

Well-indumatched trees and graphs of bounded girth

S. AKBARI

*Department of Mathematical Sciences
Sharif University of Technology, Iran
s_akbari@sharif.edu*

T. EKIM

*Department of Industrial Engineering
Bogazici University, Turkey
tinaz.ekim@boun.edu.tr*

A. H. GHODRATI

*Department of Mathematical Sciences
Sharif University of Technology, Iran
ghodrati84@gmail.com*

S. ZARE

*Department of Mathematics and Computer Science
Amirkabir University of Technology, Iran
sa.zare_f@yahoo.com*

Abstract

A graph G is called well-indumatched if all of its maximal induced matchings have the same size. In this paper, we characterize all well-indumatched trees. We provide a linear time algorithm to decide whether a tree is well-indumatched or not. Then, we characterize minimal well-indumatched graphs of girth at least 9 and show subsequently that for an odd integer $g \geq 9$ and $g \neq 11$, there is no well-indumatched graph of girth g . On the other hand, there are infinitely many well-indumatched unicyclic graphs of girth k , where $k \in \{3, 5, 7\}$ or k is an even integer greater than 2. We also show that, although the recognition of well-indumatched graphs is known to be co-NP-complete in general, one can recognize in polynomial time well-indumatched graphs, where the size of maximal induced matchings is fixed.

1 Introduction

Let G be a graph. A *matching* in G is a subset M of $E(G)$ such that no two edges of M have a common endpoint. An *induced matching* is a matching M such that no two edges of M are joined by an edge; in other words, M occurs as an induced subgraph of G . In this paper, we are interested in graphs such that all their inclusion-wise maximal induced matchings have the same size. These graphs were introduced recently in [3], where they are called *well-indumatched* graphs.

Induced matchings are also known as *2-separated matchings* [35], *strong matchings* [21] and *distance-2 matchings*. This last is used in a more general context; a matching is called a *distance- k* matching if the distance between any two edges of the matching is at least k , where the distance between two edges is the minimum of the distances (lengths of the shortest paths) between end vertices of the two edges. Clearly, an induced matching is a set of edges in which the distance between any two edges is at least 2, hence a distance-2 matching.

Given a graph G , MAXIMUM INDUCED MATCHING, called MIM for short, is the problem of finding an induced matching of maximum size. This concept was introduced by Stockmeyer and Vazirani [35], where it was called “risk-free marriage”. As pointed out in [23], MIM finds applications in secure communication networks, VLSI design and network flow problems. Besides, it is closely related to strong edge-colourings considered by Erdős [16], where every path of three edges needs three colours (see for instance [25]). It follows from its definition that a strong edge-colouring of a graph boils down to partitioning its edge set into induced matchings. MIM has recently found quite a bit of use in combinatorial commutative algebra, where the size of a maximum induced matching gives a lower bound on the regularity of the edge ideal [31]. Another relation which makes MIM an interesting problem is its direct link with the irredundancy number of a graph [22]. We observe that MIM is an important problem both for its applications and its relation to other important graph parameters. However, it is known to be NP-hard even in very restricted graph classes. For instance, it is known that MIM remains NP-hard in planar 3-regular graphs [14], planar bipartite graphs with degree 2 vertices in one part and degree 3 vertices in the other part [27], k -regular bipartite graphs for any $k \geq 3$ [11], and Hamiltonian graphs, claw-free graphs and line graphs [28]. On the other hand, MIM is polynomial-time solvable in trees [23], chordal graphs [8] (even in linear time [7]), circular arc graphs [22] and interval graphs [23]. One can refer to [11] and [14] for a more comprehensive literature review on the complexity status of MIM in various graph classes.

Another problem closely related to induced matchings, but less studied than MIM, is the problem of finding an inclusion-wise maximal induced matching of minimum size. This problem is called MINIMUM MAXIMAL INDUCED MATCHING and denoted by MMIM for short. MMIM has been shown to be NP-hard even in bipartite graphs of maximum degree 4 [32] or in graphs having all of their maximal induced matchings of size either k or $k + 1$ for some integer $k \geq 1$ [3]. The generalization of MMIM to distance- k matchings has been also considered recently [26].

When a graph parameter is hard to compute, one way to tackle this difficulty is to search for the so-called “greedy instances”, where a greedy algorithm always ensures an optimal solution. In other words, one can be interested in graphs for which the difficult task of finding a largest or smallest set having a given property becomes trivial using a greedy approach. Several such structures have been studied in the literature. These results include *well-covered* graphs, defined as graphs such that all inclusion-wise maximal independent sets have the same size (see e.g. [33]), the edge analogue of well-covered graphs called *equimatchable* graphs, where all maximal matchings have the same size (see e.g. [30]), *well-dominated* graphs having all of their minimal dominating sets of the same size [17], and *well-totally-dominated* graphs having all of their minimal total dominating sets of the same size [1, 24].

In [9], Caro, Sebó and Tarsi suggested a unified approach to study such greedy instances. Each one of the above mentioned graph classes has been extensively studied since then. For such a graph class \mathcal{G} , typical research questions considered in the literature include:

1. Structural characterizations of \mathcal{G} and/or its subclasses.
2. Complexity of recognizing a graph in the class \mathcal{G} and/or its subclasses (usually obtained using a structural characterization).
3. Forbidden subgraphs for \mathcal{G} , if any, and characterization of hereditary graphs in \mathcal{G} , namely those graphs in \mathcal{G} having all their induced subgraphs also in \mathcal{G} .
4. Complexity of various graph problems in \mathcal{G} .

As suggested in [9], the extensions of such “greedy instances”, where the size of the sets with the desired property have only two possible (consecutive or not) values have also been considered in the case of equimatchability (the related graphs are called *almost-equimatchable* [13]), and well-coveredness [15, 19].

In the same spirit, a generalization of well-covered graphs called p -equipackable graphs, has been defined. A graph is p -*equipackable* if all its maximal p -packings are of the same size, where a p -*packing* is a set of vertices such that the distance between any two distinct vertices in this set is larger than p [26]. The edge analogue of p -equipackable graphs, called p -*equimatchable* graphs, was recently introduced in [26]; a graph is p -*equimatchable* if all of its maximal distance- p matchings have the same size. We note that equimatchable graphs are exactly 1-equimatchable graphs. Although deciding whether a given graph is equimatchable or not can be done in polynomial time [12], it has been shown in [26] that the recognition of p -equimatchable graphs is co-NP-complete for any fixed $p \geq 2$. Note that 2-equimatchable graphs are exactly well-indumatched graphs, which is the focus of our paper. In [3], it has been shown that recognizing a well-indumatched graph is a co-NP-complete problem even for $(2P_5, K_{1,5})$ -free graphs. They also prove that, under the same restriction, the problem of recognizing a graph that has maximal induced matchings of at most t distinct sizes is co-NP-complete for any given $t \geq 1$. After establishing the hardness

of the recognition problem, the authors show that the decision versions of INDEPENDENT DOMINATING SET, INDEPENDENT SET and DOMINATING SET problems are all NP-complete in the class of well-indumatched graphs. Then, they note that well-indumatched graphs are not hereditary and they characterize the so-called *perfectly well-indumatched* graphs which are well-indumatched graphs such that all induced subgraphs are also well-indumatched.

