$

and S is homeomorphic to $C \times H$ (cf. [Ol81], Theorem 3.5). Moreover $\mathcal{M}$ with this causal structure is globally hyperbolic (cf. [Fa91], Théorème 4). We can choose an element $Z_o \in \mathfrak{z}(\mathfrak{k} \cap \mathfrak{h})$ defining a complex structure on $H/H \cap K$, where $\mathfrak{z}(\mathfrak{k} \cap \mathfrak{h})$ is the center of $\mathfrak{k} \cap \mathfrak{h}$ which is one dimensional since $\mathfrak{h}$ is Hermitean and simple. Let $X_o = -iZ_o$, then $\operatorname{ad} X_o$ has eigenvalues $0, \pm 1$. Thus our notation is related to the classical one, see [He78] or [Wo72], by

$$G_0 = K_{\mathbb{C}}, \qquad \mathfrak{n}_{\pm 1} = \mathfrak{p}^{\pm}, \qquad N_{\pm 1} = P^{\pm}.$$

Now $G/(G_0 N_1) \cong K/(K \cap H)$ is a compact Hermitean symmetric space Y. Let $y_o = \mathbf{1}(G_0 N_1)$. The orbit $D = H \cdot y_o \subseteq Y$ is the Borel realization of the non-compact Hermitean symmetric space $H/(K \cap H)$. We assume now that $C = C_{\max}$.

THEOREM 3.1 [Ol81]. — *Define* $\Gamma := S^{-1} = \exp(-C_{\max})H$. *Then*

$$\Gamma = \{g \in G \mid g(D) \subseteq D\}.$$
$$\Gamma^o = \{g \in G \mid g(\overline{D}) \subseteq D\}.$$

COROLLARY 3.2.

$$S \subseteq NAH.$$

Proof. — Let $g \in \Gamma$, then $gy_o \in Hy_o$, or $g \in HG_0N_1$. We write the Iwasawa decomposition of G_0,

$$G_0 = K_0 A N_0,$$

and since $K_0 \subseteq H$ and $N_0 N_1 = N$ we have $g \in HAN$ and $g^{-1} \in NAH$.

□

4. Ordered symmetric spaces.

We now describe the ordered symmetric spaces which we will use and fix the notation. For the proofs of the structure theoretic results we refer to the forthcoming book [ÓH92].

Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair, with $\mathfrak{g}$ semisimple, associated with the involution τ. Let $G_{\mathbb{C}}$ be a linear connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. We assume that τ can be lifted as a holomorphic involution of $G_{\mathbb{C}}$, and that the conjugation σ of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$ can be lifted to an antiholomorphic involution of $G_{\mathbb{C}}$. Let $G = (G_{\mathbb{C}})^{\sigma}_{o}$. Define the *c-dual* Lie algebra

$$\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q} = (\mathfrak{g}_{\mathbb{C}})^{\tau\sigma}$$

and the c-dual Lie group

$$G^c = G_{\mathbb{C}}^{\tau\sigma}.$$

We let $H = G \cap G^c$. Then $(G^{\tau})_o \subset H \subset G^{\tau}$. Let as before $\mathcal{M} = G/H$. The subspace $\mathfrak{k}^c = \mathfrak{h} \cap \mathfrak{k} + i(\mathfrak{q} \cap \mathfrak{p})$ is a maximal compact subalgebra of $\mathfrak{g}^c$. Let K^c be the maximal compact subgroup of G^c with Lie algebra $\mathfrak{g}^c$, then G^c/K^c is a Hermitean symmetric space of non-compact type.

We assume that there exists in $\mathfrak{p} \cap \mathfrak{q}$ a non-zero vector X_o which is $\mathrm{Ad}(H \cap K)$-invariant and such that the projection onto each irreducible factor is non-zero. Thus the space $\mathfrak{q}_0$ of $K \cap H$-invariant vectors is non-zero for each irreducible factor. By Section 1 there exists an H-invariant regular cone $C \subset \mathfrak{q}$ defining a causal structure on $\mathcal{M}$ such that $C^o \cap (\mathfrak{p} \cap \mathfrak{q}) \neq \varnothing$. Furthermore X_o belongs to the center of $\mathfrak{k}^c$. In fact, let $(\cdot \mid \cdot) = -B(\cdot, \theta(\cdot))$ be the usual inner product on $\mathfrak{g}$ coming from the Killing form. Then, for $X \in \mathfrak{p} \cap \mathfrak{q}$,

$$([X_o, X] \mid [X_o, X]) = (X \mid [X_o, [X_o, X]]) = 0$$

since $[X_o, X] \in \mathfrak{k} \cap \mathfrak{h}$. Thus we have $[X_o, X] = 0$.

If $(\mathfrak{g}, \mathfrak{h})$ is irreducible then there are two possible cases :

Case (1) $\mathfrak{g}^c$ is not simple, then $\mathcal{M}$ is a symmetric space of Olshanskii type.

Case (2) $\mathfrak{g}^c$ is simple, then $G_{\mathbb{C}}/G^c$ is a symmetric space of Olshanskii type. The orbit of G in $G_{\mathbb{C}}/G^c$ is isomorphic to $\mathcal{M}$.

From this, Theorem 3.1 and Corollary 3.2, we have (see [Óla90], Theorem 6.3.2)

THEOREM 4.1. — *$\mathcal{M}$ is an ordered symmetric space which is globally hyperbolic. Let S be the associated semigroup, then $S = \exp(C)H$.*

We have $H \subseteq N_{-1}G_0N_1$, and we define D as the H-orbit of the basepoint in G/G_0N_1. Then D is the "Borel realization" of H/K_0, where $K_0 = G_0 \cap H = G_0 \cap K$. Define

$$\Omega = \{\overline{n} \in N_{-1} \mid \overline{n} \cdot G_0N_1 \in D\},$$
$$\Omega = \exp \mathcal{D}, \quad \mathcal{D} \subseteq \mathfrak{n}_{-1},$$

then $\mathcal{D}$ is the "Harish Chandra realization" of H/K_0. Let $\mathcal{D}_{\mathbb{C}}$ be the Harish-Chandra realization in $(\mathfrak{n}_{-1})_{\mathbb{C}}$ of the Hermitean symmetric space G^c/K^c. Then $\mathcal{D} = \mathcal{D}_{\mathbb{C}} \cap \mathfrak{n}_{-1}$. Therefore $\mathcal{D}$ and Ω are bounded domains. We have (see [Óla90], Section 6.4)

THEOREM 4.2. — *Define* $\Gamma := S^{-1} = \exp(-C_{\max})H$, *then*

$$\Gamma = \{g \in G \mid g(D) \subseteq D\}.$$

For $g \in \Gamma^o$, $g(\overline{D}) \subset D$ *and* $g(D)$ *is relatively compact in* D. *Furthermore* $S \subseteq NAH$ *and* $S = H\exp(c_{\max})H$.

On the Lie algebra level $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{h},$$

and the map

$$N \times A \times H \ni (n, a, h) \mapsto nah \in G$$

is a diffeomorphism onto an open subset of G. From this it follows that the map

$$\begin{aligned} N \times \mathfrak{a} &\to \mathcal{M}, \\ (n, X) &\mapsto n\exp X \cdot x_o, \end{aligned}$$

is a diffeomorphism of $N \times \mathfrak{a}$ onto the open set NAx_o. For x in this set, $x = n\exp X \cdot x_o$, we let

$$A(x) = X.$$

Sometimes we will use the notation $a_H(x) := \exp A(x)$. As A and a_H are right H-invariant we can view them as functions on $NAx_0 \subset \mathcal{M}$. Note that the map A is essentially the same as the Poisson kernel for the open H-orbit of $\mathbf{1}P_{\min}$ defined in [Óla87] (see Section 4).

The following theorem due to Neeb (see [Ne91], Proposition IV.17 and Corollary IV.18) will be crucial in the proof of the convergence of various integrals over H.

