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On the zeros of some polynomials with combinatorial coefficients

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Abstract

We consider two general classes of second-order linear recurrent sequences and the polynomials whose coefficients belong to a sequence in either of these classes. We show for each such sequence $\{a_i\}_{i\geq 0}$ that the polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ always has the smallest possible number of real zeros, that is, none when the degree is even and one when the degree is odd. Among the sequences then for which this is true are the Motzkin, Riordan, Schröder, and Delannoy numbers.

Keywords: zeros of polynomials, Motzkin number, Schröder number

MSC: 11C08, 13B25

1. Introduction

Garth, Mills, and Mitchell [3] considered the Fibonacci coefficient polynomial $p_n(x) = F_1x^n + F_2x^{n-1} + \cdots + F_nx + F_{n+1}$ and showed that it has no real zeros if n is even and exactly one real zero if n is odd. Later, Mátyás [5, 6] extended this result to polynomials whose coefficients are given by more general second order recurrences (having constant coefficients), and Mátyás and Szalay [7] showed the same holds true for the Tribonacci coefficient polynomials $q_n(x) = T_2x^n + T_3x^{n-1} + \cdots + T_{n+1}x + T_{n+2}$. The latter result has been extended to k-Fibonacci polynomials by Mansour and Shattuck [4].

In the apparent absence of a general criterion for determining when a polynomial having real coefficients has the smallest possible number of real zeros, one might wonder as to what other sequences a_i for which this result holds true for the polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$. Here, we consider this question for sequences belonging to two general classes and show that it holds in all cases. Among the sequences belonging to these classes are the Motzkin [9], Riordan [1], Schröder [2], and Delannoy [10].

We note that the sequences under consideration in the current paper are all given by second order linear recurrences, but with variable instead of constant coefficients. Thus, instead of multiplying $f(x)$ by a characteristic polynomial to obtain another polynomial whose coefficients are mostly zero (as was done in [3] and in subsequent papers in the case when $a_i = F_i$ for all i), we first apply a different linear operator to f, namely, one that is of a first-order differential nature. This yields a differential equation for f which can then be used to express it in an integral form that we find more convenient.

Recall that the Motzkin numbers m_n and the Riordan numbers r_n are given by

$$(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2}, \qquad n \geq 2,$$

with $m_0 = m_1 = 1$, and by

$$(n+1)r_n = (n-1)(2r_{n-1} + 3r_{n-2}), \qquad n \geq 2,$$

with $r_0 = 1$ and $r_1 = 0$. See entries A001006 and A005043 in OEIS [8].

Recall that the (little) Schröder numbers s_n and the (central) Delannoy numbers d_n are given by

$$(n+1)s_n = 3(2n-1)s_{n-1} - (n-2)s_{n-2}, \qquad n \geq 2,$$

with $s_0 = s_1 = 1$, and by

$$nd_n = 3(2n-1)d_{n-1} - (n-1)d_{n-2}, \qquad n \geq 2,$$

with $d_0 = 1$ and $d_1 = 3$. See entries A001003 and A001850 in [8].

We will prove the following result in the next two sections.

Theorem 1.1. *If a_i denotes any one of the sequences m_i, r_i, s_i, or d_i, then the polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$, $n \geq 2$, has no real zeros if n is even and one real zero if n is odd.*

The first two parts of Theorem 1.1 are shown in the next section as special cases of a more general result, while the last two parts are shown in a comparable manner in the third section.

2. Motzkin family polynomials

Let u_n, $n \geq 0$, denote the sequence defined by the recurrence

$$(n+a)u_n = (2(n-1)+b)u_{n-1} + 3(n-1)u_{n-2}, \qquad n \geq 2, \tag{2.1}$$

with the initial values $u_0 = 1$ and $u_1 = c$, where a, b, and c are constants. Note that u_n reduces to the Motzkin sequence when $a = 2$, $b = 3$, $c = 1$ and to the Riordan sequence when $a = 1$, $b = c = 0$. Let

$$f_n(x) = \sum_{i=0}^{n} u_i x^i, \qquad n \geq 0.$$

We will need the following integral representation of $f_n(x)$.