Another complexity result in the literature follows from the links between well-covered graphs and well-indumatched graphs. As noted in [8], an induced matching in a graph G corresponds to an independent set in the square of the line graph of G , where the square of a graph is obtained by adding an edge between every pair of vertices at distance two. It follows that a graph is well-indumatched if and only if the square of the line graph is well-covered. Using the fact that the square of the line graph of a chordal graph is also chordal [8], and the characterization of well-covered chordal graphs in [34], it follows that well-indumatched chordal graphs can be recognized efficiently.

We note that the above mentioned papers focus mainly on the complexity issues for well-indumatched graphs and leave their structural properties mostly open. In this paper, we investigate the structure of well-indumatched graphs.

We start our paper with some definitions and preliminary results in Section 2. Then we proceed with the study of the structure and recognition of some subclasses of well-indumatched graphs. Note that for those graph classes \mathcal{G} , where both MIM and MMIM can be solved in polynomial time, one can decide whether a given graph in \mathcal{G} is well-indumatched or not simply by solving each one of the two problems and checking whether their optimal values coincide or not. However, in such a class \mathcal{G} , it is still interesting to find structural characterizations of well-indumatched graphs which can possibly lead to simpler recognition algorithms. This is the case for trees as both MIM and MMIM can be solved in linear time by the algorithms given in [23] and [29], respectively. In Section 3, we provide a simple characterization of well-indumatched trees which provides a much simpler linear time recognition algorithm.

Section 4 is devoted to well-indumatched graphs with bounded girth. We characterize all minimal well-indumatched graphs of girth at least 9. This result implies that for an odd integer $g \geq 9$ and $g \neq 11$, there is no well-indumatched graph of girth g . On the other hand, there are infinitely many well-indumatched trees, infinitely many well-indumatched unicyclic graphs of girth k , where $k \in \{3, 5, 7\}$ or k is an even integer greater than 2; and also infinitely many well-indumatched r -regular graphs of girth 3, where $r \geq 3$ is an arbitrary integer. A very recent paper where well-indumatched graphs of girth at least 8 are characterized has been published by Finbow, Hartnell and Plummer [18]. A polynomial time recognition algorithm follows from their characterization as well. In this paper, the authors give an alternate proof to our result stating that there is no well-indumatched graph of girth 9. They also settle our conjecture given in Section 4 which states that there are no connected well-indumatched graphs of girth exactly 11 other than the cycle of length 11. It should be mentioned that the proofs in [18] are different than ours.

Finally, Section 5 addresses well-indumatched graphs with a fixed size k of maximal induced matchings. We show that, when we consider the class of well-indumatched graphs with $k = 1$, WEIGHTED INDEPENDENT SET is polynomial time solvable, whereas DOMINATING SET is NP-complete. This latter result strengthens the known result of NP-hardness of DOMINATING SET in well-indumatched graphs by restricting the size of maximal induced matchings to 1. Recall that the recognition of well-indumatched graphs is co-NP-complete even for $(2P_5, K_{1,5})$ -free graphs. We show that deciding whether all induced matchings are of the same size k can be done in polynomial time when k is fixed.

2 Definitions and Preliminaries

All graphs in this paper are finite, simple and undirected. For a graph G , the vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, and their cardinalities n and m are called the *order* and the *size* of G , respectively. The degree of a vertex $v \in V(G)$ is the number of vertices adjacent to it and is denoted by $d(v)$. For an integer $r \geq 1$, a graph is said to be r -regular if every vertex has degree r . The *girth* of G is the length of its shortest cycle. The *path* and the *cycle* of order n are denoted by P_n and C_n , respectively. By kH we denote the disjoint union of k graphs, each one isomorphic to H . We say that a graph G is H -free whenever G does not contain H as an induced subgraph. The reader is referred to [5] for all graph theoretic definitions not given here.

The *distance* between two vertices $u, v \in V(G)$ is the length of the shortest path between u and v . The *distance* between two edges e_1, e_2 of G is defined as the minimum distance between an end-point of e_1 and an end-point of e_2 . For two edges e_1 and e_2 , we say that e_1 *covers* e_2 if the distance between e_1 and e_2 is at most 1. In particular, an edge covers itself. We say that a subset F of edges covers an edge e if there is an edge $f \in F$ such that f covers e . Note that an induced matching M is maximal if and only if M covers $E(G)$.

A graph G is called *reduced* if no two vertices of G have the same set of neighbours. Note that a tree is reduced if and only if each vertex is adjacent to at most one pendant edge (a *pendant edge* is an edge incident with a vertex of degree one). Let G be a graph. For each set of vertices with the same neighbours in G , remove all of them except one. Notice that this procedure will never create a new pair of vertices with the same set of neighbours. Therefore the resulting graph, called the *reduction* of G and denoted by $R(G)$, is a uniquely defined reduced graph.

The following remark shows that one can restrict the study of well-indumatched graphs to reduced graphs.

Remark 2.1. The graph G is well-indumatched if and only if $R(G)$ is well-indumatched.

Proof. To see this, assume without loss of generality that x and y are the only distinct vertices of G having the same neighbourhood. The general case can be shown by

repeating the following argument for all such pairs. Any matching covering all edges incident with x also covers all edges incident with y . Moreover, since an edge incident with x covers all edges incident with y and vice versa, an induced matching of G can contain at most one edge incident with x or y . So the result follows. \square

In [3], it is noted that well-indumatched graphs are not hereditary, as a P_5 is not well-indumatched but a P_7 which contains P_5 as an induced subgraph is well-indumatched. In addition, the authors provide a construction which shows that for any graph H , there is a well-indumatched graph G containing H as an induced subgraph. In other words, this certifies that there is no forbidden induced subgraph for a graph to be well-indumatched. Based on this observation, they characterize well-indumatched graphs such that their induced subgraphs are also well-indumatched, by three minimal forbidden subgraphs (see Theorem 10 in [3]). Given a graph G , they also introduce the concept of a *co-indumatched subgraph* which is a subgraph F of G obtained by the removal of the closed neighbourhood of the end-points of M for some induced matching M (possibly $M = \emptyset$) of G . This concept is then used to characterize well-indumatched graphs by forbidden co-indumatched subgraphs (Theorem 9 in [3]). Unlike this approach, we are interested in the graphs remaining after all edges covered by an induced matching are removed. The following result which will be very useful in our proofs can be seen as a natural translation of an analogous result for well-covered graphs (see e.g. [33, 34]).

