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Equitably 2-colourable even cycle systems

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Abstract

An ℓ-cycle decomposition of a graph G is said to be *equitably c-colourable* if there is a c-vertex-colouring of G such that each colour is represented (approximately) an equal number of times on each cycle. In this paper, we consider the existence of equitably 2-colourable even ℓ-cycle systems of the cocktail party graph $K_v - I$. After establishing that the problem of proving existence of equitably 2-colourable ℓ-cycle decompositions of $K_v - I$ reduces to considering ℓ-admissible values $v \in [\ell, 2\ell)$, we determine a complete existence result for equitably 2-colourable ℓ-cycle decompositions of $K_v - I$ in the cases that $v \equiv 0, 2 \pmod{\ell}$, or ℓ is a power of 2, or $\ell \in \{2q, 4q\}$ for q an odd prime power, or $\ell \leq 30$.

1 Introduction

An *ℓ-cycle decomposition* of a graph G is a partition of the edge set of G into cycles of length ℓ. The study of cycle decompositions, which are also called *cycle systems*, of the complete graph has a long history, dating to the work of Kirkman and Walecki in the 19th century. Obvious necessary conditions for the existence of an ℓ-cycle

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decomposition of the complete graph K_v are that v is odd, $3 \leq \ell \leq v$ and $\ell \mid \binom{v}{2}$. For an even number v of vertices, it is common to instead decompose $K_v - I$, the complete graph on v vertices with the edges of a 1-factor I removed (also called the *cocktail party graph*). For ease of notation in referring to various cycle decomposition results, we thus use the notation K_v^* to denote K_v if v is odd and $K_v - I$ if v is even. The existence of ℓ-cycle decompositions of K_v^* was completely solved in [3, 18]; see also [5].

Theorem 1.1 ([3, 18]) *There is an ℓ-cycle decomposition of K_v^* if and only if $3 \leq \ell \leq v$ and $\ell \mid v\lfloor \frac{v-1}{2} \rfloor$.*

For a given ℓ, we refer to an integer v satisfying the conditions of Theorem 1.1 as *ℓ-admissible*, or simply *admissible* if the value of ℓ is understood.

Suppose $\mathcal{D}$ is a cycle decomposition of G. A *c-colouring* of $\mathcal{D}$ is a function $\phi : V(G) \to \{1, \ldots, c\}$; informally, this can be though of an assignment of c colours to the vertices of G. In this paper, many of the colourings considered will have $c = 2$; we will often refer to colour 1 as *red* and colour 2 as *blue*. Given a c-colouring ϕ and a colour $i \in \{1, \ldots, c\}$, we refer to the set of vertices of colour i (i.e. the pre-image $\phi^{-1}(i)$) as *colour class i*.

When considering colourings of cycle decompositions and other designs, we generally require the function ϕ to have further properties with respect to the cycles of $\mathcal{D}$. For instance, among the most commonly studied colourings are so-called *weak colourings*, in which each cycle of $\mathcal{D}$ must have at least two vertices coloured differently. The smallest number of colours for which the cycle decomposition $\mathcal{D}$ admits a weak colouring is the *chromatic number* of $\mathcal{D}$, denoted $\chi(\mathcal{D})$. A simple counting argument shows that any 3-cycle decomposition $\mathcal{D}$ of K_v, where $v > 3$, has $\chi(\mathcal{D}) \geq 3$; see [10]. For chromatic numbers $\chi \geq 3$, de Brandes, Phelps and Rödl [9] established the existence of an integer v_χ for which any admissible order $v > v_\chi$ admits a χ-chromatic 3-cycle system of K_v. Burgess and Pike [7, 8] showed existence of even cycle systems with arbitrary chromatic number; this result was extended by Horsley and Pike [13], who showed asymptotic existence of χ-chromatic ℓ-cycle systems for any integers $\chi \geq 2$ and $\ell \geq 3$ with $(\chi, \ell) \neq (2, 3)$.