Lemma 2.1. *If $-1 < x < 0$, then*

$$f_n(x) = j(x)^{-1} \left(\int_{x_o}^{x} \frac{j(t)h_n(t)}{t(1+t)(1-3t)} dt + j(x_o) f_n(x_o) \right), \tag{2.2}$$

where $-1 < x_o < 0$ is any fixed number,

$$j(x) = |x|^a (1+x)^{\frac{3-a-b}{4}} (1-3x)^{\frac{1-3a+b}{4}},$$

and

$$h_n(x) = a + ((1+a)c - b)x - (n+a+1)u_{n+1}x^{n+1} - (3n+3)u_n x^{n+2}.$$

Proof. Let $f = f_n(x)$. By the recurrence (2.1), we have

$$\begin{aligned}
&xf' + af - 2x^2 f' - bxf - 3x^3 f' - 3x^2 f \\
&\quad = \sum_{i=1}^{n} i u_i x^i + a \sum_{i=0}^{n} u_i x^i - 2 \sum_{i=2}^{n+1} (i-1) u_{i-1} x^i - b \sum_{i=1}^{n+1} u_{i-1} x^i \\
&\qquad - 3 \sum_{i=3}^{n+2} (i-2) u_{i-2} x^i - 3 \sum_{i=2}^{n+2} u_{i-2} x^i \\
&\quad = a + ((1+a)c - b)x - ((2n+b)u_n + 3nu_{n-1})x^{n+1} - (3n+3)u_n x^{n+2} \\
&\qquad + \sum_{i=2}^{n} \left[(i+a)u_i - (2(i-1)+b)u_{i-1} - 3(i-1)u_{i-2} \right] x^i \\
&\quad = a + ((1+a)c - b)x - (n+1+a)u_{n+1}x^{n+1} - (3n+3)u_n x^{n+2},
\end{aligned}$$

where the prime denotes differentiation. The final equality may be rewritten in the form

$$f_n'(x) + \frac{a - bx - 3x^2}{x(1+x)(1-3x)} f_n(x) = \frac{h_n(x)}{x(1+x)(1-3x)}, \tag{2.3}$$

where $h_n(x)$ is as given. Note that, by partial fractions, we have

$$\frac{a - bx - 3x^2}{x(1+x)(1-3x)} = \frac{a}{x} + \frac{3-a-b}{4(1+x)} - \frac{3-9a+3b}{4(1-3x)},$$

which gives the antiderivative

$$\int \frac{a-bx-3x^2}{x(1+x)(1-3x)}dx = \log\left(|x|^a|1+x|^{\frac{3-a-b}{4}}|1-3x|^{\frac{1-3a+b}{4}}\right).$$

Formula (2.2) now follows from solving the first order linear differential equation (2.3) on the open interval $(-1,0)$ by the usual method. $\square$

In the next two lemmas, we assume that the constants a, b, and c used in defining u_n are non-negative real numbers such that $a \le b+1$ and $c \le \frac{b}{a+1}$.

Lemma 2.2. *If $n \ge 2$ is even, then the polynomial $f_n(x) = \sum_{i=0}^{n} u_i x^i$ has no zeros on the interval $(-\infty,-1]$.*

Proof. First note that $f_2(-1) > 0$ if and only if $a+5 > (a-b)c$. The latter inequality clearly holds if $a-b \le 0$. It also holds if $a-b>0$, since in this case we have

$$(a-b)c \le \frac{(a-b)b}{a+1} \le \frac{a^2}{4(a+1)} < a+5.$$

Let $k_n(x) = f_n(-x)$. Observe that

$$k_2(x) = 1 - u_1x + u_2x^2 > 0, \qquad x \ge 1,$$

since $k_2(1) > 0$ and

$$k_2'(x) = 2u_2x - u_1 \ge 2u_2 - u_1 = \frac{(2b-a+2)c+6}{a+2} \ge \frac{(b+1)c+6}{a+2} > 0.$$