Lemma 2.1. *Let G be a well-indumatched graph and $F_0 \subseteq E(G)$ be an induced matching. If F is the set of edges covered by F_0 , then $G \setminus F$ is well-indumatched.*

Proof. Let M_1 and M_2 be two maximal induced matchings of $G \setminus F$. Then $M_1 \cup F_0$ and $M_2 \cup F_0$ are maximal induced matchings of G . Since G is well-indumatched, we have $|M_1 \cup F_0| = |M_2 \cup F_0|$, which, together with the fact that $M_i \cap F_0 = \emptyset$, for $i = 1, 2$, implies that $|M_1| = |M_2|$. Therefore $G \setminus F$ is well-indumatched. \square

3 Characterization of Well-Indumatched Trees

In this section we give a simple necessary and sufficient condition for a reduced tree to be well-indumatched. Note that although this section is related to trees, Lemmas 3.3, 3.4 and 3.6 are for general well-indumatched graphs and can be useful in other contexts.

Lemma 3.1. *Let T be a tree with a longest path $P = v_1 \cdots v_k$ such that the degree of each vertex of P is at most 3, and those of degree 3 are incident to a pendant edge. Then T is well-indumatched if and only if $k \in \{1, 2, 3, 4\}$ or $k = 7$ and $d(v_4) = 2$.*

Proof. It can be checked that if $k \in \{1, 2, 3, 4\}$, or $k = 7$ and $d(v_4) = 2$, then T is well-indumatched, and if $k \in \{5, 6\}$ or $k = 7$ and $d(v_4) = 3$, then T is not well-indumatched. Now let $k \geq 8$. Then T has an edge e such that the removal of the edges covered by e leaves a component which is a tree with longest path of order

5 or 6 and such that the degree of each vertex is at most 3, and those of degree 3 are incident to a pendant edge. Since a graph is well-indumatched if and only if its connected components are well-indumatched, it follows from Lemma 2.1 that T is not well-indumatched. \square

Lemma 3.2. *The following statements hold:*

- (i) *The path P_n is well-indumatched if and only if $n \in \{1, 2, 3, 4, 7\}$.*
- (ii) *The cycle C_n is well-indumatched if and only if $n \in \{3, 4, \dots, 8, 11\}$.*

Proof. The first part follows from Lemma 3.1. For the second part, it can be seen that C_n for $n \in \{3, 4, \dots, 8, 11\}$ is well-indumatched, and that C_9 and C_{10} are not. If $n \geq 12$, then for any edge e the removal of edges covered by e results in a path with $n - 4$ vertices which is not well-indumatched by the previous part. Therefore, by Lemma 2.1, C_n is not well-indumatched. \square

A *cut-edge* is an edge of a graph whose deletion increases the graph's number of connected components.

Lemma 3.3. *Let G be a connected well-indumatched graph and $e = uv$ be a cut-edge of G . If u is an end vertex of a path of length at least 2 which does not contain v , then the component of $G \setminus e$ containing v is well-indumatched.*

Proof. Let H and K be the components of $G \setminus e$ containing u and v , respectively. Let uxy be a path in H . Extend $\{xy\}$ to a maximal induced matching M in H . The edges covered by M are $E(H) \cup \{e\}$; thus by Lemma 2.1, $K = G \setminus (E(H) \cup \{e\})$ is well-indumatched. \square

The following describes two forbidden structures in a well-indumatched graph which will be useful in several proofs.

Lemma 3.4. *Let G be a graph. Then the following statements hold:*

- (i) *If $v_1v_2v_3v_4v_5$ is a path in G such that $d(v_1) = d(v_5) = 1$ and $d(v_2) = 2$, then G is not well-indumatched.*
- (ii) *If $v_1v_2v_3v_4v_5v_6$ is a path in G such that $d(v_1) = d(v_6) = 1$ and $d(v_2) = d(v_5) = 2$, then G is not well-indumatched.*

Proof. (i) Note that every edge covered by $\{v_1v_2, v_4v_5\}$ is also covered by v_3v_4 . Thus, if M is a maximal induced matching containing v_3v_4 , then $(M \setminus \{v_3v_4\}) \cup \{v_1v_2, v_4v_5\}$ is an induced matching and is contained in a maximal induced matching M' . Clearly, $|M'| > |M|$, so G is not well-indumatched.

(ii) Since every edge covered by $\{v_1v_2, v_5v_6\}$ is also covered by v_3v_4 , we obtain the desired result exactly in the same manner as in item (i). \square

A pendant edge which is adjacent to a vertex of degree 2 is called a *good pendant edge*. To characterize well-indumatched trees, we focus on good pendent edges of

reduced well-indumatched trees. It follows from Lemmas 3.1 and 3.2 that well-indumatched trees of order at most 4 are already known; moreover, the set of good pendant edges of reduced well-indumatched trees of order at most 4 do not form an induced matching. Therefore, in what follows, we consider reduced well-indumatched trees of order at least 5.

Lemma 3.5. *Let T be a reduced well-indumatched tree of order at least 5. Then the set of good pendant edges forms a maximal induced matching for T .*

Proof. We prove the assertion by strong induction on $|V(T)|$. Lemma 3.1 implies that there is no reduced well-indumatched tree of order 5; thus the base step of the induction holds. Assume the result holds for all trees satisfying the hypothesis and with at most $n - 1$ vertices, and consider a reduced well-indumatched tree T of order n . Note that since $|V(T)| \geq 5$, the set of all good pendant edges forms an induced matching. So, it suffices to show that the set of all good pendant edges covers all edges of T . Let v be a vertex of T such that $d(v) \geq 3$. We prove the following two claims.

Claim 1. Let p be the number of pendant edges incident with v . Then at least $d(v) - 1 - p$ edges incident with v are contained in paths of length 3 starting at v . (Note that since T is reduced, $p = 0$ or $p = 1$.)

Proof of Claim 1. By definition of p , $d(v) - p$ edges incident with v are not pendant edges, and thus are contained in paths of length 2 starting at v . Suppose that two of these paths, say vxx' and $vy y'$, are not contained in paths of length 3 starting at v . This implies that $d(x') = d(y') = 1$ and by Part (a) of Lemma 3.4, x should have a neighbour z other than v and x' . Since T is reduced, z has a neighbour other than x , say z' . Now $vxzz'$ is a path of length 3, a contradiction. So at most one of the $d(v) - p$ non-pendant edges incident with v is not contained in a path of length 3 starting at v , and the claim follows.

Claim 2. If there are two edges vu_1 and vu_2 which are contained in paths of length 3 starting at v , then the set of good pendant edges covers all edges of T , unless $d(v) = 3$ and v is incident with a pendant edge.

Proof of Claim 2. For $i = 1, 2$, let T_i be the component of $T \setminus vu_i$ containing v . Then by Lemma 3.3, T_i is well-indumatched. Also, since $d(v) \geq 3$, T_i is reduced and has at least 5 vertices. Therefore, by the induction hypothesis, the set of good pendant edges of T_i , say M_i , covers all edges of T_i . Note that good pendant edges of T_1 and T_2 are good pendant edges of T , unless $d(v) = 3$ and v is incident with a pendant edge. Since we have excluded the case $d(v) = 3$ and v is incident with a pendant edge in the statement of Claim 2, $M_1 \cup M_2$ is a set of good pendant edges which covers all edges of T and the claim is proved.