THEOREM 4.3. — *If* $a \in \exp(c_{\max})$, $\quad \overline{n} \in \overline{N} \cap NAH$, *then*

$$A(a^{-1}\overline{n}a) - A(\overline{n}) \in c_{\min}$$

and

$$A(\overline{n}) \in -c_{\min}.$$

The following fact will be used later on. Let W_0 be the Weyl group associated with the root system Δ_0. It can be identified with $N_H(\mathfrak{a})/Z_H(\mathfrak{a})$ and the cone $c_{\max}$ can be reconstructed from the Weyl chamber

$$\mathfrak{a}^- = \{X \in \mathfrak{a} \mid \forall \alpha \in \Delta^+,\ \alpha(X) < 0\}$$

using W_0,

$$c_{\max} = \overline{W_0 \mathfrak{a}^-} \tag{4.1}$$

5. Spherical functions.

We come to the proper subject of the paper. Let $\mathcal{M} = G/H$ be an ordered symmetric space as in the last section. We assume that the ordering is associated with the cone $C_{\max}$,

$$S = \exp(C_{\max})H.$$

A *spherical function* is a function φ defined on the interior S^o of S such that for all $x, y \in S^o$,

$$H \ni h \mapsto \varphi(xhy) \in \mathbb{C}$$

is integrable, and

$$\int_H \varphi(xhy)dh = \varphi(x)\varphi(y),$$

where dh is a Haar-measure on H. We remark here that we will use the same normalization of measures as in [He84], p.449, and [Óla87]. By Theorem 4.2. $hx \in S \subset NAH$ for $h \in H$ and $x \in S$. Thus $a_H(hx)$ is defined. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, let φ_λ be the function defined on S^o by

$$\varphi_\lambda(x) = \int_H e^{\langle \rho - \lambda, A(hx)\rangle} dh = \int_H a_H(hx)^{\rho-\lambda} dh, \tag{5.1}$$

provided the integral converges. Here $a^\lambda = e^{\langle \lambda, X\rangle}$ for $a = \exp X \in A$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Let $\mathcal{E}$ be the set of λ in $\mathfrak{a}^*$ such that, for all x in S^o, $h \mapsto a_H(hx)^{\rho-\lambda}$ is integrable over H. For the proof of Theorem 5.2 below we need a lemma :

LEMMA 5.1. — *Let $M = Z_H(A)$ then $M = Z_K(A)$ and M is compact.*

Proof. — As $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ it follows that $Z_H(A) = Z_{K\cap H}(A) \subset Z_K(A)$. Furthermore $Z_H(A)_o = Z_K(A)_o$ as $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$. Let

$$F := K \cap \exp_{G_\mathbf{C}} i\mathfrak{a}.$$

Then every element of F has order 2 and thus $F \subset G^\tau$. Furthermore $F \subset G^{\tau\sigma}$. It follows that $F \subset H$. As in [KnZu82], p. 400, it follows, that $Z_K(A) = F(Z_K(A))_o \subset H$ and the claim follows. □

THEOREM 5.2. — *For $\Re\lambda \in \mathcal{E}$, the function φ_λ is spherical.*

Proof. — For $g \in NAH$ one writes

$$g = na_H(g)h(g),$$

with $n \in N$ and $h(g) \in H$. We prove first

$$a_H(xy) = a_H(x)a_H(h(x)y), \tag{5.2}$$

for $x, y \in NAH$ such that $xy \in NAH$. Notice that in this case $h(x)y$ is also in NAH. Let $x = n_1 a_H(x)h(x)$ then :

$$\begin{aligned} xy &= n_1 a_H(x)h(x)y \\ &= n_1 a_H(x)n_2 a_H(h(x)y)h(h(x)y) \\ &= n_3 a_H(x)a_H(h(x)y)h(h(x)y), \end{aligned}$$

with $n_3 = n_1(a_H(x)n_2 a_H(x)^{-1}) \in N$. Therefore

$$a_H(xhy)^{\rho-\lambda} = a_{H\cdot}(x)^{\rho-\lambda}a_H(h(x)hy)^{\rho-\lambda}$$

for $x, y \in S^o$. By integrating with respect to h one obtains

$$\int_H a_H(xhy)^{\rho-\lambda}dh = a_H(x)^{\rho-\lambda}\varphi_\lambda(y).$$

Now by replacing x by $h'x$, integrating with respect to h' and using Fubini's theorem to change the order of the integrations we see that φ_λ is spherical. □

PROPOSITION 5.3. — *Let $\mu \in \mathfrak{a}^*$. Then $\mu \in \mathcal{E}$ if and only if*

$$\int_{K\cap NAH} a_H(k)^{\rho+\mu}dk < \infty,$$

and, for $\Re\lambda \in \mathcal{E}$,

$$\varphi_\lambda(x) = \int_{K\cap NAH} a_H(kx)^{\rho-\lambda} a_H(k)^{\rho+\lambda} dk.$$

For $\Re\lambda \in \mathcal{E}$, *the spherical function* φ_λ *is continuous on* S^0.

This shows that the convergence of the integral defining $\varphi_\lambda(x)$ only depends on λ and not on x, a remark which will be used in Section 8.

Proof. — We use the following fact which follows from Lemma 5.1 above and Lemma 1.3 in [Óla87]. A function f on H is integrable if and only if $f\big(h(k)\big)a_H(k)^{2\rho}$ is integrable over $K \cap NAH$, and

$$\int_H f(h)dh = \int_{K\cap NAH} f\big(h(k)\big) a_H(k)^{2\rho} dk. \tag{5.3}$$

Therefore, for $\Re\lambda \in \mathcal{E}$,

$$\varphi_\lambda(x) = \int_{K\cap NAH} a_H(kx)^{\rho-\lambda} a_H(k)^{\rho+\lambda} dk.$$

Let $x \in S^0$, $g = x^{-1} \in \Gamma^0$. By Theorem 4.2, $g(\overline{D}) \subset D$. Let $L = \overline{K \cap NAH}$. Since L^{-1}/G_0N_1 is contained in $\overline{D}$, then gL^{-1}/G_0N_1 is contained in D, or $gL^{-1} \subset HG_0N_1 = HAN$, and Lx is a compact set contained in HAN. For a compact set $B \subset S^0$, $\mu \in \mathfrak{a}^*$, there are constants $a, b > 0$ such that

$$\forall k \in L,\ a \leqslant a_H(kx)^{\rho+\mu} \leqslant b.$$

Therefore $\mu \in \mathcal{E}$ if and only if

$$\int_{K\cap NAH} a_H(k)^{\rho+\mu} dk < \infty.$$

By the dominated convergence theorem of Lebesgue, this shows that, for $\Re\lambda \in \mathcal{E}$, the spherical function φ_λ is continuous on S^0. □

COROLLARY 5.4.

$$\{\mu \in \mathfrak{a}^* \mid \forall\alpha \in \Delta_+,\ \langle \rho + \mu, \alpha\rangle < 0\} \subset \mathcal{E}.$$

Proof. — Assume that, for all $\alpha \in \Delta_+$, $\langle\rho + \mu, \alpha\rangle < 0$. Then, by Lemma 4.1 in [Óla87], the function $g \mapsto a_H(g)^{\rho+\mu}$ is continuous on G. Therefore it is bounded on the compact set $K \cap NAH$, hence integrable over $K \cap NAH$. □

By considering the orbit $\mathcal{M}_0 = G_0 \cdot x_o$ in $\mathcal{M}$ one obtains an imbedding of the Riemannian symmetric space $\mathcal{M}_0 = G_0/K_0$ in $\mathcal{M}$. Every element x in $NA \cdot x_o$ can be written $x = n_1 y$ with $n_1 \in N_1$ and $y \in \mathcal{M}_0$ in a unique way. We write

$$x = n_1 y(x).$$

For $g \in G_0$, $a_H(g)$ coincides with the Iwasawa projection relative to the Iwasawa decomposition $G_0 = N_0 A K_0$ of the reductive group G_0. Thus $y \in \mathcal{M}_0$ can be written in a unique way as

$$y = n_0 a_H(y) x_o.$$

Then, for x in NAx_o,

$$a_H(x) = a_H(y(x)).$$

Furthermore, for k in K_0,

$$y(kx) = ky(x),$$

since N_1 is normalized by K_0. By integrating first with respect to K_0 in the integral defining the spherical function φ_λ, we obtain

$$\varphi_\lambda(x) = \int_{K_0\backslash H} \left(\int_{K_0} a_H(ky(hx))^{\rho-\lambda} dk \right) d\dot{h},$$

where $d\dot{h}$ is a suitably normalized H-invariant measure on $K_0\backslash H$. The spherical functions for the Riemannian symmetric space $\mathcal{M}_0$ are given by

$$\varphi^0_\lambda(y) = \int_{K_0} a_H(ky)^{\lambda+\rho_0} dk.$$

Therefore we have proved

THEOREM 5.5.

$$\varphi_\lambda(x) = \int_{K_0\backslash H} \varphi^0_{\rho_1-\lambda}\big(y(hx)\big) d\dot{h}.$$

The Weyl group W_0 associated with the root system Δ_0 is the Weyl group for the Riemannian symmetric space G_0/K_0. As $w\rho_1 = \rho_1$ for all $w \in W_0$, the Weyl group invariance of the spherical functions in the Riemannian case yields

COROLLARY 5.6. — *For $w \in W_0$ we have $\varphi_{w\lambda} = \varphi_\lambda$.*

6. Convergence of integrals and asymptotics.

To study the convergence set $\mathcal{E}$ and the asymptotic behaviour of the spherical function φ_λ we will carry the integral defining φ_λ to an integral over $\overline{N} \cap NAH$.