In this paper, we consider a stricter type of colouring, requiring each colour to be represented (approximately) an equal number of times on each cycle. Specifically, in an *equitable c-colouring* of an ℓ-cycle decomposition, we have that in each cycle C of the decompositions, each colour appears on $\lfloor \ell/c \rfloor$ or $\lceil \ell/c \rceil$ of the vertices of C. Clearly for $c \geq 2$, every equitably c-colourable cycle system is weakly c-colourable; however the converse is false. We note that the term *equitable colouring* also occurs in the literature to mean colourings in which the cardinalities of the colour classes differ by at most one (see, for instance [10]); here, we follow the terminology of [1, 2] in using the term *equitable* to mean that there is even representation of colours on each cycle. We also note that the terminology of equitable colouring has been used in the context of block colourings, see for example [16].

Equitable colourings of cycle systems of the complete and cocktail party graphs were considered in [1] and [2], while the papers [15] and [20] consider equitable colourings of complete multipartite graphs. The main results of these papers restrict their attention to the case that the number of colours is 2 or 3, and the cycle length is small (at most 6). In particular, in [1] and [2], the authors completely determine the existence of equitably 2- or 3-colourable ℓ-cycle systems of K_v and $K_v - I$ when the cycle length ℓ is 4, 5 or 6. Specifically, the main results of these two papers are as follows.

Theorem 1.2 ([1]) *Let $\ell \in \{4, 5, 6\}$ and let v be ℓ-admissible.*

1. *If ℓ is even and v is odd, there is no equitably 2-colourable ℓ-cycle decomposition of K_v.*
2. *If $\ell = 5$ and v is odd, there is an equitably 2-colourable 5-cycle decomposition of K_v. Moreover, if $v > 5$, there also exists a 5-cycle decomposition of K_v which is not equitably 2-colourable.*
3. *If v is even, there is an equitably 2-colourable ℓ-cycle decomposition of $K_v - I$.*

Theorem 1.3 ([2]) *Let $\ell \in \{4, 5, 6\}$ and let v be ℓ-admissible.*

1. *If $\ell = 4$, then there is an equitably 3-colourable 4-cycle decomposition of K_v if and only if $v = 9$, and an equitably 3-colourable 4-cycle decomposition of $K_v - I$ if and only if $v \in \{4, 6, 8, 10, 12, 18\}$.*
2. *If $\ell = 5$, then there is an equitably 3-colourable 5-cycle decomposition of K_v^*.*
3. *If $\ell = 6$, then there is an equitably 3-colourable 6-cycle decomposition of K_v^* if and only if $3 \mid v$.*

It is notable that while asymptotic existence of both χ-chromatic cycle systems and balanced incomplete block designs is known [13, 14], the problems of equitably colouring cycle decompositions and BIBDs are quite different. In a 2016 paper, Luther and Pike [17] determined that there exists an equitably c-colourable nontrivial BIBD(v, k, λ) if and only if either $c \geq v$ (so that the design is trivially equitably colourable) or else $v = k + 1$ and $k \equiv c - 1 \pmod{c}$. However, subject to additional necessary conditions when $k \mid c$ or $k \leq c - 1$, which will be discussed in Section 2, Theorems 1.2 and 1.3 suggest that an equitably c-colourable ℓ-cycle system may exist for any ℓ-admissible order.

In this paper, we consider the existence of equitably 2-colourable ℓ-cycle decompositions of K_v^*. In Section 2, we discuss basic results on equitably colourable cycle decompositions and methods for constructing cycle decompositions. In Section 3, we construct equitably 2-colourable ℓ-cycle decompositions of complete bipartite graphs, which are subsequently employed in Section 4 to show that the problem of proving existence of equitably 2-colourable ℓ-cycle decompositions of $K_v - I$ reduces to considering ℓ-admissible values $v \in [\ell, 2\ell)$. Finally, in Section 5, we determine a complete existence result for equitably 2-colourable ℓ-cycle decompositions of K_v^* in each of the following cases:

- $v \equiv 0$ or $2 \pmod{\ell}$
- ℓ is a power of 2,
- $\ell \in \{2q, 4q\}$ for q an odd prime power, and
- $\ell \leq 30$.

2 Preliminaries

2.1 Cycle decompositions and equitable colourings

In this section, we present some basic properties of equitably coloured cycle decompositions.

We begin by noting that any Hamiltonian cycle system can easily be equitably coloured. While we state the result for decompositions of K_v^*, we note that the result holds more generally for any graph G of order ℓ which decomposes into Hamiltonian cycles.