Now let v be an arbitrary vertex of T . If $d(v) \geq 4$, then by Claim 1 there are two edges incident with v which are contained in paths of length 3 starting at v , and so Claim 2 implies the result. Therefore we can assume that for every $v \in V(T)$,

$d(v) \leq 3$, and if $d(v) = 3$, then by Claim 2 either v is incident with a pendant edge or there is at most one path of length 3 starting at v .

Let $P = v_1v_2 \cdots v_k$ be a longest path in T . Therefore $d(v_1) = d(v_k) = 1$. If there is a vertex $x \in V(T) \setminus V(P)$ such that $v_{k-1}x \in E(T)$, then since T is reduced, there is a vertex $y \in V(T) \setminus (V(P) \cup \{x\})$ such that $xy \in E(T)$. Now $v_1 \cdots v_{k-1}xy$ is a path longer than P , which is impossible. Thus, $d(v_{k-1}) = 2$, and similarly $d(v_2) = 2$. By Lemma 3.4, there is no path of length 2 in $T \setminus E(P)$ starting at v_3 or v_{k-2} , and since P is a longest path of T , there is no such path of length at least 3. So $d(v_3), d(v_{n-2}) \in \{2, 3\}$ and if $d(v_3) = 3$ (respectively, $d(v_{n-2}) = 3$) then v_2 (respectively, v_{n-2}) is incident with a pendant edge. Also, for $3 < i < n - 2$, since there are two paths of length at least 3 starting at v_i , either $d(v_i) = 2$ or $d(v_i) = 3$ and v_i is incident with a pendant edge. Therefore by Lemma 3.1, $k \in \{1, 2, 3, 4\}$ or $k = 7$ and $d(v_4) = 2$. In all of these cases, it is not hard to see that the set of good pendant edges covers all edges of T . \square

Lemma 3.6. *Let G be a graph and M be a matching of G which satisfies the following properties:*

- (i) *each edge of G is covered by exactly one edge of M ;*
- (ii) *if $e_1, e_2 \in E(G)$ are covered by $e_0 \in M$, then e_1 covers e_2 .*

Then G is well-indumatched.

Proof. Let M' be a maximal induced matching of G . We construct a one-to-one correspondence between M' and M . Let $e \in M'$. By (i), there is exactly one $e_0 \in M$ which covers e . Define $f(e) = e_0$. We claim that $f : M' \rightarrow M$ is a bijection from M' onto M . If $e_1, e_2 \in M'$ and $e_1 \neq e_2$, but $f(e_1) = f(e_2)$, then by (ii), e_1 covers e_2 which contradicts the fact that M' is an induced matching. So, f is injective. On the other hand, for each $e_0 \in M$, since M' is a maximal induced matching, there is $e \in M'$ which covers e_0 , so by the definition $f(e) = e_0$. Thus f is onto. Therefore, the size of any maximal induced matching of G equals $|M|$ which implies that G is well-indumatched. \square

Theorem 3.1. *Let T be a reduced tree of order at least 5, and M be the set of good pendant edges of T . Then, T is well-indumatched if and only if each edge of T is covered by exactly one edge of M .*

Proof. Assume that every edge of T is covered by exactly one edge of M . If $e = xy \in M$ and z is the other neighbour of y , then the set of edges covered by e consists of e and all edges incident with z . So the second condition of Lemma 3.6 also holds and therefore T is well-indumatched. Conversely, let T be well-indumatched. By Lemma 3.5, M is a maximal induced matching of T , so every edge of T is covered by at least one edge of M . If two edges of M cover an edge of T , then T has either a path $v_1v_2v_3v_4v_5$ such that $d(v_1) = d(v_5) = 1$ and $d(v_2) = d(v_4) = 2$ or a path $v_1v_2v_3v_4v_5v_6$ such that $d(v_1) = d(v_6) = 1$ and $d(v_2) = d(v_5) = 2$ and by Lemma 3.4, T is not well-indumatched, a contradiction. \square

In [23], a linear time algorithm for finding the maximum size of an induced matching in a tree is presented. Besides, a linear time algorithm is given for finding the size of a minimum maximal induced matching in a tree in [29]. These two algorithms provide a linear time algorithm for recognizing whether a tree is well-indumatched; the input graph is well-indumatched if and only if the two algorithms provide optimal solutions with the same value. However, this does not inform us about the structure of well-indumatched trees. Besides implying a simple linear time recognition algorithm, our structural characterization of well-indumatched trees in Theorem 3.1 also provides a structural insight about well-indumatched trees.

Corollary 3.1. *Given a tree T , it can be decided in linear time whether T is well-indumatched.*

Proof. First, obtain $T' = R(T)$ in linear time. Let M be the set of all good pendant edges of T' . Clearly, M can be formed in linear time just by checking the degrees of the parents of the leaves. Now, for each edge e of T' , determine the edge(s) of M covering e ; this can be done by checking whether a neighbour of an end-point of e appears in $V(M)$, thus requiring a time proportional to the sum of the degrees, which is linear. By Theorem 3.1, T' is well-indumatched if and only if each edge of T' is covered exactly once. We conclude the result by Remark 2.1, since T is well-indumatched if and only if T' is well-indumatched. \square

4 Well-indumatched Graphs of Bounded Girth

In this section we study well-indumatched graphs with lower bounded girth. We characterize all minimal well-indumatched graphs of girth at least 9. An important consequence of this characterization is that for an odd integer $g \geq 9$ and $g \neq 11$, there is no well-indumatched graph of girth g . It turns out that minimal well-indumatched graphs of girth at least 9 are (in particular) unicyclic. A *unicyclic* graph is a connected graph with a unique cycle. Unlike for odd girth at least 9 (and not equal to 11), we show that there are infinitely many well-indumatched unicyclic graphs of even girth, or odd girth smaller than 9. We also show that there are infinitely many well-indumatched trees and infinitely many well-indumatched r -regular graphs containing a triangle (of girth 3), where $r \geq 3$ is an arbitrary integer.

Let G be a unicyclic graph and C be the unique cycle of G . For each $v \in V(C)$, the rooted tree in $G \setminus E(C)$ with root v is denoted by T_v . If T is a rooted tree with root v , then the *depth* of T is the longest length of a path starting at v . In Figure 1, two types of rooted trees which are encountered in the following results are shown. A graph G is said to be *minimal* well-indumatched with property \mathcal{P} if G is a well-indumatched graph with property \mathcal{P} and has no proper well-indumatched subgraph with property \mathcal{P} . This definition of minimality implies the following.