PROPOSITION 6.1. — *Let $\Omega \subset N_{-1}$ be as in Theorem 4.1. Then $\overline{N} \cap NAH = \overline{N}_0 \cdot \Omega$.*

Proof. — By definition of Ω and $G_0 = K_0AN_0$ we get

$$\Omega G_0 N_1 = HG_0N_1 = HAN.$$

Therefore $\Omega\overline{N}_0 = \overline{N} \cap HAN$. Thus $\overline{N} \cap NAH = \overline{N}_0\Omega^{-1}$. As $\theta\tau X = -X$ for all $X \in \overline{\mathfrak{n}}_{-1}$ and HG_0N_1 is $\theta\tau$-stable it follows that $\Omega^{-1} = \Omega$. □

We consider the following integral for $\lambda \in \mathfrak{a}^*_{\mathfrak{C}}$,

$$c_\Omega(\lambda) = \int_\Omega a_H(\overline{n}_1)^{\rho+\lambda} d\overline{n}_1. \tag{6.1}$$

Let

$$c^\vee_{\min} = \{\lambda \in \mathfrak{a}^* \mid \forall X \in c_{\min},\ \lambda(X) \geqslant 0\}$$

be the dual cone of $c_{\min}$ in $\mathfrak{a}^*$. We also define

$$\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^* \mid \forall \alpha \in \Delta_0^+,\ \langle \alpha, \lambda \rangle > 0\}.$$

PROPOSITION 6.2. — *The integral defining $c_\Omega(\lambda)$ converges if $\Re\lambda + \rho \in c^\vee_{\min}$.*

Proof. — Since Ω is bounded, it is enough to prove that

$$\overline{n} \mapsto a_H(\overline{n})^{\rho+\lambda}$$

is bounded on Ω. By Theorem 4.3 $A(\overline{n}) \in -c_{\min}$, therefore

$$\Re\langle \rho + \lambda, A(\overline{n}) \rangle \leqslant 0,$$

and

$$|e^{\langle \rho+\lambda, A(\overline{n})\rangle}| \leqslant 1. \qquad \square$$

THEOREM 6.3. — *The domain of convergence $\mathcal{E}$ of the integral defining the spherical function φ_λ is given by*

$$\mathcal{E} = \{\lambda \in \mathfrak{a}^* \mid c_\Omega(\lambda) < \infty\}.$$

Proof. — By Proposition 5.3 we know that $\lambda \in \mathcal{E}$ if and only if $k \mapsto a_H(k)^{\rho+\lambda}$ is integrable over $K \cap NAH$. For $x \in G$ we use the Iwasawa decomposition $G = NAK$ for defining $a(x) \in A$ and $k(x) \in K$ by $x \in Na(x)k(x)$. By Lemma 1.3 in [Óla87] this is the case if and only if $\overline{n} \mapsto a_H\left(k(\overline{n})\right)^{\rho+\lambda} a(\overline{n})^{2\rho}$ is integrable on $\overline{N} \cap NAH$ and in that case

$$\int_{K\cap NAH} a_H(k)^{\rho+\lambda} dk = \int_{\overline{N}\cap NAH} a_H(k(\overline{n}))^{\rho+\lambda} a(\overline{n})^{2\rho} d\overline{n}.$$

By Proposition 6.1 $\overline{N} \cap NAH = \overline{N}_0\Omega$. It is also obvious that $d\overline{n} = d\overline{n}_0 d\overline{n}_1$. Thus since

$$a_H(k(\overline{n})) = a_H(\overline{n}) a(\overline{n})^{-1}$$

and

$$\int_{\overline{N}_0} a(\overline{n}_0)^{2\rho} d\overline{n}_0 = 1,$$

we obtain

$$\begin{aligned}\int_{K\cap NAH} a_H(k)^{\rho+\lambda} dk &= \int_{\overline{N}\cap NAH} a_H(k(\overline{n}))^{\rho+\lambda} a(\overline{n})^{2\rho} d\overline{n} \\ &= \int_{\overline{N}_0\Omega} a_H(\overline{n})^{\rho+\lambda} a(\overline{n})^{\rho-\lambda} d\overline{n} \\ &= \int_\Omega a_H(\overline{n})^{\rho+\lambda} d\overline{n} \\ &= c_\Omega(\lambda)\end{aligned}$$

as $a_H(x) = a(x)$, for $x \in G_0$. $\square$

COROLLARY 6.4.

$$-\rho + c_{\min}^{\vee} \subseteq \mathcal{E}.$$

PROPOSITION 6.5. — *For* $\Re\lambda \in \mathfrak{a}_+^* \cap \mathcal{E}$,

$$\int_{\overline{N}\cap NAH} a_H(\overline{n})^{\rho+\lambda} d\overline{n} = c_0(\lambda)c_\Omega(\lambda),$$

where

$$c_0(\lambda) = \int_{\overline{N}_0} a_H(\overline{n}_0)^{\rho_0+\lambda} d\overline{n}_0$$

is the c-function of the Riemannian symmetric space G_0/K_0.

Proof. — One uses once more $a_H(\overline{n}_0\overline{n}_1) = a_H(\overline{n}_0)a_H\left(h(\overline{n}_0)\overline{n}_1\right)$, and notices that $h(\overline{n}_0) \in K_0$. For $k_0 \in K_0$, $a_H(k_0\overline{n}_1) = a_H(k_0\overline{n}_1k_0^{-1})$, and the measure $d\overline{n}_1$ is K_0-invariant. This proves the claim since $a_H(\overline{n}_0)^{\rho_1} = 1$.

□

PROPOSITION 6.6. — *For* a *in* $A \cap S^o$,

$$\varphi_\lambda(a) = a^{\rho-\lambda} \int_{\overline{N}\cap NAH} a_H(a^{-1}\overline{n}a)^{\rho-\lambda} a_H(\overline{n})^{\rho+\lambda} d\overline{n}.$$

Proof. — First we notice that for all $f \in \mathbf{L}^1(H)$ right M-invariant we have

$$\int_H f(h)dh = \int_{\overline{N}\cap NAH} f\left(h(\overline{n})\right) a_H(\overline{n})^{2\rho} d\overline{n}$$

by Theorem 4.5 in [Óla85] or by [Óla87]. The proposition now follows since

$$\begin{aligned} A(h(\overline{n})a) &= A(\overline{n}a) - A(\overline{n}) \\ &= \log a + A(a^{-1}\overline{n}a) - A(\overline{n}). \end{aligned}$$

□

LEMMA 6.7. — *Let* Q *be a compact set in* Ω, *and let* $\lambda \in \mathfrak{a}^*$. *There exist positive constants* $M_1(Q,\lambda)$ *and* $M_2(Q,\lambda)$ *such that*

$$M_1(Q,\lambda)e^{\langle\lambda, A(\overline{n}_0)\rangle} \leqslant e^{\langle\lambda, A(\overline{n}_0\overline{n}_1)\rangle} \leqslant M_2(Q,\lambda)e^{\langle\lambda, A(\overline{n}_0)\rangle}$$

for $\overline{n}_1 \in Q$, $\overline{n}_0 \in \overline{N}_0$.

Proof. — Using the Iwasawa decomposition $G_0 = N_0AK_0$, one writes $\overline{n}_0 = n_0 \exp A(\overline{n}_0)k_0$, where $k_0 = h(\overline{n}_0) \in K_0$, and we have

$$\begin{aligned} A(\overline{n}_0\overline{n}_1) &= A(\overline{n}_0) + A(k_0\overline{n}_1) \\ &= A(\overline{n}_0) + A(k_0\overline{n}_1k_0^{-1}). \end{aligned}$$

Notice that $k_0\overline{n}_1k_0^{-1} \in \mathrm{Ad}(K_0)Q =: Q_1$ which is a compact set in Ω. Let

$$M_1(Q,\lambda) = \inf_{\overline{n}_1 \in Q_1} e^{\langle \lambda, A(\overline{n}_1)\rangle},$$
$$M_2(Q,\lambda) = \sup_{\overline{n}_1 \in Q_1} e^{\langle \lambda, A(\overline{n}_1)\rangle}. \qquad \square$$

Let $A_- = \exp(\mathfrak{a}^-)$ (cf. Section 4) and introduce the notation $a \to \infty$ to mean that $a \in A_-$ and for all $\alpha \in \Delta^+$ we have $\lim a^\alpha = 0$.