Lemma 2.1 *Let ℓ and c be positive integers with $\ell \geq 3$. Any ℓ-cycle decomposition of K_ℓ^* is equitably c-colourable.*

Proof: Let $\ell = qc + r$, where $0 \leq r < c$. Partitioning $V(K_\ell^*)$ into r colour classes of cardinality $q+1$, and $c-r$ of cardinality q gives the required equitable colouring. □

Clearly, if there is an equitably c-colourable ℓ-cycle decomposition of K_v^*, then v is ℓ-admissible. We now describe further necessary conditions, first recalling the following result from [2].

Lemma 2.2 ([2]) *Let v be odd. If there is an equitably $(\ell-1)$-colourable ℓ-cycle decomposition of K_v, then $v \leq (\ell-1)^2$.*

In the case of two colours, this result implies that there can be no equitably 2-colourable 3-cycle decomposition of K_v where $v > 3$. (This fact can also be deduced from the nonexistence of a weakly 2-colourable nontrivial Steiner triple system; see [10].) While the result of Lemma 2.2 does not apply in general for even orders v, we are able to determine the spectrum of equitably 2-colourable 3-cycle decompositions of $K_v - I$.

Theorem 2.3 *There is an equitably 2-colourable 3-cycle decomposition of K_v^* if and only if $v \in \{3, 6, 8\}$.*

Proof: If v is odd, the result follows from Lemma 2.2 for $v > 3$, with the existence of an equitably 2-colourable 3-cycle decomposition of K_3 being trivial.

Now, let v be even and suppose there exists an equitably 2-colourable 3-cycle decomposition $\mathcal{D}$ of $K_v - I$. Adding a vertex ∞ and the 3-cycles (∞, x_1, x_2) for each edge $\{x_1, x_2\} \in I$ gives a 3-cycle decomposition $\mathcal{D}'$ of K_{v+1}. Note that $\mathcal{D}'$ is 3-chromatic, and a weak 3-colouring is given by assigning ∞ a new colour. Proposition 1.2 of [12] states that the two largest colour classes in a weak c-colouring of any 3-cycle decomposition of K_u can contain at most $\frac{4}{5}u + 1$ vertices. Consequently, the size of the smallest colour class in any weak 3-colouring of such a decomposition is at least $\frac{1}{5}u - 1$, and so if $u > 10$, there can be no colour class of size 1. Hence it must be that $v \leq 10$, leaving the 3-admissible values $v = 6, 8$.

For $v = 6$, the following 3-cycles decompose $K_6 - I$, and can be equitably 2-coloured by colouring vertices 1, 3 and 5 blue, and vertices 2, 4 and 6 red:

$$(1, 2, 4), (1, 6, 5), (2, 3, 5), (3, 4, 6).$$

For $v = 8$, the following 3-cycles decompose $K_8 - I$, and can be equitably 2-coloured by colouring vertices 1, 2, 4, 8 blue, and vertices 3, 5, 6, 7 red:

$$(1, 3, 8), (1, 4, 7), (1, 5, 6), (2, 3, 7), (2, 4, 6), (2, 5, 8), (3, 4, 5), (6, 7, 8).$$

□

We now consider the case where the number of colours is a divisor of the cycle length. The following result is used in [1, 2] in the case $c \in \{2, 3\}$; however, we prove it here more generally for the sake of completeness.

Lemma 2.4 *Suppose there is an equitably c-colourable ℓ-cycle system of K_v^*, where $c \mid \ell$. Then $c \mid v$, and each colour class has size v/c.*

Proof: Let the colour classes be $X_1, X_2, \ldots, X_c$. Note that each vertex appears in $r = \lfloor (v-1)/2 \rfloor$ cycles, and hence for $i \in \{1, 2, \ldots, c\}$, the total number of appearances of colour i in all cycles is $r|X_i|$. But each colour appears equally often in every cycle, so must have the same number of total appearances. □

Our main focus in this paper is the existence of equitably 2-colourable ℓ-cycle systems for even ℓ. One immediate consequence of Lemma 2.4 is the following.