Lemma 4.1. *If G is a minimal well-indumatched graph of girth $g \geq 3$ and C is a cycle of girth g in G , then every edge of G is covered by some edge of C .*

Proof. Assume there is an edge $e \in E(G)$ which is covered by no edge of C . Then also e covers no edge of C , and the graph obtained by removing e and all edges covered by e is well-indumatched (by Lemma 2.1) of girth g and with fewer edges than G , contradicting the minimality of G . \square

Lemma 4.2. *If G is a minimal well-indumatched graph of girth $g \geq 9$, then G is a reduced unicyclic graph. Moreover, if the unique cycle of G is $C = v_1 \cdots v_g$, then every T_{v_i} is of one of the Types (i) or (ii) in Figure 1.*

Proof. Let $C = v_1 v_2 \cdots v_g v_1$ be a cycle in G . By Lemma 4.1, every edge of G is covered by some edge of C . We will show that the vertices v_1, v_2, \dots, v_g belong to different components in $G \setminus E(C)$. Suppose this is not true, and let P be a path of minimum length in $G \setminus E(C)$ between two vertices of C . Let v_i and v_j be the end vertices of P and uv_i and wv_j be the first and the last edges of P . Since P has minimum length, it has no common vertex with C except v_i and v_j . Suppose that there is an edge $e = xy$ between $x \in V(P)$ and $y \in V(C)$ other than uv_i and wv_j . If $x \notin \{u, w\}$, then there is a shorter path in $G \setminus E(C)$ between two vertices of C , which contradicts the choice of P . Also, if $x = u$ or $x = w$, then G has a cycle of size smaller than g . So there is no edge between $V(P)$ and $V(C)$ except the first and the last edges of P . Let Q be a shortest path on C between v_i and v_j . Thus the length of Q is at most $\lfloor \frac{g}{2} \rfloor$. Now since $P \cup Q$ is a cycle, the length of P is at least $\lceil \frac{g}{2} \rceil \geq 5$. But then P has an edge which is not covered by any edge of C , a contradiction to Lemma 4.1.

So let H_i be the component of $G \setminus E(C)$ containing v_i , $1 \leq i \leq g$. If H_i has a cycle, which should be of length at least 9, then H_i has a path of length at least 7 which does not contain v_i , say $u_1, u_2 \cdots u_8$. Since each edge of G should be covered by at least one edge of C , for each j , there is at least one edge between v_i and $\{u_j, u_{j+1}\}$, which implies that H_i has a cycle of length at most 4 (by considering three consecutive edges $u_1 u_2, u_2 u_3, u_3 u_4$), a contradiction to the girth assumption. Thus each H_i is a tree and G is unicyclic.

If G is not reduced, then by Remark 2.1, $R(G)$ is a proper well-indumatched subgraph of G whose girth is g , a contradiction to the minimality of G . So G is reduced. For some $v \in V(C)$, if the depth of T_v is at least 3 and $vx_1 x_2 x_3$ is a path in T_v , then the edge $x_2 x_3$ is covered by no edge of C , a contradiction to Lemma 4.1. Thus, for each $v \in V(C)$ the depth of T_v is at most 2.

Now we prove that for each i , $T_{v_i} \neq P_2$. If $T_{v_i} = P_2$, for some i , then remove $v_{i+5} v_{i+6}$ and all edges covered by $v_{i+5} v_{i+6}$. By Lemma 2.1, the resulting graph is a well-indumatched forest. Let T be the component of this forest containing v_i . By Theorem 3.1, the single edge of T_{v_i} should be covered by a good pendant edge of $R(T)$. If $g \geq 11$, then it is clearly impossible. Let $g = 9$. For the single edge of T_{v_i} to be covered by a good pendant edge of $R(T)$, it is necessary that $T_{v_{i-1}} = P_2$. By a similar argument, $T_{v_{i-2}} = P_2$, and continuing in this way, we conclude that T_{v_j} is P_2 for all j . However, it is not hard to see that the resulting graph is not well-indumatched. Now let $g = 10$. For the single edge of T_{v_i} to be covered by a good pendant edge of $R(T)$, it is necessary that $T_{v_{i-2}} = P_1$ and $T_{v_{i-1}} = P_1$ or P_2 . Similarly,

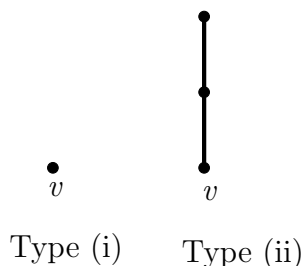


Figure 1: Two types of rooted trees

if one removes $v_{i-5}v_{i-6}$ and the edges covered by it, then it yields $T_{v_{i+2}} = P_1$ and $T_{v_{i+1}} = P_1$ or P_2 . Since $v_{i+1}v_{i+2}$ should be covered by a good pendant edge of $R(T)$, $T_{v_{i+3}} = P_2$ (if $T_{v_{i+3}} = P_1$, then the edge $v_i v_{i+1}$ is covered by two good pendant edges of $R(T)$, which contradicts Theorem 3.1). By repeating this argument we conclude that $T_{v_{i+3k}} = P_2$, for every k . So each T_{v_j} is P_2 . However, it can be checked that the resulting graph is not well-indumatched, a contradiction. Now the minimality of G implies that for each i , T_{v_i} is of Type (i) or (ii) shown in Figure 1. \square

Lemma 4.3. *The following statements hold:*

- (i) *Let G be a reduced well-indumatched unicyclic graph of girth 11 with the unique cycle $C = v_1v_2 \cdots v_{11}v_1$, such that for each i , T_{v_i} is one of the two rooted trees shown in Figure 1; then $G = C_{11}$.*
- (ii) *There is no well-indumatched graph of girth 9.*

Proof. (i) Consider the indices modulo 11. Assume for a contradiction that there is an index j such that $T_{v_j} \neq P_1$.

We note that there is no index i such that T_{v_i} and $T_{v_{i+1}}$ are of Type (ii), because this would induce the forbidden structure in Lemma 3.4 (ii). It follows that for each i , at least one of T_{v_i} and $T_{v_{i+1}}$ is of Type (i). Now T_{v_j} is of Type (ii), and $T_{v_{j+1}}$ is of Type (i). By removing $v_{j-3}v_{j-2}$ and the edges covered by it, we obtain a well-indumatched forest. If T is the component of this forest which contains $v_{j+1}v_{j+2}$, then $R(T)$ has at least five vertices v_j, \dots, v_{j+4} and by Theorem 3.1, every edge of $R(T)$, in particular edge $v_{j+1}v_{j+2}$, should be covered by exactly one good pendant edge of $R(T)$. This implies that $T_{v_{j+2}}$ is of Type (ii), and therefore $T_{v_{j+3}}$ is of Type (i). Continuing in this way, we conclude that $T_{v_{j+5}}, T_{v_{j+7}}, T_{v_{j+9}}, T_{v_{j+11}} = T_{v_j}$ is of Type (i), a contradiction. Thus $G = C_{11}$.

(ii) Assume that G is a minimal well-indumatched graph of girth 9. Then, by Lemma 4.2, G is a reduced unicyclic graph. Moreover, if $C = v_1v_2 \cdots v_9v_1$ is its unique cycle, then for each index i , T_{v_i} is one of the Types (i) or (ii) shown in Figure 1.