THEOREM 6.8. — *Assume $\Re\lambda \in \mathfrak{a}_+^* \cap \mathcal{E}$. Then*

$$\lim_{a\to\infty} a^{\lambda-\rho}\varphi_\lambda(a) = c_0(\lambda)c_\Omega(\lambda).$$

Proof. — For $\overline{n} \in \overline{N}$,

$$\lim_{a\to\infty} A(a^{-1}\overline{n}a) = 0,$$

thus by Proposition 6.3 and Proposition 6.6 it suffices to show that one can apply the dominated convergence theorem of Lebesgue. For $0 < \delta < 1$ we set

$$A(\delta) = \{a \in A_- \mid \forall\alpha \in \Delta^+,\ a^\alpha \leqslant \delta\},$$
$$\omega = \bigcup_{a\in A(\delta)} a^{-1}\Omega a.$$

By Theorem 4.2 $a^{-1}\Omega a$ is relatively compact since $a^{-1} \in \Gamma^0$. If $a^\alpha \leqslant b^\alpha$ for all $\alpha \in \Delta^+$, then $ba^{-1} \in \Gamma$ and $a^{-1}\Omega a \subset b^{-1}\Omega b$. It follows that ω is relatively compact in Ω. From Lemma 6.7 it follows that, for $\mu = \Re\lambda$, $\overline{n} = \overline{n}_0\overline{n}_1$, $a \in A(\delta)$,

$$|e^{\langle \rho-\lambda, A(a^{-1}\overline{n}a)\rangle}| \leqslant M_2(\overline{\omega}, \rho-\mu)e^{\langle \rho-\mu, A(a^{-1}\overline{n}_0a)\rangle}.$$

There exists ϵ, $0 < \epsilon < 1$, such that

$$\rho_0 - \epsilon\mu \in \mathfrak{a}_+^*.$$

Using the properties

$$A(\overline{n}_0) \in -{}^+\overline{\mathfrak{a}}$$
$$A(a\overline{n}_0a^{-1}) - A(\overline{n}_0) \in {}^+\overline{\mathfrak{a}}$$

for $a \in A^+$ (cf. [He84], p. 439), one shows that

$$\langle \mu - \rho_0, A(a^{-1}\overline{n}_0a)\rangle \geqslant (1-\epsilon)\langle \mu, A(a^{-1}\overline{n}_0a)\rangle \geqslant (1-\epsilon)\langle \mu, A(\overline{n}_0)\rangle.$$

It follows that

$$|e^{\langle\rho-\lambda,A(a^{-1}\overline{n}a)\rangle}e^{\langle\rho+\lambda,A(\overline{n})\rangle}| \leqslant M_2(\overline{\omega},\rho-\mu)e^{\langle\epsilon\mu+\rho,A(\overline{n}_0)\rangle}e^{\langle\mu+\rho,A(h(\overline{n}_0)\overline{n}_1)\rangle}$$

and the function on the right handside is integrable over $\overline{N}_0\times\Omega$. In fact, for $k_0\in K_0$,

$$A(k_0\overline{n}_1)=A(k_0\overline{n}_1k_0^{-1})$$

and

$$\int_\Omega e^{\langle\mu+\rho,A(k_0\overline{n}_1k_0^{-1})\rangle}d\overline{n}_1=\int_\Omega e^{\langle\mu+\rho,A(\overline{n}_1)\rangle}d\overline{n}_1.$$

Furthermore, for $\mu\in\mathfrak{a}_+^*$,

$$\int_{\overline{N}_0} e^{\langle\mu+\rho_0,A(\overline{n}_0)\rangle}d\overline{n}_0=c_0(\mu)<\infty. \qquad \square$$

In a completely analogous way we obtain

$$\lim_{t\to\infty} e^{\langle\lambda-\rho,tX_o\rangle}\varphi_\lambda(a\exp tX_o)=\varphi^0_{\lambda-\rho_1}(a)c_\Omega(\lambda), \tag{6.1}$$

for $a\in A$ and $\lambda\in\mathcal{E}$.

From (6.1) and Corollary 5.6 we deduce the following corollary using the W_0-invariance of the c-function c_0 for G_0/K_0.

COROLLARY 6.9.

$$c_\Omega(s\lambda)=c_\Omega(\lambda)$$

for all $s\in W_0$.

7. Spherical functions and spherical distributions.

The spherical functions φ_λ are related to H-spherical distributions associated with principal series representations of G. More precisely we will show that φ_λ is the restriction to S^0 of a H-spherical distribution Θ_λ.

We start by giving some definitions and results (cf [Ba88], [Óla87]). Recall that M denotes the centralizer of A in H which is also the centralizer of A in K (Lemma 5.1). For $\lambda\in\mathfrak{a}_\mathbb{C}^*$ let (π_λ,I_λ) be the representation of the principal series induced by the character $man \mapsto a^\lambda$ of the minimal parabolic subgroup MAN,

$$I_\lambda = \{f \in \mathcal{C}^\infty(G) \mid f(xman) = a^{\lambda-\rho} f(x)\},$$
$$\pi_\lambda(g)f(x) = f(g^{-1}x).$$

The formula

$$\langle f_1, f_2\rangle = \int_K f_1(k) f_2(k) dk$$

defines an invariant pairing on $I_\lambda \times I_{-\lambda}$. Let ξ_λ be the function defined on G by

$$\xi_\lambda(g) = 0, \text{ if } g \notin HAN,$$
$$\xi_\lambda(han) = a^{\lambda-\rho}.$$

LEMMA 7.1. — *For $\Re\lambda \in \mathcal{E}$, the function $k \mapsto \xi_{-\lambda}(k)$ is integrable over K. If $f \in I_\lambda$ then $h \mapsto f(h)$ is integrable over H and*

$$\int_K f(k)\xi_{-\lambda}(k) dk = \int_H f(h) dh.$$

Proof. — For $g \in HAN$,

$$\xi_\lambda(g) = a_H(g^{-1})^{\rho-\lambda}.$$

Therefore the first statement follows from Proposition 5.3. For $k \in K \cap HAN$, by writing

$$k^{-1} = n a_H(k^{-1}) h(k^{-1}),$$

we obtain

$$f(k)\xi_{-\lambda}(k) = a_H(k^{-1})^{2\rho} f\big(h(k^{-1})^{-1}\big).$$

By formula (5.3) the second statement follows. □

For $\Re\lambda \in \mathcal{E}$ the linear form

$$f \mapsto \langle f, \xi_{-\lambda}\rangle = \int_K f(k)\xi_{-\lambda}(k) dk$$

defines an element of $I'_{-\lambda}$, the dual of I_λ. This element still denoted by $\xi_{-\lambda}$ is H-invariant. The function $\lambda \mapsto \xi_{-\lambda} \in I'_{-\lambda}$ has a meromorphic continuation for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. For $f \in C_c^\infty(G)$, $\pi'_\lambda(f)\xi_\lambda \in I_\lambda$, where π'_λ denotes the contragredient representation of $\pi_{-\lambda}$. One defines, for $f \in C_c^\infty(G)$,

$$\Theta_\lambda(f) = \langle \pi'_\lambda(f)\xi_\lambda, \xi_{-\lambda} \rangle.$$

Then Θ_λ is an H-spherical distribution on G : it is an H-biinvariant distribution and there exists a character χ_λ of the algebra $\mathbb{D}(\mathcal{M})$ of invariant differential operators on $\mathcal{M}$ such that

$$D\Theta_\lambda = \chi_\lambda(D)\Theta_\lambda, \ D \in \mathbb{D}(\mathcal{M}).$$

THEOREM 7.2. — If $\Re\lambda \in \mathcal{E}$, and if $\operatorname{supp}(f) \subset S^0$, then

$$\Theta_\lambda(f) = \int_G f(x)\varphi_\lambda(x)dx.$$

Proof. — If $\Re(\langle \lambda - \rho, \alpha \rangle) > 0$, for $\alpha \in \Delta_+$, then ξ_λ is a continuous function on G (cf Lemma 4.1 in [Óla87]), and for $f \in C_c^\infty(G)$, $\pi'_\lambda(f)\xi_\lambda$ is the continuous function given by

$$\pi'_\lambda(f)\xi_\lambda(x) = \int_G f(y)\xi_\lambda(y^{-1}x)dy.$$

Assume that $\operatorname{supp}(f) \subset S^0$. For $x \in \Gamma = S^{-1}$, $x^{-1} \cdot \operatorname{supp}(f) \subset S \subset NAH$ (Theorem 4.2), and

$$\pi'_\lambda(f)\xi_\lambda(x) = \int_G f(y)a_H(x^{-1}y)^{\rho-\lambda}dy.$$

This integral is defined and analytic for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Therefore, by Lemma 7.1, if $\Re\lambda \in \mathcal{E}$,

$$\Theta_\lambda(f) = \int_H \Big(\int_G f(y)a_H(hy)^{\rho-\lambda}dy \Big) dh.$$