Lemma 2.5 ([1]) *If ℓ is even, then there is no equitably 2-colourable ℓ-cycle decomposition of K_v.*

In this paper, we will thus consider equitably 2-colourable cycle decompositions of $K_v - I$. Among the tools we use to construct such decompositions are auxiliary decompositions of the complete bipartite graph. We recall the following result of Sotteau [19], which gives necessary and sufficient conditions for ℓ-cycle decomposition of $K_{a,b}$.

Theorem 2.6 ([19]) *There is an ℓ-cycle decomposition of $K_{a,b}$ if and only if a, b and ℓ are all even, $\ell \leq \min\{2a, 2b\}$ and $\ell \mid ab$.*

Note that any ℓ-cycle decomposition of $K_{a,b}$ can be equitably 2-coloured: colour the vertices in one part with one colour, and the vertices in the other part with another. However, we will require colourings which satisfy a stronger property.

Definition 2.7 Let $\mathcal{D}$ be an ℓ-cycle decomposition of $K_{a,b}$. A c-colouring ϕ of $\mathcal{D}$ is called *doubly equitable* if

1. ϕ is an equitable c-colouring, i.e. each cycle of $\mathcal{D}$ contains $\lfloor \ell/c \rfloor$ or $\lceil \ell/c \rceil$ vertices of each colour, and
2. ϕ equitably colours the parts of $K_{a,b}$, i.e. for each colour i, there are $\lfloor a/c \rfloor$ or $\lceil a/c \rceil$ vertices of colour i in the part of size a, and $\lfloor b/c \rfloor$ or $\lceil b/c \rceil$ vertices of colour i in the part of size b.

Example 2.8 We identify $V(K_{4,4})$ with $\mathbb{Z}_4 \times \{0, 1\}$, and write for brevity x_i rather than (x, i). Now consider the 4-cycle decomposition

$$\mathcal{C} = \{(0_0, 0_1, 1_0, 1_1), (0_0, 2_1, 1_0, 3_1), (2_0, 0_1, 3_0, 1_1), (2_0, 2_1, 3_0, 3_1)\}.$$

Colouring vertex x_i red if x is even and blue otherwise yields a doubly equitable 2-colouring.

2.2 Cyclic and 2-pyramidal cycle systems: difference methods

In what follows, we will construct and colour cycle systems exhibiting some regularity properties, namely cyclic and 2-pyramidal cycle systems. To build these systems, we shall use difference methods.

We briefly recall here some definitions and results useful in these constructions.

First, a cycle system regular under the action of the cyclic group is said to be *cyclic*. Cyclic cycle systems have been well studied, and can be constructed using the method of *partial differences*, see for instance [6].

Definition 2.9 Let $C = (x_0, x_1, \ldots, x_{\ell-1})$ be an ℓ-cycle with vertices in $\mathbb{Z}_v$ and let d be the order of the stabilizer of C under the natural action of $\mathbb{Z}_v$, that is, $d = |\{g \in \mathbb{Z}_v : C + g = C\}|$. The multisets

$$\begin{aligned} \Delta C &= \{\pm(x_{h+1} - x_h) \mid 0 \leq h < \ell\}, \\ \partial C &= \{\pm(x_{h+1} - x_h) \mid 0 \leq h < \ell/d\}, \end{aligned}$$

where the subscripts are taken modulo ℓ, are called, respectively, the *list of differences* from C and the *list of partial differences* from C.

If $\mathscr{C}$ is a set of ℓ-cycles with vertices in $\mathbb{Z}_v$, by $\Delta\mathscr{C}$ and $\partial\mathscr{C}$ we mean the union (counting multiplicities) of all multisets ΔC and ∂C respectively, where $C \in \mathscr{C}$.

We will use the following notation: let $x_0, x_1, \ldots, x_{r-1}, x$ be elements of $\mathbb{Z}_v$, with x of order d. The closed trail represented by the concatenation of the sequences

$$\begin{gathered}[x_0, x_1, \ldots, x_{r-1}],\\ [x_0 + x, x_1 + x, \ldots, x_{r-1} + x],\\ [x_0 + 2x, x_1 + 2x, \ldots, x_{r-1} + 2x],\\ \vdots\\ [x_0 + (d-1)x, x_1 + (d-1)x, \ldots, x_{r-1} + (d-1)x]\end{gathered}$$

will be denoted by

$$[x_0, x_1, \ldots, x_{r-1}]_x. \tag{2.1}$$

Example 2.10 In $K_{20} - I$, with $[0, 1, 19]_{15}$, we mean the closed trail (a 12-cycle in this case) $C = (0, 1, 19, 15, 16, 14, 10, 11, 9, 5, 6, 4)$; its list of partial differences is $\partial C = \pm\{1, 2, 4\}$.