By Lemma 3.2, we know that C_9 is not well-indumatched. So, assume that there is an index j such that $T_{v_j} \neq P_1$. Now take the indices modulo 9. As in Part (i), there is no index i such that T_{v_i} and $T_{v_{i+1}}$ are of Type (ii), because this would induce the forbidden structure in Lemma 3.4 (ii). So, for each i , at least one of T_{v_i} and $T_{v_{i+1}}$ is of Type (i). Since T_{v_j} is of Type (ii), $T_{v_{j+1}}$ is of Type (i).

By removing $v_{j-3}v_{j-2}$ and the edges covered by it, we obtain a well-indumatched forest. If T is the component of this forest which contains $v_{j+1}v_{j+2}$, then $R(T)$ has at least five vertices v_j, \dots, v_{j+4} and by Theorem 3.1, every edge of $R(T)$, in particular edge $v_{j+1}v_{j+2}$, should be covered by exactly one good pendant edge of $R(T)$. Since the girth is 9, we note that the unique good pendant edge of $R(T)$ covering $v_{j+1}v_{j+2}$ can also be the edge $v_{j+3}v_{j+4}$, unlike for girth 11. If $v_{j+3}v_{j+4}$ is the unique good pendant edge of $R(T)$ which covers $v_{j+1}v_{j+2}$, then $T_{v_{j+4}}, T_{v_{j+3}}, T_{v_{j+2}}, T_{v_{j+1}}$ are all of Type (i). Now remove edge $v_{j-4}v_{j-3}$ (from the original graph G) and all edges covered by it. This leaves a well-indumatched forest; let T' be its component containing v_{j+3} . Also let $x_1x_2v_j$ be the path of length 2 in T_{v_j} . Then $x_1, x_2, v_j, v_{j+1}, v_{j+2}, v_{j+3}$ induce a P_6 which is forbidden for being well-indumatched by Lemma 3.4 (ii), a contradiction. Therefore edge $v_{j+1}v_{j+2}$ is covered by a good pendant edge in $T_{v_{j+2}}$ which is of Type (ii), and we obtain a contradiction as for girth 11. We conclude that there is no well-indumatched graph of girth 9. \square

Lemma 4.4. *Let $G \neq C_{11}$ be a reduced well-indumatched unicyclic graph of girth $g \geq 10$ with the unique cycle $C = v_1v_2 \cdots v_gv_1$, such that for each i , T_{v_i} is one of the two rooted trees shown in Figure 1. Then, for each i (modulo 9), the trees T_{v_i} and $T_{v_{i+1}}$ are alternately of Type (i) and (ii). In particular, g is even and the size of any maximal induced matching is $\frac{g}{2}$.*

Proof. Let i be an integer, $1 \leq i \leq g$. By removing the edge $v_{i+6}v_{i+7}$ and the edges covered by $v_{i+6}v_{i+7}$, we obtain a well-indumatched forest. Let T be the component of that forest containing v_iv_{i+1} . Since T contains a path of order 6 induced by v_{i-1}, \dots, v_{i+4} , $R(T)$ is a reduced well-indumatched tree with at least 5 vertices. Then by Theorem 3.1, every edge of $R(T)$ is covered by exactly one of the good pendant edges of $R(T)$. In particular, the edge v_iv_{i+1} has to be covered by exactly one good pendant edge.

If $g \geq 12$, it follows that exactly one of T_{v_i} and $T_{v_{i+1}}$ should be of Type (ii) and the other of Type (i). This yields that for each i modulo 9, the trees T_{v_i} and $T_{v_{i+1}}$ are alternately of Type (i) and (ii) and consequently g is even.

By Lemma 4.3 (i), C_{11} is the only graph of girth 11 with the desired properties, and thus $g \neq 11$.

If $g = 10$ and $v_{i-1}v_i$ is a good pendant edge of $R(T)$ covering v_iv_{i+1} (thus also $v_{i+1}v_{i+2}$), then $T_{v_{i-1}}, T_{v_i}, T_{v_{i+1}}, T_{v_{i+2}}$ are of Type (i) and $v_{i+3}v_{i+4}$ is not a good pendant edge of $R(T)$. Since $v_{i+2}v_{i+3}$ should be covered by one good pendant edge, $T_{v_{i+3}}$ is of Type (ii) and $T_{v_{i+4}}$ is of Type (i). Now, removing the edge $v_{i+7}v_{i+8}$ and the edges covered by $v_{i+7}v_{i+8}$, we obtain a well-indumatched forest. However, its component containing v_{i+3} has a P_6 induced by v_i, v_{i+1}, v_{i+2} and three vertices of $T_{v_{i+3}}$ which is forbidden for being well-indumatched by Lemma 3.4 (ii), a contradiction. It follows

that, in order to cover the edge $v_i v_{i+1}$ by exactly one good pendant edge of $R(T)$, exactly one of T_{v_i} and $T_{v_{i+1}}$ should be of Type (ii), and the other of Type (i). Then we conclude as in the case $g \geq 12$.

We complete the proof by noting that the set of good pendant edges of G forms an induced matching of size $\frac{g}{2}$. Moreover, it covers all edges, and thus it is maximal. Since G is well-indumatched, it follows that all maximal induced matchings have size $\frac{g}{2}$. \square

Corollary 4.1. *The only minimal well-indumatched graph of girth 11 is C_{11} .*

We are now ready to characterize all minimal well-indumatched graphs of girth at least 9.

Theorem 4.1. *The graph G is a minimal well-indumatched graph of girth at least 9 if and only if either $G = C_{11}$ or G is a reduced unicyclic graph of even girth $g \geq 10$ with the unique cycle $C = v_1 v_2 \cdots v_g v_1$, such that for each i , T_{v_i} is alternately of Type (i) and Type (ii).*

Proof. Let G be a minimal well-indumatched graph of girth $g \geq 9$. By Lemma 4.2, G is a reduced unicyclic graph. Moreover, if $C = v_1 v_2 \cdots v_g v_1$ is its unique cycle, then every T_{v_i} is of one of the Types (i) or (ii). By Lemma 4.3 (ii), there is no well-indumatched graph of girth 9. So we have $g(G) \geq 10$. Then either $G = C_{11}$ by Lemma 4.3 (i), or the length of the unique cycle is even and for each i , T_{v_i} is alternately of Type (i) and Type (ii) by Lemma 4.4.

Now let G be a reduced unicyclic graph of even girth g at least 10 with the unique cycle $C = v_1 v_2 \cdots v_g v_1$, such that for each i , T_{v_i} is alternately of Type (i) and Type (ii). Also let M be a matching of G containing all good pendant edges. Then each edge of G is covered by exactly one edge of M . Moreover, if two edges of G are covered by the same edge of M , then these two edges cover each other. It follows from Lemma 3.6 that G is well-indumatched. Let us now show that G is also a minimal well-indumatched graph of girth g . Indeed, any subgraph G' of G of the same girth g is obtained by removing (at least) one edge from $E(G) \setminus E(C)$. If $R(G')$ has $T_{v_i} = P_2$ for some i , then by Lemma 4.2, G' is not well-indumatched. So assume that every T_{v_i} in $R(G')$ is of one of the Types (i) or (ii) with both v_i and v_{i+1} of degree 2 (in $R(G')$), for some i . Then G' is not well-indumatched, by Lemma 4.4. \square

It is not hard to see that given a graph G , one can check in time $O(m)$ whether G is reduced and unicyclic; and if it is unicyclic, whether for each i , T_{v_i} is alternately of Type (i) and (ii) (and therefore whether it has even girth at least 10). It follows from Theorem 4.1 that one can decide in time $O(m)$ whether a given graph is minimal well-indumatched of girth at least 9. However, unfortunately, this result does not imply a polynomial time algorithm to recognize well-indumatched graphs of girth at least 9. Indeed, this is due to the fact that the property of being well-indumatched is not hereditary, as we already noted in Section 2.