Using recurrence (2.1) and the assumption $a-b \le 1$, one can show by induction that $u_n > u_{n-1}$ if $n \ge 4$. If $x \ge 1$, then

$$\begin{aligned} k_n(x) &= (1-u_1x+u_2x^2) + \sum_{i=2}^{\frac{n}{2}}(u_{2i}x^{2i} - u_{2i-1}x^{2i-1}) \\ &\ge (1-u_1x+u_2x^2) + \sum_{i=2}^{\frac{n}{2}}(u_{2i}-u_{2i-1})x^{2i-1} > 0, \end{aligned}$$

being the sum of positive terms, whence $f_n(x) > 0$ for $x \le -1$. $\square$

Lemma 2.3. *If $n \ge 3$ is odd and $f_n(-1) \ge 0$, then $f_n(x)$ has exactly one negative zero.*

Proof. Let $k_n(x) = f_n(-x)$, $x > 0$. Note that as in the previous proof, we have $u_n \ge u_{n-1}$ if $n \ge 3$, with equality possible only when $n=3$. Therefore, if $0 < x < 1$, we have

$$u_{2i+1}x^{2i+1} - u_{2i}x^{2i} = u_{2i+1}x^{2i}(x-1) + x^{2i}(u_{2i+1}-u_{2i}) < u_{2i+1} - u_{2i}, \qquad i \ge 1,$$

which implies for $n \geq 3$ odd that

$$
\begin{aligned}
k_n(x) &= (u_0 - u_1 x + u_2 x^2 - u_3 x^3) + \sum_{i=2}^{\frac{n-1}{2}} (u_{2i} x^{2i} - u_{2i+1} x^{2i+1}) \\
&> (u_0 - u_1 + u_2 - u_3) + \sum_{i=2}^{\frac{n-1}{2}} (u_{2i} - u_{2i+1}) = k_n(1) \geq 0.
\end{aligned}
$$

Thus, $k_n(x) > 0$ if $0 < x < 1$.

If $x \geq 1$, then

$$
k_n'(x) = -u_1 + \sum_{i=1}^{\frac{n-1}{2}} (2iu_{2i} x^{2i-1} - (2i+1) u_{2i+1} x^{2i}) < 0,
$$

since $2iu_{2i} < (2i+1)u_{2i+1}$ for $i \geq 1$. Then $k_n(x)$ has one zero for $x \geq 1$ since $k_n(1) \geq 0$ and $k_n'(x) < 0$, which implies $f_n(x)$ has one negative real zero. □

We now prove the main result of this section.

Theorem 2.4. *Suppose a, b, and c are non-negative real numbers such that $a \leq b+1$ and $c \leq \frac{b}{a+1}$. If $n \geq 2$, then the polynomial $f_n(x) = \sum_{i=0}^{n} u_i x^i$ has no real zeros if n is even and one real zero if n is odd.*

Proof. Clearly, $f_n(x)$ has no positive zeros since it has non-negative coefficients. First suppose n is even. By Lemma 2.2, we may restrict our attention to the case $-1 < x < 0$. By Lemma 2.1, we have

$$
f_n(x) = j(x)^{-1} \alpha_n(x), \qquad -1 < x < 0, \tag{2.4}
$$

where

$$
\alpha_n(x) = \int_{x_o}^{x} \frac{j(t) h_n(t)}{t(1+t)(1-3t)} dt + j(x_o) f_n(x_o), \qquad -1 < x < 0,
$$

$x_o \in (-1, 0)$ is fixed, and $j(x), h_n(x)$ are as above. By (2.4), to complete the proof in the even case, it suffices to show that $\alpha_n(x) > 0$ for $-1 < x < 0$ as $j(x) > 0$ on this interval. Since $\alpha_n(x) = j(x) f_n(x)$, with $f_n(0), f_n(-1) > 0$, we first see that $\alpha_n(x) > 0$ for all x sufficiently close to either -1 or 0.