A cyclic cycle system of $K_v - I$ is completely determined by a set of *base cycles*, that is a system of representatives for the orbits of its cycles under the action of $\mathbb{Z}_v$. The next result (see for instance Theorem 3.3 in [6]) shows how to use partial differences to build a set of base cycles.

Theorem 2.11 *A set $\mathscr{C}$ of ℓ-cycles is a set of base cycles of a cyclic cycle system of $K_v - I$ if and only if $\partial\mathscr{C} = \mathbb{Z}_v - \{0, v/2\}$.*

Remark 2.12 Note that $[x_0, x_1, \ldots, x_{r-1}]_x$ is a (dr)-cycle if and only if the elements x_i, for $i = 0, \ldots, r-1$, belong to pairwise distinct cosets of the subgroup $\langle x\rangle$ in $\mathbb{Z}_v$. Also, if $C = [x_0, x_1, \ldots, x_{r-1}]_x$ is a (dr)-cycle then

$$\partial C = \{\pm(x_i - x_{i-1}) \mid i = 1, \ldots, r-1\} \cup \{\pm(x_0 + x - x_{r-1})\}.$$

We point out that in the case of cyclic ℓ-cycle system of $K_v - I$, we have that $dr = \ell$ and the order of $Stab(C)$ is d; the length of the $\mathbb{Z}_v$-orbit of C is v/d.

Example 2.13 In $K_{20} - I$, a set $\mathscr{C}$ of base cycles for a cyclic cycle system of the graph $K_{20} - I$ is given by the two cycles

$$C_1 = [0, 1, 19]_{15} = (0, 1, 19, 15, 16, 14, 10, 11, 9, 5, 6, 4)$$

and

$$C_2 = [0, 3, -3, 4, -4, 5]_{10} = (0, 3, 17, 4, 16, 5, 10, 13, 7, 14, 6, 15),$$

since $\partial C_1 = \pm\{1, 2, 4\}$ and $\partial C_2 = \pm\{3, 5, 6, 7, 8, 9\}$.

We will also consider 2-*pyramidal cycle systems* for the graph $K_v - I$ (see for instance [4, 11]).

Definition 2.14 An ℓ-cycle system $\mathcal{D}$ of the graph $K_v - I$ is 2-*pyramidal* under the action of $\mathbb{Z}_{v-2}$ if the set of vertices of $K_v - I$ is identified with $\mathbb{Z}_{v-2} \cup \{\infty_1, \infty_2\}$, the removed 1-factor I is formed by the edges $\{(0,(v-2)/2)+g, g \in \mathbb{Z}_{v-2}\} \cup \{(\infty_1, \infty_2)\}$, and we have $C + g \in \mathcal{D}$ for all $g \in \mathbb{Z}_{v-2}$, with the assumption that $\infty_i + g = \infty_i$, $i = 1, 2$.

Also in this case, if $C = (x_0, x_1, \ldots, x_{\ell-1})$ is an ℓ-cycle with vertices in $\mathbb{Z}_{v-2}$, and if d is the order of the stabilizer of C under the natural action of $\mathbb{Z}_{v-2}$, then the list of differences (resp. partial differences) is $\Delta C = \{\pm(x_{h+1} - x_h) \mid 0 \le h < \ell\}$ (resp. $\partial C = \{\pm(x_{h+1} - x_h) \mid 0 \le h < \ell/d\}$).

Let $v - 2 = 2n$; if $P = x_0, x_1, \ldots, x_p$, with $x_i \in \mathbb{Z}_{2n}$, is a path in $K_v - I$, we denote with $\sigma(P)$ the path $x_0 + n, x_1 + n, \ldots, x_p + n$. Let C_P be the cycle formed from the edges of P and $\sigma(P)$ together with the edges $x_0\infty_1$, $\infty_1(x_0 + n)$, $x_p\infty_2$ and $\infty_2(x_p + n)$. For a cycle C_P, set $\partial C_P = \{\pm(x_{h+1} - x_h) \mid 0 \le h < p\}$.