Theorem 4.1 also implies the following, which is of interest on its own.

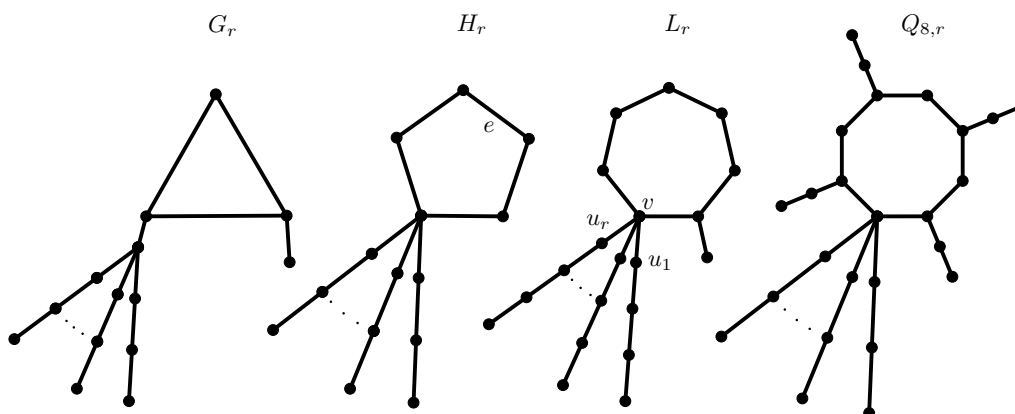


Figure 2: Well-indumatched unicycle graphs of girths 3, 5, 7 and 8

Corollary 4.2. *For an odd integer $g \geq 9$ and $g \neq 11$, there is no well-indumatched graph of girth g .*

Unlike this negative result, we show in what follows that there are infinitely many well-indumatched unicyclic graphs of even girth or small odd girth, that is 3, 5 and 7.

Let $r \geq 1$ and $k \geq 0$ be integers, and let $S_{r,k}$ be a tree obtained by subdividing each edge of $K_{1,r}$ by k vertices. Consider the disjoint union of C_3 and $S_{r,2}$. Join the vertex of degree r in $S_{r,2}$ to a vertex of C_3 and add a new vertex and join it to another vertex of C_3 . Denote this graph by G_r and note that the order of this graph is $3r + 5$. Consider the disjoint copy of C_5 and $S_{r,2}$ and identify the vertex of degree r in $S_{r,2}$ with a vertex of C_5 and denote the resulting graph by H_r . The order of this graph is $3r + 5$. Also, identify a vertex of C_7 and the vertex of degree r in $S_{r,3}$ and add a new vertex and join it to a neighbour of the identified vertex in C_7 . Denote this graph by L_r . The order of L_r is $4r + 8$. Finally, for any even integer $k \geq 4$, consider the disjoint copy of C_k with the vertex set $\{v_1, \dots, v_k\}$ and $S_{r,2}$ and identify the vertex of degree r in $S_{r,2}$ with v_1 . Also, add $\frac{k}{2}$ copies of P_2 and join one vertex of each of them to v_2, v_4, \dots, v_k . We denote the resulting graph by $Q_{k,r}$ (see Figure 2).

Theorem 4.2. *Let r be a positive integer. Then the following statements hold:*

- (i) $S_{r,2}$ is a well-indumatched tree.
- (ii) G_r is a well-indumatched unicyclic graph of girth 3.
- (iii) H_r is a well-indumatched unicyclic graph of girth 5.
- (iv) L_r is a well-indumatched unicyclic graph of girth 7.
- (v) For every even integer $k \geq 4$, $Q_{k,r}$ is a well-indumatched unicyclic graph of girth k .

Thus there are infinitely many well-indumatched trees and infinitely many well-indumatched unicyclic graphs of girth k , where $k \in \{3, 5, 7\}$ or k is an even integer greater than 2.

Proof. (i), (ii), (v) Let G be one of the graphs $S_{r,2}$, G_r or $Q_{k,r}$. The pendant edges of G form a matching and each edge of G is covered by exactly one of these edges. Moreover, any two edges of G which are covered by the same pendant edge cover each other. Therefore Lemma 3.6 implies that G is well-indumatched.

(iii) Let M be the set of pendant edges of H_r together with e (see Figure 2). M is a matching of H_r and each edge of H_r is covered by exactly one edge of M . Moreover, any two edges of H_r which are covered by the same edge of M , cover each other. Therefore, by Lemma 3.6, H_r is well-indumatched.

(iv) For $i = 1, \dots, r$, denote the path of length 4 in L_r which starts at v and contains u_i by P_{u_i} . Let M be a maximal induced matching of L_r . If M contains vu_i , for some i , then M should contain two edges of P_{u_i} , one edge of each P_{u_j} , for $j \neq i$, and one edge of C_7 . So $|M| = r + 2$. If M does not contain vu_i for any i , then M should contain one edge of each P_{u_j} and two edges of C_7 . Thus $|M| = r + 2$. Therefore the size of every maximal induced matching of L_r is $r + 2$ and L_r is well-indumatched. \square

The following result gives an infinite family of well-indumatched graphs of girth 3, with the additional property of being regular.

Theorem 4.3. *For every positive integer $r \geq 3$, there are infinitely many well-indumatched r -regular graphs of girth 3.*

Proof. Let t be a positive integer and consider t disjoint copies of complete graphs of order $r + 1$, say $K_{r+1}^1, K_{r+1}^2, \dots, K_{r+1}^t$, and let $x_i y_i, u_i v_i \in E(K_{r+1}^i)$ be two disjoint edges, $i = 1, \dots, t$. Now, join u_i to x_{i+1} and v_i to y_{i+1} , for $i = 1, \dots, t - 1$, and then remove $x_i y_i$ and $u_i v_i$, for $i = 1, \dots, t$, and call the resulting graph G_t . Also, let $H_t = G_t + x_1 y_1$ and $L_t = G_t + x_1 y_1 + u_t v_t$. Note that L_t is r -regular and has girth 3.

