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Enhanced power graphs of groups are weakly perfect

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Abstract

A graph is weakly perfect if its clique number and chromatic number are equal. We show that the enhanced power graph of a finite group G is weakly perfect: its clique number and chromatic number are equal to the maximum order of an element of G. The proof requires a combinatorial lemma. We give some remarks about related graphs.

1 Introduction

The *directed power graph* of a finite group G, introduced in [10], has the elements of G as vertices, with an arc from x to y if $y = x^n$ for some integer n. This relation is reflexive and transitive, hence is a *partial preorder*. The (undirected) *power graph*, defined in [6], is obtained by ignoring directions: that is, x and y are joined if one is a power of the other. This graph is thus the comparability graph of a partial preorder; a small extension of Dilworth's theorem shows that it is *perfect*, that is, every induced subgraph has clique number equal to chromatic number. This was first proved by Alireza, Ahmad and Abbas [2] and Feng, Ma and Wang [9]. (*Dilworth's theorem* states in part that the *comparability graph* of a partially ordered set P, whose vertices are the elements of P, joined if they are comparable in the order, is perfect. To extend this to a partial preorder, note that the relation $\equiv$ defined by $x \equiv y$ if $x \to y$ and $y \to x$, is an equivalence relation; imposing a total order on each equivalence class gives a partial order with the same comparability graph.)

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Both these graphs were first defined for semigroups, but most work on them has concerned groups.

According to the *strong perfect graph theorem*, proved by Chudnovsky, Robertson, Seymour and Thomas [7], a graph is perfect if and only if it has no induced subgraph which is a cycle of odd length greater than 3 or the complement of one.

The *enhanced power graph* of G, defined in [1], again has vertex set G, with x and y joined if there is an element z such that both x and y are powers of z. (Since subgroups of cyclic groups are cyclic, an equivalent statement is that x and y are joined if and only if the group they generate is cyclic.) It is shown in [4] that enhanced power graphs of finite groups are *universal*, that is, every finite graph occurs as an induced subgraph of such a graph. Thus, these graphs are not in general perfect.

Our purpose here is to show that enhanced power graphs are *weakly perfect*, that is, they have chromatic number equal to clique number. Indeed our result is not restricted to finite groups, but applies to groups in which all elements have finite and bounded order.

Theorem 1 *Let G be a finite group, or a torsion group of bounded exponent. Then the clique number and the chromatic number of G are both equal to the maximal order of an element of G.*

The result for clique number is known, and the proof is straightforward; the result for chromatic number requires the following purely combinatorial result. We note that the proof is constructive, so gives an easy algorithm for colouring the enhanced power graph.

Theorem 2 *For every natural number n, there exist subsets $A_1, A_2, \ldots, A_n$ of $\{1, 2, \ldots, n\}$ with the properties*

- $|A_q| = \phi(q)$ *for $q \in \{1, \ldots, n\}$, where ϕ is Euler's totient;*
- *if* $\mathrm{lcm}(q, q') \leq n$, *then* $A_q \cap A_{q'} = \emptyset$, *where* lcm *denotes the least common multiple.*

These theorems will be proved in the next two sections. In the final section we give some concluding remarks.

Many further properties of power graphs and enhanced power graphs can be found in [12] and [16].

2 Proof of Theorem 2

Let D be the set of fractions p/q (in their lowest terms) in $(0, 1]$, for $1 \leq q \leq n$. We define a function $f \colon D \to \{1, 2, \ldots, n\}$ by

$$f(p/q) = \lceil np/q \rceil.$$

We make the key observation that

If $p/q \neq p'/q'$ and $f(p/q) = f(p'/q')$, then $\mathrm{lcm}(q, q') > n$.

For, if $f(p/q) = f(p'/q')$, then there exists m such that

$$m - 1 < np/q, np'/q' \leq m.$$

Thus $|p/q - p'/q'| < 1/n$. On the other hand, $|p/q - p'/q'|$ is a rational number whose numerator is at least 1 (since $p/q \neq p'/q'$), and the denominator is $\mathrm{lcm}(q, q')$. So we have

$$\frac{1}{n} > \left| \frac{p}{q} - \frac{p'}{q'} \right| \geq \frac{1}{\mathrm{lcm}(q, q')},$$

and so $\mathrm{lcm}(q, q') > n$, as required.

Now we let D_q be the set of fractions in D with denominator q, so that $|D_q| = \phi(q)$, and let $A_q = f(D_q) \subseteq \{1, \ldots, n\}$. By our key observation we see that

- the restriction of f to D_q is injective, so $|A_q| = \phi(q)$;
- if $q \neq q'$ and $\mathrm{lcm}(q, q') \leq n$, then $A_q \cap A_{q'} = \emptyset$.

So the theorem is proved.

For example, here are the sets generated for $n = 12$ by the above procedure.

$A_1 = \{12\}$, $A_2 = \{6\}$, $A_3 = \{4, 8\}$,
$A_4 = \{3, 9\}$, $A_5 = \{3, 5, 8, 10\}$, $A_6 = \{2, 10\}$,
$A_7 = \{2, 4, 6, 7, 9, 11\}$, $A_8 = \{2, 5, 8, 11\}$, $A_9 = \{2, 3, 6, 7, 10, 11\}$,
$A_{10} = \{2, 4, 9, 11\}$, $A_{11} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $A_{12} = \{1, 5, 7, 11\}$.