If $\mu = \Re\lambda$,

$$\int_H \int_G |f(y)|a_H(hy)^{\rho-\mu}dydh = \int_G |f(y)|\varphi_\mu(y)dy < \infty,$$

since φ_μ is continuous on S^0 (Theorem 5.3). By Fubini's theorem, it follows that

$$\Theta_\lambda(f) = \int_G f(y)\varphi_\lambda(y)dy. \qquad \square$$

8. Invariant Volterra kernels and the spherical Laplace transform.

We recall some definitions and results from [Fa91]. Let $\mathcal{M} = G/H$ be an ordered semisimple symmetric space. A *causal* kernel or *Volterra kernel* on $\mathcal{M}$ is a function on $\mathcal{M} \times \mathcal{M}$ which is continuous on $\{(x,y) \mid x \geqslant y\}$ and zero outside this set. We compose two such kernels F_1 and F_2 via the formula

$$F_1 \diamond F_2(x,y) = \int_{\mathcal{M}} F_1(x,z)F_2(z,y)dz,$$

where dz is an invariant measure on $\mathcal{M}$. This definition makes sense because $\mathcal{M}$ is globally hyperbolic. With respect to this multiplication the set $V(\mathcal{M})$ becomes an algebra, called the *Volterra algebra* of $\mathcal{M}$. A Volterra kernel is said to be *invariant* if

$$F(gx, gy) = F(x,y), \quad \forall g \in G.$$

The space $V(\mathcal{M})^\sharp$ of all invariant Volterra kernels is a commutative subalgebra of $V(\mathcal{M})$ by [Fa91], Théorème 1. An invariant kernel is determined by the function

$$f(x) = F(x, x_o), \quad x \in \mathcal{M}.$$

The function f is continuous on $S \cdot x_o$, H-invariant and supported on $S \cdot x_o$,

$$S \cdot x_o = \{x \in \mathcal{M} \mid x \geqslant x_o\}.$$

Conversely if f is an H-invariant continuous function on $S \cdot x_o$, we can define an invariant Volterra kernel F by, for $a, b \in G$,

$$\begin{aligned} F(a \cdot x_o, b \cdot x_o) &= f(b^{-1}a \cdot x_o), \text{ if } b^{-1}a \in S, \\ &= 0, \text{ otherwise.} \end{aligned}$$

With this identification the product $\diamond$ corresponds to the 'convolution'

$$f_1 \diamond f_2(x) = \int_{G/H} f_1(g^{-1}x) f_2(g \cdot x_o) d\dot{g}.$$

So the algebra $V(\mathcal{M})^\sharp$ becomes the algebra of continuous H-invariant functions on $S \cdot x_o$ with the above 'convolution' product.

The *spherical Laplace transform* of an invariant Volterra kernel F is defined by

$$\mathcal{L}F(\lambda) = \int_{\mathcal{M}} F(x, x_o) e^{\langle \rho - \lambda, A(x)\rangle} dx = \int_{\mathcal{M}} F(x, x_o) a_H(x)^{\rho - \lambda} dx,$$

whenever the integral converges. Note here that $F(x, x_o) \neq 0$ only for $x \geqslant x_o$, i.e., for $x \in S/H$. The corresponding formula for the H-invariant function on $S \cdot x_o$ that we will use in the following is

$$\mathcal{L}f(\lambda) = \int_{\mathcal{M}} f(x) e^{\langle \rho - \lambda, A(x) \rangle} dx = \int_{\mathcal{M}} f(x) a_H(x)^{\rho - \lambda} dx.$$

Let $\mathcal{D}(f)$ be the set of λ for which the integral converges absolutely.

PROPOSITION 8.1. — *Let $f_1, f_2 \in V(\mathcal{M})^\sharp$ be invariant causal kernels. Then $\mathcal{D}(f_1) \cap \mathcal{D}(f_2) \subseteq \mathcal{D}(f_1 \diamond f_2)$. For $\lambda \in \mathcal{D}(f_1) \cap \mathcal{D}(f_2)$ we have*

$$\mathcal{L}(f_1 \diamond f_2)(\lambda) = \mathcal{L}f_1(\lambda) \mathcal{L}f_2(\lambda).$$

Proof. — We will only prove the second statement as the first one follows in the same way noticing that

$$\int_{\mathcal{M}} |f_1 \diamond f_2(x) a_H(x)^{\rho - \lambda}| dx \leqslant \int_{\mathcal{M}} |f_1| \diamond |f_2|(x) a_H(x)^{\rho - \Re\lambda} dx.$$

Let $g \in S$ then a change of variables $x' = g^{-1}x$ together with (5.2) yields

$$\int_{\mathcal{M}} f(g^{-1}x) a_H(x)^{\rho - \lambda} dx = a_H(g)^{\rho - \lambda} \mathcal{L}f(\lambda).$$

Now we calculate using Fubini's Theorem

$$\begin{aligned}\mathcal{L}(f_1 \diamond f_2)(\lambda) &= \int_{\mathcal{M}} \int_{\mathcal{M}} f_1(g^{-1}x) f_2(g \cdot x_0) d(gH) a_H(x)^{\rho - \lambda} dx \\ &= \mathcal{L}f_1(\lambda) \int_{\mathcal{M}} f_2(g \cdot x_0) a_H(g)^{\rho - \lambda} d(gH) \\ &\mathcal{L}f_1(\lambda) \mathcal{L}f_2(\lambda).\end{aligned}$$

□

Next we want to compute the spherical Laplace transform using "polar coordinates", i.e., the decomposition HAH. For that we use the fact that the map

$$H/M \times A^- \ni (hM, a) \mapsto ha \cdot x_o \in S^o \cdot x_o$$

is a diffeomorphism onto a dense open subset and $S \cdot x_o \setminus HA^- \cdot x_o$ has measure zero. In what follows we may also replace A^- by $\exp c^o_{\max}$ and notice that the map is now a w_o-covering, where w_o is the order of the Weyl group W_0. Define

$$\tilde{\delta}(X) = \prod_{\alpha \in -\Delta^+} (\mathrm{sh}\langle \alpha, X\rangle)^{m_\alpha}, \quad X \in \mathfrak{a}^-$$

and

$$\delta(a) = \tilde{\delta}(\log a), \ a \in A^-$$

where m_α is the multiplicity of the root α. Replacing the Cartan involution by τ in the proof of [He84], Theorem I.5.8, one can prove the following integration formula :

$$\text{(8.1}a) \qquad \int_{S/H} f(x)dx = c\int_H \int_{\mathfrak{a}^-} f(h \exp XH)\tilde{\delta}(X)dXdh$$

$$\text{(8.1}b) \qquad = c\int_H \int_{A^-} f(haH)\delta(a)dadh,$$

where c is some positive constant depending only on the normalization of the measures. Since the above map is a diffeomorphism onto a dense open subset, whose complement has measure zero, this does hold for every integrable function f on $S \cdot x_o$.

PROPOSITION 8.2. — *Let $c > 0$ be the constant defined above. Let $f: S \cdot x_0 \to \mathbb{C}$ be continuous and H-invariant. If $\lambda \in \mathcal{D}(f)$ then φ_λ exists and*

$$\mathcal{L}f(\lambda) = c\int_{A^-} f(a)\varphi_\lambda(a)\delta(a)da.$$

Proof. — By the above it follows that

$$\int_{H/M \times A^-} |f(a)a_H(ha)^{\rho-\lambda}|\delta(a)d(hM)da < \infty.$$

By Fubini's theorem $h \mapsto a_H(ha)^{\rho-\lambda}$ has to be integrable for almost all $a \in A^-$. By our remark after Proposition 5.3 this shows that φ_λ exists. The Proposition now follows from (8.1). □

We now calculate the spherical Laplace transform in "horospherical coordinates" and relate it to the Abel transfom. For that we need the following integration formula (see Theorem 1.2 in [Óla87]) :

LEMMA 8.3. — *Let $f \in L^1(\mathcal{M})$ such that f is zero outside $NA \cdot x_o$. Then*

$$\int_{\mathcal{M}} f(m)dm = \int_N \int_A f(na \cdot x_o)a^{-2\rho}dadn.$$

We now define the *Abel transform* $\mathcal{A}f\colon A \to \mathbb{C}$ of an H-invariant function f on $S \cdot x_o$ by

$$\mathcal{A}(f)(a) = a^{-\rho} \int_N f(na)dn,$$

whenever the integral exists.

LEMMA 8.4. — *Let f be a continous H-invariant function on $S \cdot x_o$ (extended by zero outside $S \cdot x_o$) such that $n \mapsto f(na)$ is integrable on N for all $a \in A$. Let $L \subset c_{\max}$ be the convex hull of* $\log(\mathrm{Supp}(f|_{S\cap A}))$. *Then*

$$\log\big(\mathrm{Supp}(\mathcal{A}f)\big) \subseteq L + c_{\min}.$$

Proof. — Let $X \in c_{\max}$, $X \neq 0$ and let conv $(W_0 X)$ be the convex hull of $W_0 \cdot X$. Then the convexity theorem of Neeb ([Ne91], Theorem IV.14) states that

$$A(H \exp X) = \text{conv }(W_0 X) + c_{\min}.$$

Let now $a \in A$ be such that $\mathcal{A}f(a) \neq 0$. Then we can find $n \in N$, $h \in H$ and $b \in \mathrm{Supp}(f|_{S\cap A})$ such that

$$na \cdot x_o = hb \cdot x_o.$$

This follows from the fact that $\mathrm{Supp}(f) = H\mathrm{Supp}(f|_{S\cap A})$. But from this we get $a = a_H(hb)$ or

$$\log a \in A\big(\mathrm{Supp}(f)\big) \subseteq L + c_{\min}. \qquad \square$$

By Lemma 8.3, Lemma 8.4 and the left N-invariance of $a_H(x)^{\rho-\lambda}$ we get :

PROPOSITION 8.5. — *Let f be an H-invariant function on $S \cdot x_0$ and $\lambda \in \mathcal{D}(f)$. Then*

$$\mathcal{L}(f)(\lambda) = \int_{\exp c_{\max}} a^{-\lambda}\mathcal{A}f(a)da = \mathcal{L}_A(\mathcal{A}f)(\lambda),$$

where $\mathcal{L}_A$ is the Euclidean Laplace transform on A with respect to the cone $c_{\max}$.