We say that a set $\mathscr{C} = \{A_1, A_2, \ldots, A_s, B\}$ of ℓ-cycles is a set of *base cycles of type* σ for a 2-pyramidal cycle system of $K_{2n+2} - I$ if $B = C_P$ for some path P of length $(\ell-4)/2$, $V(A_i) \subseteq \mathbb{Z}_{2n}$, and $\partial\mathscr{C} = \mathbb{Z}_{2n} - \{0, n\}$. We remark that 2-pyramidal cycle systems of type σ are extensively used in [3], but are not referred to explicitly using this terminology.

The proof of the following lemma is a straightforward application of standard difference methods.

Lemma 2.15 *If* $\mathscr{C} = \{A_1, A_2, \ldots, A_s, B\}$ *of* ℓ*-cycles is a set of base cycles of type* σ*, then* $\mathcal{C} = \cup_{C \in \mathscr{C}} Orb(C)$ *is a* 2*-pyramidal cycle system of* $K_{2n+2} - I$*.*

Example 2.16 Let $\ell = 8$ and $v = 10$; a set of base cycles of type σ is given by the pair $A = (0, 1, 2, 3, 4, 5, 6, 7) = [0]_1$, with $\partial A = \pm\{1\}$, and $B = C_P$ with $P = 0, 2, 7$, so that $B = (\infty_1, 0, 2, 7, \infty_2, 3, 6, 4)$ and $\partial B = \pm\{2, 3\}$.

3 Equitably colourable decompositions of the complete bipartite graph

In the proof of the reduction step in Section 4, we will use doubly equitable 2-colourable cycle systems for the graphs $K_{\ell,\ell+r}$, with ℓ and r even and $r < \ell$; in this section we prove the existence of these decompositions.

First, let us denote by Π_0 the part of size ℓ and by Π_1 that of size $\ell + r$; we will identify Π_0 with $\mathbb{Z}_\ell \times \{0\}$ and Π_1 with $\mathbb{Z}_{\ell+r} \times \{1\}$, and write for brevity x_i rather than (x, i). In the discussion that follows, by the *parity* of vertex x_i, we mean the parity of x; this is well defined since ℓ and r are both even.

Theorem 3.1 *Let* ℓ *be even and let* r *be an even integer with* $0 \le r < \ell$*. Then there exists an* ℓ*-cycle system of* $K_{\ell,\ell+r}$ *which admits a doubly equitable* 2*-colouring.*

Proof: We will first prove the result for $\ell \equiv 0 \pmod 4$, and then for $\ell \equiv 2 \pmod 4$.

Case 1. $\ell \equiv 0 \pmod 4$

Let $\ell = 4s$, and note that $K_{4s,4s+r}$ decomposes into two subgraphs, each isomorphic to $K_{2s,4s+r}$, which share no vertices in the part of size $4s$; thus it suffices to prove the result for $K_{2s,4s+r}$, where r is even and $0 \leq r \leq 4s-2$.

Let the parts of $K_{2s,4s+r}$ be $\Pi_0' = \mathbb{Z}_{2s} \times \{0\}$ and $\Pi_1 = \mathbb{Z}_{4s+r} \times \{1\}$. Form a starter cycle as follows

$$C_0 = (0_0, 0_1, 1_0, 1_1, \ldots, (2s-1)_0, (2s-1)_1).$$

Note that in C_0 every vertex x_0 with $x \neq 0$ is adjacent to x_1 and $(x-1)_1$, while 0_0 is adjacent to 0_1 and $(2s-1)_1$. In particular, every vertex in Π_0' is adjacent to two vertices in Π_1 with different parities.

For $i = 1, 2, \ldots, 2s + r/2 - 1$, form C_i by adding $2i$ to the first coordinate of every vertex in Π_1. It is easy to see that the cycles $C_0, C_1, \ldots, C_{2s+r/2-1}$ decompose $K_{2s,4s+r}$.