When n is even, note that the polynomial $h_n(x)$ has one negative zero, by Descartes' rule of signs and the assumption $c \leq \frac{b}{a+1}$. Since $h_n(0) \geq 0$, we must have either (i) $h_n(x) > 0$ if $-1 < x < 0$, or (ii) $h_n(x) > 0$ if $r < x < 0$ and $h_n(x) < 0$ if $-1 < x < r$, for some $r \in (-1, 0)$. Note that

$$
\alpha_n'(x) = \frac{j(x) h_n(x)}{x(1+x)(1-3x)}, \qquad -1 < x < 0.
$$

If (i) occurs, then $\alpha_n'(x) < 0$, which implies $\alpha_n(x) > 0$ for $-1 < x < 0$, since it is positive for all x sufficiently close to either endpoint of this interval. If (ii) occurs, then $\alpha_n'(x) > 0$ for $-1 < x < r$ and $\alpha_n'(x) < 0$ for $r < x < 0$, which again implies $\alpha_n(x) > 0$ for $-1 < x < 0$, since in this case the minimum value of $\alpha_n(x)$ on the interval is achieved as x approaches one of the endpoints.

Now suppose n is odd. We'll show in this case that $f_n(x)$ possesses exactly one negative zero. By Lemma 2.3, we may assume $f_n(-1) < 0$. Note that $f_n(-1) < 0$ implies $f_n(x) < 0$ for all $x \le -1$ since $f_n'(x) > 0$ if $x \le -1$ and n is odd. Thus, we may again restrict attention to when $-1 < x < 0$, and we'll show in this case that $\alpha_n(x)$, and thus $f_n(x)$, possesses exactly one zero. Note first that $\alpha_n(x)$ is positive for all x near zero and negative for all x near -1 since $f_n(0) > 0$ and $f_n(-1) < 0$.

We claim that $h_n(x)$ must possess at least one zero on the interval $(-1, 0)$ when $a > 0$. Suppose that this is not the case. By Descartes' rule and the assumption $c \le \frac{b}{a+1}$, the polynomial $h_n(x)$ when n is odd has either two negative zeros or none at all. Then $h_n(0) \ge 0$ and $\lim_{x\to-\infty} h_n(x) = \infty$ would imply $h_n(x) > 0$ if $-1 < x < 0$ and thus $\alpha_n'(x) < 0$. But this would contradict the fact that $\alpha_n(x)$ is negative for x near -1 and positive for x near 0. Thus, $h_n(x)$ possesses two negative zeros and at least one of these zeros lies in the interval $(-1, 0)$ when $a > 0$. Therefore, we must have either (a) $h_n(x) > 0$ if $r < x < 0$ and $h_n(x) < 0$ if $-1 < x < r$ for some $-1 < r < 0$, or (b) $h_n(x) > 0$ if $-1 < x < r$ or $s < x < 0$, with $h_n(x) < 0$ if $r < x < s$ for some $-1 < r < s < 0$.

If (a) occurs, then $\alpha_n(x)$ initially increases going to the right from $x = -1$ and crosses the x-axis before it decreases in its approach to $x = 0$ from the left. If (b) occurs, then $\alpha_n(x)$ traces out a similar curve in going from $x = -1$ to $x = 0$ except that it initially decreases some from its negative value near $x = -1$ before it starts to increase. In each case, we see that $\alpha_n(x)$, and thus $f_n(x)$, possesses exactly one zero for $-1 < x < 0$ when $a > 0$.