The Abel transform can be split up further according to the semidirect product decomposition $N = N_1 N_0$. Set

$$\mathcal{A}_1 f(g_0) = a_H(g_0)^{-\rho_1} \int_{N_1} f(n_1 g_0) dn_1$$

for $g_0 \in G_0$. Then obviously $\mathcal{A}_1 f$ is K_0-biinvariant and

$$\mathcal{A}f(a) = a^{-\rho_o} \int_{N_0} \mathcal{A}_1(f)(n_0 a) dn_0.$$

Denote by $\mathcal{A}_0$ the Abel transform with respect to the *Riemannian* symmetric space G_0/K_0. Then

$$\mathcal{A}f(a) = \mathcal{A}_0(\mathcal{A}_1 f)(a)$$

for all continuous, H-invariant functions $f: S \cdot x_0 \to \mathbb{C}$, whenever the above integrals converge, and all $a \in A$. Since it is well known how to invert the transform $\mathcal{A}_0$, at least for "good" functions, the inversion of the Abel transform associated to the ordered space reduces to invert the transform $\mathcal{A}_1$.

PROPOSITION 8.6. — *Let $f: S \cdot x_0 \to \mathbb{C}$ be continuous, H-invariant and such that the Abel transform exists. Then its Abel transform is invariant under W_0,*

$$\mathcal{A}f(sa) = \mathcal{A}f(a) \quad \forall a \in A, s \in W_0.$$

Proof. — This follows from $\mathcal{A}_1 f(sa) = \mathcal{A}_1 f(a)$, since W_0 is the Weyl group of G_0/K_0. □

COROLLARY 8.7.

(i) $\mathcal{L}f(s\lambda) = \mathcal{L}f(\lambda)$ *for all H-invariant functions $f: S \cdot x_0 \to \mathbb{C}$ and $s \in W_0$.*

(ii) $\varphi_{s\lambda} = \varphi_\lambda$ *for all $s \in W_0$.*

Proof.

(i) This is an immediate consequence of Propositions 8.5 and 8.6.

(ii) As φ_λ is continuous and H-invariant, we only have to show that $\varphi_\lambda(a) = \varphi_{s\lambda}(a)$ for all $a \in A^-$. Let $f \in C_c^\infty(A^-)$. Then we can extend

f to a H-invariant function on $S^o \cdot x_o$ by $F(ha \cdot x_o) = f(a \cdot x_o)$. Apply Proposition 8.2 to F and notice that $F|_{A^-} = f$. The claim follows now from the fact that δ vanishes nowhere in $\mathfrak{a}^-$. □

Thus the Laplace transform yields another proof of Corollary 5.6.

9. Inversion formulas for symmetric spaces of Olshanskii type.

In this section we assume that $\mathcal{M} = G/H$ is a symmetric space of Olshanskii type, G is a complex group, H a real form of G (cf. Section 3). We will prove an inversion formula for the spherical Laplace transform and for the Abel transform. By Theorem 7.2 φ_λ is the restriction to S^0 of an H-spherical distribution Θ_λ . In the present case Delorme gave an explicit formula for Θ_λ ([De90], Théorème 3), which can be stated in the following way :

THEOREM 9.1. — *Define $\Delta : A \to \mathbb{R}$ by*

$$\Delta(a) = \prod_{\alpha \in -\Delta^+} \operatorname{sh}\langle \alpha, \log a \rangle.$$

For $w \in W$ let $\epsilon(w)$ be the determinant of w as a linear transformation on $\mathfrak{a}$. Then, for $\lambda \in \mathcal{E} + i\mathfrak{a}^*$, $a \in S^o \cap A$

$$\varphi_\lambda(a) = 2^{\#\Delta^+} \gamma \frac{\sum_{w \in W_0} \epsilon(w) a^{-w\lambda}}{\left(\prod_{\alpha \in \Delta^+} \langle \lambda, \alpha \rangle\right) \Delta(a)}, \tag{9.1}$$

with a constant γ depending only on $\mathcal{M}$.

In the special case $G = \mathrm{GL}(n, \mathbb{C})$, $H = \mathrm{U}(p, q)$, the preceding formula has been proved by direct computation of the integral defining φ_λ ([Fa87], Théorème 7).

As $G_0/K \cap H = K_{\mathbb{C}}/K$, the Harish-Chandra c-function for this space is well known to be

$$c_0(\lambda) = \frac{\gamma_0}{\prod_{\alpha \in \Delta_0^+} \langle \lambda, \alpha \rangle}$$

cf. [He84], p. 432. This, Theorem 6.8 and the theorem of Delorme gives us the c-functions related to $\mathcal{M}$:

LEMMA 9.2. — *Let* $\gamma_1 = \gamma/\gamma_0$ *and* $c(\lambda) = c_\Omega(\lambda)c_0(\lambda)$. *Then*

$$c(\lambda) = \frac{\gamma}{\prod_{\alpha\in\Delta^+}\langle\lambda,\alpha\rangle}, \qquad c_\Omega(\lambda) = \frac{\gamma_1}{\prod_{\alpha\in\Delta_1}\langle\lambda,\alpha\rangle}.$$

We also get

COROLLARY 9.3. — *The function*

$$(\lambda, a) \mapsto c(\lambda)^{-1}\varphi_\lambda(a)$$

extends to a function on $\mathfrak{a}_{\mathbb{C}}^* \times S^o \cap A$, *holomorphic in* λ *and analytic in* a.

Denote by $\mathcal{C}_c^\infty(H\backslash S^o/H)$ the space of H-biinvariant smooth functions f on S^o, such that $f|_{S^o\cap A}$ has compact support. We view these functions also as H-invariant functions on $\mathcal{M}$ that are left H-invariant. Let $c_1 = 2^{\#\Delta^+}c$, c being the constant in the integral formula (8.1a).

THEOREM 9.4. — *Let* $f \in \mathcal{C}_c^\infty(H\backslash S^o/H)$ *be such that* $\mathrm{Supp}(f|_{S^o\cap A})$ *is contained in a ball of radius* $R > 0$. *Then*

$$\lambda \mapsto c(\lambda)^{-1}\mathcal{L}(f)(\lambda)$$

extends to a holomorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ *given by*

$$c(\lambda)^{-1}\mathcal{L}(f)(\lambda) = c_1\int_{S\cap A} f(a)\Delta(a)a^{-\lambda}da. \tag{9.2}$$

Furthermore for all $N \in \mathbb{N}$ *there exists a constant* $C_N > 0$ *such that, for* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,

$$|c(\lambda)^{-1}\mathcal{L}(f)(\lambda)| \leqslant C_N(1+|\lambda|)^{-N}e^{R|\mathrm{Re}\lambda|}. \tag{9.3}$$

Proof. — Obviously the last statement follows from the formula for $c(\lambda)^{-1}\mathcal{L}(f)(\lambda)$ by using classical estimates. It is clear that in the notation of Section 8 we have

$$\Delta(a)^2 = \delta(a)$$

since in this case all the multiplicities are two. Furthermore

$$\Delta(w\cdot a) = \epsilon(w)\Delta(a),\ \ w \in W_0.$$

By Proposition 8.2 and Theorem 9.1, for $\lambda \in \mathcal{E} + i\mathfrak{a}^*$,

$$\begin{aligned} c(\lambda)^{-1}\mathcal{L}f(\lambda) &= c_1 \sum_{w\in W_0} \int_{A^-} f(a)\Delta(a)\epsilon(a)a^{-w\lambda}da \\ &= c_1 \sum_{w\in W_0} \int_{A^-} f(wa)\Delta(wa)a^{-w\lambda}da \\ &= c_1 \int_{\exp c_{\max}} \Delta(a)f(a)a^{-\lambda}da. \end{aligned}$$

The theorem now follows since the last integral is holomorphic in λ. □

THEOREM 9.5. — *Let f be an H-invariant smooth function on $S^o \cdot x_o$ such that $f|_{S^o\cap A}$ has compact support. Then there exists a constant $c_2 > 0$ only depending on the normalization of the measures such that for $a \in S^o \cap A$*

$$f(a) = c_2 \int_{\mathfrak{a}^*} \mathcal{L}(f)(i\lambda)\varphi_{-i\lambda}(a)\frac{d\lambda}{c(i\lambda)c(-i\lambda)}.$$