It remains to show that this decomposition has a doubly equitable 2-colouring. We colour a vertex x_i (where $i \in \{0,1\}$) red if x is even and blue otherwise. Clearly each part has an equal number of red and blue vertices. Since each cycle has $2s$ consecutive vertices (an even number) in each part, the colouring is equitable.

Case 2. $\ell \equiv 2 \pmod 4$

This is a variation of Sotteau's construction [19]. Let $\ell = 4s+2$, and once more set $\Pi_0 = \mathbb{Z}_\ell \times \{0\}$ and $\Pi_1 = \mathbb{Z}_{\ell+r} \times \{1\}$.

The cycle decomposition will consists of two sets of $(\ell+r)/2$ ℓ-cycles, say $A_0, A_1, \ldots, A_{(\ell+r)/2-1}$ and $B_0, B_1, \ldots, B_{(\ell+r)/2-1}$ with

$$A_i = (2i_1, 0_0, (2i+1)_1, 1_0, (2i+2)_1, 2_0, \ldots, (2s-1)_0, (2i+2s)_1, (e_i)_0)$$

and

$$B_i = ((2i+1)_1, (2s)_0, (2i+2)_1, (2s+1)_0, (2i+3)_1, \ldots, \\ (\ell-4)_0, (2i+2s)_1, (d_i)_0, (2i+2s+1)_1, (e_i)_0).$$

It remains to choose the values of $(d_i)_0$ and $(e_i)_0$ in $\{(\ell-3)_0, (\ell-2)_0, (\ell-1)_0\}$ for $i \in \{0, \ldots, (\ell+r)/2-1\}$ in such a way as to obtain a cycle decomposition. We begin by suitably choosing values for the $(e_i)_0$, ensuring that $(e_i)_0$ and $(e_{i+s})_0$ are distinct. (Note that we compute the subscript of $i+s$ modulo $(\ell+r)/2$.) We then take as $(d_i)_0$ the third element of $\{(\ell-3)_0, (\ell-2)_0, (\ell-1)_0\}$, distinct from $(e_i)_0$ and $(e_{i+s})_0$.

It is not difficult to see that such a choice is always possible: one possibility is as follows. Note that $2s+1 \leq (\ell+r)/2 \leq 4s+1$. If $(\ell, r) \neq (6,4)$, consider the euclidean division $(\ell+r)/2 = qs + \rho$; we have $2 \leq q \leq 4$. Let us take (we omit the subscript 0 for readability)

$$e_i = \begin{cases} \ell-3 & \text{if } 0 \le i \le s-1, \\ \ell-2 & \text{if } s \le i \le 2s-1, \\ \ell-1 & \text{if } 2s \le i \le 3s-1, \text{ or } 2s \le i < 2s+\rho \text{ when } q=2, \\ \ell-2 & \text{if } 3s \le i \le 4s-1, \text{ or } 3s \le i < 3s+\rho \text{ when } q=3, \\ \ell-1 & \text{if } i = 4s. \end{cases}$$

If $(\ell, r) = (6, 4)$, then we have $s = 1$ and $(\ell + r)/2 = 4s + 1$. In this case, we take

$$(e_0, e_1, e_2, e_3, e_4) = (3, 4, 5, 4, 5).$$

This decomposition has a doubly equitable 2-colouring: in the part Π_0 colour the vertices $(\ell-3)_0, (\ell-2)_0, (\ell-1)_0, 0_0, 1_0, \ldots (s-2)_0$ and $(2s)_0, (2s+1)_0, \ldots (3s-2)_0$ blue. In Π_1, colour the vertices 0_1, 2_1, 4_1, …, $(\ell+r-2)_1$ blue. Colour the remaining vertices red. □

Example 3.2 A 6-cycle system for $K_{6,10}$ is the following.