If $a = 0$, then a similar argument applies if $c < b$. If $a = 0$ and $c = b$, then $h_n(x)$ possesses exactly one negative zero, which we will denote by t. Note that $h_n(x) < 0$ if $t < x < 0$ and $h_n(x) > 0$ if $x < t$ since $h_n(0) = 0$ and $\lim_{x\to-\infty} h_n(x) = \infty$. If $t \le -1$, then $h_n(x) < 0$ on $(-1, 0)$ and thus $\alpha_n(x)$ is increasing on $(-1, 0)$, which implies it has a single zero there. If $-1 < t < 0$, then $\alpha_n(x)$ is decreasing on $(-1, t)$ and increasing on $(t, 0)$, which yields the same conclusion. This completes the odd case and the proof. □

Taking $a = 2, b = 3, c = 1$ and $a = 1, b = c = 0$ in the prior theorem gives the first two parts of Theorem 1.1 above concerning the Motzkin and the Riordan sequences.

3. Schröder family polynomials

Let v_n, $n \ge 0$, denote the sequence defined by the recurrence

$$(n + a)v_n = 3(2n - 1)v_{n-1} - (n - 2 + b)v_{n-2}, \qquad n \ge 2, \tag{3.1}$$

with the initial values $v_0 = 1$ and $v_1 = c$, where a, b, and c are constants. Note that v_n reduces to the (little) Schröder sequence when $a = 1, b = 0, c = 1$ and to the (central) Delannoy sequence when $a = 0, b = 1, c = 3$. Let

$$g_n(x) = \sum_{i=0}^{n} v_i x^i, \qquad n \geq 0.$$

We will need the following integral representation of $g_n(x)$.

Lemma 3.1. *If $x < 0$, then*

$$g_n(x) = j(x)^{-1} \left(\int_{x_o}^{x} \frac{j(t)h_n(t)}{t(1-6t+t^2)} dt + j(x_o)g_n(x_o) \right), \tag{3.2}$$

where $x_o < 0$ is any fixed number,

$$j(x) = |x|^a (1-6x+x^2)^{\frac{b-a}{2}} \left(\frac{x-3-2\sqrt{2}}{x-3+2\sqrt{2}} \right)^{\frac{3(a+b-1)}{4\sqrt{2}}},$$

and

$$h_n(x) = a + ((1+a)c-3)x - (n+a+1)v_{n+1}x^{n+1} + (n+b)v_n x^{n+2}.$$

Proof. Let $g = g_n(x)$. By (3.1), we have

$$\begin{aligned} &xg' + ag - 6x^2g' - 3xg + x^3g' + bx^2g \\ &\quad = a + ((1+a)c-3)x - (3(2n+1)v_n - (n-1+b)v_{n-1})x^{n+1} + (n+b)v_n x^{n+2} \\ &\qquad + \sum_{i=2}^{n} \left[(i+a)v_i - 3(2i-1)v_{i-1} + (i-2+b)v_{i-2} \right] x^i \\ &\quad = a + ((1+a)c-3)x - (n+1+a)v_{n+1}x^{n+1} + (n+b)v_n x^{n+2}. \end{aligned}$$

This may be rewritten in the form

$$g_n'(x) + \frac{a-3x+bx^2}{x(1-6x+x^2)} g_n(x) = \frac{h_n(x)}{x(1-6x+x^2)}, \qquad x < 0, \tag{3.3}$$

where $h_n(x)$ is as given. Note that, by partial fractions, we have

$$\begin{aligned} \frac{a-3x+bx^2}{x(1-6x+x^2)} &= \frac{a}{x} + \frac{(b-a)(x-3)}{1-6x+x^2} \\ &\quad + \frac{3(a+b-1)}{4\sqrt{2}} \left(\frac{1}{x-3-2\sqrt{2}} - \frac{1}{x-3+2\sqrt{2}} \right), \end{aligned}$$

which gives the antiderivative

$$\int \frac{a-3x+bx^2}{x(1-6x+x^2)}dx = \log\left(|x|^a|1-6x+x^2|^{\frac{b-a}{2}}\left|\frac{x-3-2\sqrt{2}}{x-3+2\sqrt{2}}\right|^{\frac{3(a+b-1)}{4\sqrt{2}}}\right).$$