3 Proof of Theorem 1

We begin with the observation that if a finite subset of a group has the property that any two of its elements generate a cyclic group, then the whole subset generates a cyclic group. A proof can be found in [1, Lemma 32]. It follows that a maximal clique in the enhanced power graph is a maximal cyclic subgroup of G, and the clique number is equal to the order of the largest cyclic subgroup, say n.

In order to find a colouring with n colours, we take $\{1, 2, \ldots, n\}$ to be the set of colours, with the subsets A_q given by Theorem 2. We will use the set A_q to colour elements of order q. If two elements of order q are joined, they lie in the same cyclic subgroup of order q; this subgroup has $\phi(q)$ generators, so we have enough colours to give them all different colours. Other elements of order q are not joined to these ones, so we may re-use the same set of colours for them. Now, if two elements of different orders q and q' are joined, they generate a cyclic group of order $\mathrm{lcm}(q, q')$, which is at most n; so the sets of colours assigned to them are disjoint. Thus, we obtain a proper colouring.

4 Further remarks

Our combinatorial lemma can deal with any set of element orders, as long as the largest order n is given. Now there are groups in which the set of element orders is $\{1, \ldots, n\}$ for some n. (For example, the orders of elements in the alternating group A_7 are $1, 2, 3, 4, 5, 6, 7$.) But, as we show below, this can only occur for finitely many values of n. So, at first glance, it seems we may be able to simplify the argument for most groups by using the fact that not all orders occur. We have not attempted to do so, and indeed our proof is simple enough that any substantial simplification seems unlikely.

Proposition 3 *There are only finitely many values of n for which there exists a finite group in which the set of element orders is $\{1, \ldots, n\}$.*

Proof We use the *Gruenberg–Kegel graph* of a group G (sometimes called the *prime graph*): the vertices are the prime divisors of $|G|$, with vertices p and q joined if G contains an element of order pq. Gruenberg and Kegel described this graph in an unpublished manuscript on the decomposition of the augmentation ideal of the integral group ring of a finite group; their main theorem, a description of the groups whose Gruenberg–Kegel graph is disconnected, was published by Gruenberg's student Williams [15] and refined by later authors, notably Kondrat'ev [11].

We use the fact that the Gruenberg–Kegel graph of a finite group has at most 6 connected components. (This follows from the cited results of [15, 11]. However, these papers contain some errors, subsequently fixed. A corrected result is given in [5, Table 1]. The only group with six components in its GK graph is the Janko group J_4.)

Now suppose that G is a group in which the element orders are $\{1, 2, \ldots, n\}$. If p is a prime in the interval $(n/2, n]$, then p is an isolated vertex in the Gruenberg–Kegel graph of G; so there can be at most five such primes. But, in a strengthening of Bertrand's postulate, Erdős [8] showed that the number of primes in this interval tends to ∞ with n. The result is proved.

The weak perfect graph theorem asserts that a graph is perfect if and only if its complement is perfect. This does not hold for weakly perfect graphs. However, we note that Jitender Kumar Parveen has recently posted on the arXiv a paper showing (among other things) that the complement of the enhanced power graph of a finite group is weakly perfect [13].

A related graph is the difference of the enhanced power graph and power graph of the group G, which we will denote by $\Delta(G)$: x and y are joined in this graph if they are joined in the enhanced power graph but not in the power graph.

For a group G, let $\Omega(G)$ denote the set of orders of elements of G. For a positive integer n, let $\alpha(n)$ denote the size of the largest antichain in the lattice of divisors of n. De Bruijn et al. [3] showed that, if n has m prime factors (counted with multiplicity), then a maximum-size antichain consists of all divisors with $m/2$ prime factors if m is even, and either all divisors with $\lfloor m/2 \rfloor$ prime factors or all with $\lceil m/2 \rceil$ prime factors if m is odd. (This is a generalisation of Sperner's theorem [14].)

Proposition 4 *For a finite group G, the clique number of $\Delta(G)$ is equal to* $\max\{\alpha(n) : n \in \Omega(G)\}$.

Proof A clique S in $\Delta(G)$ is a clique in the enhanced power graph, and so is contained in a cyclic group C. Now a cyclic group has the property that if x and y are two elements for which the order of x divides the order of y, then x is a power of y. It follows that the elements of S all have different orders, and these form an antichain in the lattice of divisors of $|C|$.

However, $\Delta(G)$ is not always weakly perfect:

Proposition 5 *Let G be the symmetric group S_8 on 8 letters. Then $\Delta(G)$ is not weakly perfect.*

Proof We have $\Omega(G) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 15\}$; so the clique number of $\Delta(G)$ is equal to 2. But $\Delta(G)$ is not bipartite, since

$$\{(1, 2), (3, 4, 5), (6, 7), (1, 2, 3), (4, 5, 6, 7, 8)\}$$

induces a 5-cycle.

We leave the description of groups G for which $\Delta(G)$ is weakly perfect as an open problem.

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