Proof. — By viewing f as a smooth function on A that vanishes outside $S^o \cap A$ we can write (9.2) as

$$c(\lambda)^{-1}\mathcal{L}f(\lambda) = c_1 \int_A f(a)\Delta(a)a^{-\lambda}da.$$

By the Fourier inversion formula for the abelian group A we get

$$cf(a)\Delta(a) = \int_{\mathfrak{a}^*} c(i\lambda)^{-1}\mathcal{L}f(i\lambda)a^{i\lambda}d\lambda. \tag{9.4}$$

Now $c(w\lambda) = \epsilon(w)c(\lambda)$ for all $w \in W_0$, as follows easily from the formula for the c-function. As $\mathcal{L}(f)(w\lambda) = \mathcal{L}(f)(\lambda)$, we now get :

$$\begin{aligned} &\Delta(a)\int_{\mathfrak{a}^*} \mathcal{L}(f)(i\lambda)\varphi_{-i\lambda}(a)\frac{d\lambda}{c(i\lambda)c(-i\lambda)} \\ &= c_3 \sum_{w\in W_0} \epsilon(w) \int_{\mathfrak{a}^*} c(i\lambda)^{-1}\mathcal{L}(f)(i\lambda)a^{iw\lambda}d\lambda \\ &= c_3|W_0| \int_{\mathfrak{a}^*} c(i\lambda)^{-1}\mathcal{L}(f)(i\lambda)a^{i\lambda}d\lambda \\ &= c_4 f(a)\Delta(a). \end{aligned}$$

Now the theorem follows with $c_2 = \dfrac{1}{c_4}$. □

As $c(\lambda)^{-1}$ is a polynomial function on $\mathfrak{a}_{\mathbb{C}}^*$ it determines a differential operator Λ on A given by

$$\Lambda a^\lambda = c(\lambda)^{-1} a^\lambda,$$

i.e., $\Lambda = \gamma^{-1} \prod_{\alpha \in \Delta^+} H_\alpha$, where H_α is the coroot corresponding to α, $\langle \lambda, H_\alpha \rangle = \langle \lambda, \alpha \rangle$, interpreted as a differential operator on A.

THEOREM 9.6. — *Let $f \in C_c^\infty(H \backslash S^o / H)$. Then*

$$c\Delta(a) f(a) = \Lambda \mathcal{A} f(a).$$

Proof. — By Proposition 8.5 we have $\mathcal{L}(f)(\lambda) = \int_A a^{-\lambda} \mathcal{A} f(a) da$. Thus

$$\mathcal{A} f(a) = \int_{\mathfrak{a}^*} \mathcal{L}(f)(\mu + i\lambda) a^{\mu + i\lambda} d\lambda$$

for all $\mu \in \mathcal{E}$. Now (9.3) shows that we are allowed to move the integration path in (9.4) to get

$$\begin{aligned} cf(a)\Delta(a) &= \int_{\mathfrak{a}^*} c(i\lambda)^{-1} \mathcal{L}(f)(i\lambda) a^{i\lambda} d\lambda \\ &= \int_{\mathfrak{a}^*} c(\mu + i\lambda)^{-1} \mathcal{L}(f)(\mu + i\lambda) a^{\mu + i\lambda} d\lambda. \end{aligned}$$

Take $\mu \in \mathcal{E}$ such that $< \alpha, \mu > \neq 0$ for all α. Then, as $c(\mu + i\lambda)^{-1}$ is a non-zero polynomial, we get by (9.3)

$$\forall N,\ \exists C(N),\ \forall \lambda \in \mathfrak{a}^*,\ |\mathcal{L}(f)(\mu + i\lambda)| \leqslant C(N)(1 + |\mu + i\lambda|)^{-N} e^{R|\mu|}.$$

This shows that we are allowed to exchange the order of differentiation with respect to a and integration. From this we get

$$\begin{aligned} \Lambda \mathcal{A} f(a) &= \int_{\mathfrak{a}^*} \mathcal{L}(f)(\mu + i\lambda) \Lambda a^{\mu + i\lambda} d\lambda \\ &= \int_{\mathfrak{a}^*} c(\mu + i\lambda)^{-1} \mathcal{L}(f)(\mu + i\lambda) a^{\mu + i\lambda} d\lambda \\ &= cf(a)\Delta(a). \end{aligned}$$

□

10. Inversion formulas for spaces of rank 1.

In this section we consider the symmetric space

$$\mathcal{M} = \mathrm{SO}_o(1,n)/\mathrm{SO}_o(1,n-1) = \mathrm{SO}(1,n)/\mathrm{SO}(1,n-1), \quad n \geqslant 2.$$

The involution τ is given by

$$\tau(g) = \mathbf{1}_{n,1} g \mathbf{1}_{n,1},$$

where

$$\mathbf{1}_{n,1} = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -1 \end{pmatrix}.$$

The manifold $\mathcal{M}$ can be identified with the hyperboloid with one sheet defined by the equation

$$-x_0^2 + x_1^2 + \ldots + x_n^2 = 1,$$

with the base point $\mathbf{1}H$ identified with $(0, \ldots, 0, 1)$.

Let C be the cone in $\mathfrak{q}$

$$\mathfrak{q} = \left\{ \begin{pmatrix} 0 & v \\ v^\top & 0 \end{pmatrix} \middle| v \in \mathbb{R}^n \right\}$$

defined by

$$v_0^2 - v_1^2 - \ldots - v_{n-1}^2 \geqslant 0, \quad v_0 \geqslant 0.$$

This cone defines on $\mathcal{M}$ a global invariant causal structure, and, for the corresponding ordering, $x \geqslant y$ if and only if

$$-x_0 y_0 + x_1 y_1 + \ldots + x_n y_n \geqslant 1, \quad x_0 \geqslant y_0.$$

The Cartan involution θ defined by

$$\theta(g) = (g^{-1})^\top$$

commutes with τ, and

$$\mathfrak{a} = \mathfrak{p} \cap \mathfrak{q} = \left\{ X_t = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\},$$

$$A = \left\{ a_t = \begin{pmatrix} \operatorname{ch} t & 0 & \operatorname{sh} t \\ 0 & \mathbf{1}_{n-1} & 0 \\ \operatorname{sh} t & 0 & \operatorname{ch} t \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

We choose the positive root

$$\alpha(X_t) = -t.$$

Then

$$\mathfrak{n} = \mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & v^\top & 0 \\ v & 0 & v \\ 0 & -v^\top & 0 \end{pmatrix} \middle| v \in \mathbb{R}^{n-1} \right\},$$

$$N = \left\{ n(v) = \begin{pmatrix} 1 + \frac{1}{2}||v||^2 & v^\top & \frac{1}{2}||v||^2 \\ v & \mathbf{1}_{n-1} & v \\ -\frac{1}{2}||v||^2 & -v^\top & 1 - \frac{1}{2}||v||^2 \end{pmatrix} \middle| v \in \mathbb{R}^{n-1} \right\}.$$

We identify $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}$ by $\mathbb{C} \ni z \mapsto -z\alpha \in \mathfrak{a}_\mathbb{C}^*$, and then $\rho = -(n-1)/2$.

We will give an explicit formula for the spherical function

$$\varphi_\lambda(a_t) = \int_H a_H(ha_t)^{\rho-\lambda} dh.$$

If $g = n(v)a_t h$ then

$$t = A(g) = \log(x_n + x_0), \qquad a_H(g)^{\rho-\lambda} = (x_n + x_0)^{-\frac{n-1}{2}-\lambda},$$

if $gH = (x_0, ..., x_n)$. Note that

$$K_0 = K \cap H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| k \in \mathrm{SO}(n-1) \right\},$$

hence we have $A(hk_0a_t) = A(ha_t)$ for all $k_0 \in K_0$. The space

$$\mathfrak{b} = \left\{ \begin{pmatrix} 0 & \theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$$

is a maximal abelian subspace in $\mathfrak{h} \cap \mathfrak{p}$ and induces a Cartan decomposition of H,

$$H = K_0 \overline{B^+} K_0,$$

where

$$B = \left\{ b_\theta = \begin{pmatrix} \operatorname{ch}\theta & \operatorname{sh}\theta & 0 \\ \operatorname{sh}\theta & \operatorname{ch}\theta & 0 \\ 0 & 0 & \mathbf{1}_{n-1} \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}, \qquad B^+ = \{b_\theta \mid \theta > 0\}.$$

Now it follows from [He84], Theorem I.5.8 that

$$\begin{aligned}\varphi_\lambda(a_t) &= \int_{B^+} e^{\langle\rho-\lambda, A(b_\theta a_t)\rangle} (\operatorname{sh}\theta)^{n-2}\, d\theta \\ &= \int_0^\infty (\operatorname{ch} t + \operatorname{sh} t \operatorname{ch}\theta)^{-\frac{n-1}{2}-\lambda} (\operatorname{sh}\theta)^{n-2}\, d\theta.\end{aligned} \tag{10.1}$$

From [Er53a] p. 155, we now get :

PROPOSITION 10.1. — *The integral converges for $t > 0$, $\Re\lambda > \frac{n-3}{2} = -\rho - 1$ and*

$$\varphi_\lambda(a_t) = \gamma_n \frac{\Gamma(\lambda - \frac{n-3}{2})}{\Gamma(\lambda + \frac{n-1}{2})(\operatorname{sh} t)^{\frac{n}{2}-1}} Q^{\frac{n}{2}-1}_{\lambda-\frac{1}{2}}(\operatorname{ch} t) \tag{10.2a}$$

$$= 2^{n-2}\Gamma\left(\frac{n-1}{2}\right) \frac{\Gamma\left(\lambda - \frac{n-3}{2}\right)}{\Gamma\left(\lambda+1\right)} (2cht)^{-\lambda-\frac{n-1}{2}} \cdot {}_2F_1\left(\frac{\lambda+\frac{n+1}{2}}{2}, \frac{\lambda+\frac{n-1}{2}}{2}, \lambda+1, \frac{1}{\operatorname{ch}^2 t}\right), \tag{10.2b}$$

where Γ is the usual Γ-function, Q^μ_ν is the usual Legendre function of the second kind, ${}_2F_1$ the hypergeometric function and γ_n is a constant depending only on n.