By strong induction on t , we prove that the size of all maximal induced matchings of G_t , H_t and L_t is t , and therefore these graphs are well-indumatched. For $t = 1$ the assertion is clear. Let the result hold for G_j, H_j, L_j , $j = 1, \dots, t - 1$. Let M be a maximal induced matching of L_t . If for all i , $1 \leq i \leq t - 1$, none of the $u_i x_{i+1}$ and $v_i y_{i+1}$ are in M , then it is not hard to see that $|M| = t$. If there is some i , $1 \leq i \leq t - 1$, such that one of the $u_i x_{i+1}$ or $v_i y_{i+1}$ is in M , then both of these edges should be in M . By removing the edges covered by $u_i x_{i+1}, v_i y_{i+1}$, we obtain a graph with two components which are isomorphic to G_{i-1} and G_{t-i-1} . Since the restriction of M to each of these graphs is a maximal induced matching, by the induction hypothesis, M has $i - 1$ edges in the component isomorphic to G_{i-1} and $t - i - 1$ edges in the other component. Thus, $|M| = 2 + (i - 1) + (t - i - 1) = t$. The proof for H_t and G_t is similar. \square

We conclude this section by noting that our results settle the existence of well-indumatched graphs with all possible girths except girth 11. In other words, for all girth $g \neq 11$ and at least 3, either we establish that there is no well-indumatched graph of girth g or we exhibit an infinite family of well-indumatched graphs of girth g .

The only exception to this dichotomy is $g = 11$ for which we only know that the only minimal well-indumatched graph of girth 11 is C_{11} . We conjecture the following:

Conjecture 4.1. The cycle C_{11} is the only connected well-indumatched graph of girth 11.

5 Well-indumatched graphs with maximal induced matchings of fixed size

Let us call a graph *k-well-indumatched* if all of its maximal induced matchings have size k . In this section we focus on k -well-indumatched graphs with fixed k . Let K_n be a clique on n vertices. We start with a general observation:

Remark 5.1. If G is a k -well-indumatched graph then it is $(k + 1)K_2$ -free.

On the other hand, as expected, if ℓ is the smallest integer such that G is ℓK_2 -free, then G is not necessarily $(\ell - 1)$ -well-indumatched. For instance, a P_5 is $3K_2$ -free, but it is not 2-well-indumatched. However, if we restrict to 1-well-indumatched graphs, then the converse becomes true as well. Indeed, if G is not 1-well-indumatched, then G is either k -well-indumatched with $k \geq 2$, or it is not well-indumatched. In both cases, G has an induced matching of size 2 which induces a $2K_2$. So we have the following:

Remark 5.2. A graph G is 1-well-indumatched if and only if G is a non-empty $2K_2$ -free graph.

Remark 5.2 is a forbidden subgraph characterization for 1-well-indumatched graphs. This implies directly that 1-well-indumatched graphs form a hereditary class of graphs, that is a class of graphs closed under taking induced subgraphs. Note that this is in contrast with the non-hereditary nature of k -well-indumatched graphs starting from already $k \geq 2$; recall the example of a P_7 which is 2-well-indumatched but contains a P_5 which is not well-indumatched.

Remarks 5.1 and 5.2 have some consequences from the computational complexity point of view. Note that INDEPENDENT SET and DOMINATING SET problems are shown to be NP-complete in the class of well-indumatched graphs [3]. In contrast to this hardness result, Remark 5.1 and the fact that WEIGHTED INDEPENDENT SET is polynomial time solvable in kK_2 -free graphs, for any fixed k [2], implies the following:

Corollary 5.1. WEIGHTED INDEPENDENT SET is polynomial time solvable, when restricted to k -well-indumatched graphs for any positive k .

On the other hand, the NP-completeness of the DOMINATING SET problem in well-indumatched graphs can be strengthened using Remark 5.2 and the fact that DOMINATING SET is NP-complete in split graphs [6, 10]. A graph is *split*, if its vertex set can be partitioned into a clique and an independent set. It is known that G is a split graph, if and only if it contains no $2K_2$, C_4 or C_5 as induced subgraph [20].

Thus, split graphs are $2K_2$ -free and therefore 1-well-indumatched graphs contain the class of split graphs. This implies that DOMINATING SET is NP-complete in well-indumatched graphs, even if every maximal induced matching has size 1.

Corollary 5.2. *DOMINATING SET is NP-complete in 1-well-indumatched graphs.*

Note that there is no containment relationship between k -well-indumatched graphs and k' -well-indumatched graphs for $k' > k$ (they are actually disjoint sets partitioning the set of all well-indumatched graphs) and therefore, Corollary 5.2 has no consequence on the NP-completeness of DOMINATING SET in k -well-indumatched graphs for $k > 1$.

Remark 5.2 is also an important intermediary result which makes the recognition of k -well-indumatched graphs polynomial time solvable, whenever k is fixed. Note that this is in contrast with the co-NP-completeness of the recognition problem in general [3].

Theorem 5.1. *Given a graph G , it can be decided in time $O(m^{k-1}n^4)$ whether G is k -well-indumatched or not, where n and m are the order and the size of G , respectively.*

Proof. We note that a graph G is k -well-indumatched if and only if for every edge $e \in E(G)$, the graph $G \setminus C(e)$, where $C(e)$ is the set of edges covered by e , is $(k - 1)$ -well-indumatched. This is indeed a necessary condition for G being k -well-indumatched. Besides, if for all $e \in E(G)$, every maximal induced matching of $G \setminus C(e)$ has size $k - 1$, then every maximal induced matching of G has size k , thus G is k -well-indumatched. Now, repeat recursively $k - 1$ times the removal of an edge e together with $C(e)$. The above equivalence implies that G is well-indumatched if and only if this recursive procedure yields a 1-well-indumatched graph for any choice of $k - 1$ edges throughout the recursive procedure. Since there are at most $O(m^{k-1})$ such choices and whether the remaining graph is 1-well-indumatched or not can be checked in time $O(n^4)$ (by Remark 5.2, simply by checking if any possible subset of four vertices induces a $2K_2$), the overall procedure takes time $O(m^{k-1}n^4)$. \square

6 Conclusion

Well-indumatched graphs were introduced to the literature very recently. Consequently, the structure of well-indumatched graphs is not yet well understood and seems to be a very promising research area. In this work, we have characterized well-indumatched trees and studied well-indumatched graphs of bounded girth. We have established several structural results on well-indumatched graphs of bounded girth and conjectured that there is no connected well-indumatched graph of girth 11 other than C_{11} .

As a future research topic, it would be interesting to characterize those well-indumatched graphs in special graph classes and to derive polynomial time recognition algorithms. Our characterization of minimal well-indumatched graphs of girth at least 9 (in Theorem 4.1) does not seem to imply directly a polynomial time

recognition algorithm for well-indumatched graphs of girth at least 9. It would be interesting to exploit this characterization to develop such a recognition algorithm, or to investigate the recognition of well-indumatched graphs of bounded girth more broadly. Some other graph classes that could be investigated in this direction are interval graphs, claw-free graphs or equimatchable graphs.

Another research direction would be the study of graphs having a bounded gap (1 or some fixed k) between the size of a maximum induced matching and minimum maximal induced matching. This approach has been applied to well-covered graphs and yielded several significant results (see e.g. [4, 15]), and more recently to equimatchable graphs [13].

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