Formula (3.2) now follows from solving (3.3) for $x<0$ by the usual method. □

Theorem 3.2. *Suppose a, b, and c satisfy $0 \le a \le 2$, $0 \le b \le 7-a$, and $1 \le c \le \frac{3}{a+1}$. If $n \ge 2$, then the polynomial $g_n(x) = \sum_{i=0}^{n} v_i x^i$ has no real zeros if n is even and one real zero if n is odd.*

Proof. Using (3.1) and the assumptions $c \ge 1$ and $a+b \le 7$, one can show by induction that $v_n \ge v_{n-1}$ for all $n \ge 1$, with equality possible only when $n=1$ or $n=2$. Then $g_n(x)$ clearly has no positive zeros since it has positive coefficients.

Suppose n is even. If $x \le -1$, then

$$g_n(x) = 1 + \sum_{i=1}^{\frac{n}{2}} x^{2i-1}(v_{2i-1}+v_{2i}x) > 0,$$

so we may restrict attention to the case $-1<x<0$. By Lemma 3.1, we have

$$g_n(x) = j(x)^{-1}\beta_n(x), \qquad x<0, \tag{3.4}$$

where

$$\beta_n(x) = \int_{x_o}^{x} \frac{j(t)h_n(t)}{t(1-6t+t^2)}dt + j(x_o)g_n(x_o), \qquad x<0,$$

x_o is a fixed negative number, and $j(x), h_n(x)$ are as stated in this lemma.

Since $j(x)>0$ for $x<0$, to complete the proof in the even case, it suffices to show that $\beta_n(x)>0$ for $x<0$, by (3.4). Since $\beta_n(x) = j(x)g_n(x)$, with $g_n(0), g_n(-1) > 0$, we see that $\beta_n(x)>0$ for $x=-1$ and all x sufficiently close to 0. Next observe that

$$\beta_n'(x) = \frac{j(x)h_n(x)}{x(1-6x+x^2)}, \qquad x<0,$$

and that $h_n(x)>0$ if $x<0$ for n even, by the assumption $c \le \frac{3}{a+1}$. Thus $\beta_n'(x)<0$, which implies $\beta_n(x)>0$ for $-1<x<0$, since it is positive at $x=-1$ and for all x near 0. This completes the even case.

Now suppose n is odd. We'll show in this case that $g_n(x)$ possesses exactly one negative zero. Note first that $g_n(-1)<0$, which implies $g_n(x)<0$ for all $x \le -1$ since $g_n'(x)>0$ if $x \le -1$ and n is odd. To complete the proof, we will show that $\beta_n(x)$, and thus $g_n(x)$, possesses exactly one zero on the interval $(-1,0)$. Observe that $\beta_n(x)$ is positive for all x sufficiently close to zero and that $\beta_n(-1)<0$ since $g_n(0)>0$ and $g_n(-1)<0$.

By Descartes' rule, the polynomial $h_n(x)$ has one negative zero when n is odd and $a > 0$. Reasoning as in the proof of Theorem 2.4 above then shows that this zero must belong to the interval $(-1, 0)$. Since $h_n(0) > 0$, it must be the case that $h_n(x) > 0$ if $r < x < 0$ and $h_n(x) < 0$ if $-1 < x < r$ for some $-1 < r < 0$. Reasoning now as in the proof of Theorem 2.4 shows that $\beta_n(x)$, and thus $g_n(x)$, possesses exactly one zero on the interval $(-1, 0)$ when $a > 0$. A similar argument applies to the case when $a = 0$ and $c < 3$. If $a = 0$ and $c = 3$, then $h_n(x) < 0$ if $x < 0$, which implies $\beta_n(x)$ is increasing on $(-1, 0)$ and thus has one zero there. This completes the odd case and the proof. □

Taking $a = 1, b = 0, c = 1$ and $a = 0, b = 1, c = 3$ in the prior theorem gives the last two parts of Theorem 1.1 above concerning the Schröder and Delannoy sequences.

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