In particular, for $n = 2$,

$$\varphi_\lambda(a_t) = Q_{\lambda-\frac{1}{2}}(\operatorname{ch} t),$$

and, for $n = 3$,

$$\varphi_\lambda(a_t) = \frac{1}{\lambda}\frac{1}{\operatorname{sh} t} e^{-\lambda t}.$$

In this case $c_0 = 1$. (10.2b) then gives

$$c_\Omega(\lambda) = 2^{n-2}\Gamma\left(\frac{n-1}{2}\right) \frac{\Gamma\left(\lambda - \frac{n-3}{2}\right)}{\Gamma\left(\lambda+1\right)}. \tag{10.3}$$

The c-function can also be computed by using the integral formula (6.1). The subgroup $\overline{N}$ can be described as

$$\overline{N} = \left\{ \overline{n}(v) = \begin{pmatrix} 1+\frac{1}{2}\|v\|^2 & v^\top & -\frac{1}{2}\|v\|^2 \\ v & \mathbf{1}_{n-1} & -v \\ \frac{1}{2}\|v\|^2 & v^\top & 1-\frac{1}{2}\|v\|^2 \end{pmatrix} \middle| v \in \mathbb{R}^{n-1} \right\}.$$

The domain Ω is the unit ball $\Omega = \{\overline{n}(v) \mid \|v\| < 1\}$. According to the normalization of Proposition 6.6 we get

$$c_\Omega(\lambda) = \frac{2^{n-1}}{\omega_{n-1}} \int_{\|v\|<1} (1 - \|v\|^2)^{\lambda - \frac{n-1}{2}} dv, \tag{10.4}$$

where ω_{n-1} denotes the volume of the unit sphere in $\mathbb{R}^{n-1}$. The integral converges for $\Re\lambda > \dfrac{n-3}{2} = -\rho - 1$, and

$$\begin{aligned} c_\Omega(\lambda) &= 2^{n-2} B\left(\lambda - \frac{n-3}{2}, \frac{n-1}{2}\right) \\ &= 2^{n-2}\Gamma\left(\frac{n-1}{2}\right) \frac{\Gamma\left(\lambda - \frac{n-3}{2}\right)}{\Gamma(\lambda+1)}. \end{aligned}$$

The above formula for φ_λ in terms of the hypergeometric function gives

COROLLARY 10.2. — *The function $c_\Omega(\lambda)^{-1}\varphi_\lambda(a_t)$ extends to a function on $\mathfrak{a}_\mathbb{C}^* \times \exp c_{\max}$, holomorphic in λ for $\Re\lambda > -1$ and analytic in t.*

An H-invariant function on $\{x \in \mathcal{M} \mid x \geqslant \mathbf{1}H\}$ can be written

$$f(x) = f^\sharp(x_n),$$

where $f^\sharp$ is a function defined on $[1, \infty[$, and the corresponding invariant causal kernel F on $\mathcal{M}$ can be written, for $x \geqslant y$,

$$F(x, y) = f^\sharp(-x_0y_0 + x_1y_1 + \ldots + x_ny_n).$$

Then the spherical Laplace transform of f takes the following form

$$\mathcal{L}f(\lambda) = \omega_{n-1} \int_0^\infty \varphi_\lambda^\sharp(\operatorname{ch} t) f^\sharp(\operatorname{ch} t)(\operatorname{sh} t)^{n-1} dt. \tag{10.5}$$

For computing the Abel transform of f we note that

$$f(n(v)a_tH) = f^\sharp\left(\operatorname{ch} t - \frac{1}{2}\|v\|^2 e^t\right),$$

and hence

$$\begin{aligned} \mathcal{A}f(a_t) &= e^{\frac{n-1}{2}t} \int_{\|v\| \leqslant 1 - e^{-t}} f^\sharp\left(\operatorname{ch} t - \frac{1}{2}\|v\|^2 e^t\right) dv \\ &= \omega_{n-1} \int_0^{2\operatorname{sh}\frac{t}{2}} f^\sharp\left(\operatorname{ch} t - \frac{1}{2}r^2\right) r^{n-2} dr \\ &= \omega_{n-1} \int_0^t f^\sharp(\operatorname{ch}\tau)(2\operatorname{ch} t - 2\operatorname{ch}\tau)^{\frac{n-3}{2}} \operatorname{sh}\tau d\tau. \end{aligned}$$

For inverting the Abel transform, we consider the Riemann-Liouville transform

$$I_\alpha f(r) = \frac{1}{\Gamma(\alpha)} \int_0^r f(s)(r-s)^{\alpha-1} ds$$

for $\alpha > 0$. It satisfies the following properties (cf. [Er53b], Chapter XIII) :

$$I_\alpha \circ I_\beta = I_{\alpha+\beta}, \quad \forall \alpha, \beta > 0, \tag{10.6a}$$

$$\left(\frac{d}{dr}\right)^m (I_m \phi) = \phi \quad \forall m = 0, 1, 2, \dots . \tag{10.6b}$$

Note that for $t \geqslant 0$

$$c_n \mathcal{A}f(a_t) = I_{\frac{n-1}{2}} \phi(\operatorname{ch} t - 1),$$

where $\phi(r) = f^\sharp(r+1)$, and $c_n^{-1} = 2^{\frac{n-1}{2}} \omega_{n-1}$. Now (10.6b) implies, for $n = 2m+1$,

$$f^\sharp(\operatorname{ch} t) = c_n \left(\frac{1}{\operatorname{sh} t} \frac{d}{dt}\right)^m \mathcal{A}f(a_t), \tag{10.7a}$$

and for $n = 2m$

$$f^\sharp(\operatorname{ch} t) = c_n \frac{1}{\sqrt{\pi}} \left(\frac{1}{\operatorname{sh} t} \frac{d}{dt}\right)^m \int_0^t \mathcal{A}f(a_\tau)(\operatorname{ch} t - \operatorname{ch} \tau)^{-\frac{1}{2}} \operatorname{sh} \tau d\tau. \tag{10.7b}$$

If $\mu = \Re\lambda > \frac{n-3}{2}$ then $e^{-\mu t}\mathcal{A}f(a_t)$ is a Schwartz function. By Proposition 8.5

$$\mathcal{L}f(\mu + i\lambda) = \int_0^\infty e^{-i\lambda t} e^{-\mu t} \mathcal{A}f(a_t) dt.$$

Thus

$$\mathcal{A}f(a_t) = \frac{1}{2\pi} \int_0^\infty \mathcal{L}(f)(\mu + i\lambda) e^{\mu + i\lambda} d\lambda.$$

Combined with (10.7) this gives the following theorem :

THEOREM 10.3. — *Let $f \in C_c^\infty(H\backslash S^o/H)$. Let $\mu > \frac{n-3}{2}$. Then*

$$f(a_t) = f^\sharp(\operatorname{ch} t) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{L}(f)(\mu + i\nu) \psi_n(t, \mu + i\nu) d\nu,$$

where

$$\psi_{2m+1}(t,\lambda) = C_{2m+1}\left(\frac{1}{\operatorname{sh} t}\frac{d}{dt}\right)^m e^{\lambda t}$$

and

$$\psi_{2m}(t,\lambda) = C_{2m}\left(\frac{1}{\operatorname{sh} t}\frac{d}{dt}\right)^m \int_0^t e^{\lambda t}(\operatorname{ch} t - \operatorname{ch}\tau)^{-\frac{1}{2}} \operatorname{sh}\tau d\tau$$

for suitable constants C_n.

Earlier versions of this theorem can be found in the literature. A Laplace transform associated with the Legendre functions of the second kind has been introduced in [CK72]. In [Vi80] this transform is related to the harmonic analysis of the unit disc. A more general Laplace-Jacobi transform associated with the Jacobi functions of the second kind is studied in [Mi83].

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