$$\begin{array}{ll} A_0 = (0_1, 0_0, 1_1, 1_0, 2_1, 3_0), & B_0 = (1_1, 2_0, 2_1, 5_0, 3_1, 3_0), \\ A_1 = (2_1, 0_0, 3_1, 1_0, 4_1, 4_0), & B_1 = (3_1, 2_0, 4_1, 3_0, 5_1, 4_0), \\ A_2 = (4_1, 0_0, 5_1, 1_0, 6_1, 5_0), & B_2 = (5_1, 2_0, 6_1, 3_0, 7_1, 5_0), \\ A_3 = (6_1, 0_0, 7_1, 1_0, 8_1, 4_0), & B_3 = (7_1, 2_0, 8_1, 3_0, 9_1, 4_0), \\ A_4 = (8_1, 0_0, 9_1, 1_0, 0_1, 5_0), & B_4 = (9_1, 2_0, 0_1, 4_0, 1_1, 5_0). \end{array}$$

Colouring the vertices $3_0, 4_0, 5_0, 0_1, 2_1, 4_1, 6_1, 8_1$ blue, and the remaining vertices red gives a doubly equitable 2-colouring.

Example 3.3 The following cycles form a 10-cycle decomposition of $K_{10,12}$.

$$\begin{array}{ll} A_0 = (0_1, 0_0, 1_1, 1_0, 2_1, 2_0, 3_1, 3_0, 4_1, 7_0), & B_0 = (1_1, 4_0, 2_1, 5_0, 3_1, 6_0, 4_1, 9_0, 5_1, 7_0), \\ A_1 = (2_1, 0_0, 3_1, 1_0, 4_1, 2_0, 5_1, 3_0, 6_1, 7_0), & B_1 = (3_1, 4_0, 4_1, 5_0, 5_1, 6_0, 6_1, 9_0, 7_1, 7_0), \\ A_2 = (4_1, 0_0, 5_1, 1_0, 6_1, 2_0, 7_1, 3_0, 8_1, 8_0), & B_2 = (5_1, 4_0, 6_1, 5_0, 7_1, 6_0, 8_1, 7_0, 9_1, 8_0), \\ A_3 = (6_1, 0_0, 7_1, 1_0, 8_1, 2_0, 9_1, 3_0, 10_1, 8_0), & B_3 = (7_1, 4_0, 8_1, 5_0, 9_1, 6_0, 10_1, 7_0, 11_1, 8_0), \\ A_4 = (8_1, 0_0, 9_1, 1_0, 10_1, 2_0, 11_1, 3_0, 0_1, 9_0), & B_4 = (9_1, 4_0, 10_1, 5_0, 11_1, 6_0, 0_1, 8_0, 1_1, 9_0), \\ A_5 = (10_1, 0_0, 11_1, 1_0, 0_1, 2_0, 1_1, 3_0, 2_1, 9_0), & B_5 = (11_1, 4_0, 0_1, 5_0, 1_1, 6_0, 2_1, 8_0, 3_1, 9_0). \end{array}$$

Colouring the vertices $0_0, 4_0, 7_0, 8_0, 9_0$ and $0_1, 2_1, 4_1, 6_1, 8_1, 10_1$ blue and the remaining vertices red gives a doubly equitable 2-colouring.

4 Reduction step

A key step in the proof given in [3] of existence of ℓ-cycle systems of $K_v - I$ was to reduce the problem to orders v in the interval $[\ell, 2\ell)$. In this section, we give an analogous reduction step for equitably 2-colourable cycle decompositions, which will allow us to obtain in Section 5 a complete solution for the existence of an equitably 2-colourable ℓ-cycle system when ℓ is a power of 2, $\ell \in \{2q, 4q\}$ for q an odd prime power, or $\ell \leq 30$.

Theorem 4.1 *Let $\ell \geq 4$ be even and r an even integer with $0 \leq r < \ell$. If $K_{\ell+r} - I$ admits an equitably 2-colourable ℓ-cycle decomposition, then $K_v - I$ admits an equitably 2-colourable ℓ-cycle decomposition for any $v \equiv r \pmod{\ell}$ with $v \geq \ell$.*

PROOF: Let $v = q\ell + r$, where $q \geq 1$. Let the vertex set of $K_v - I$ be $\cup_{i=0}^{q-1} \Pi_i$, where $\Pi_0 = \mathbb{Z}_{\ell+r} \times \{0\}$ and $\Pi_i = \mathbb{Z}_\ell \times \{i\}$ if $i \geq 1$. For each $i \in \{0, \ldots, q-1\}$ colour half of the vertices of Π_i red and the other half blue.

We decompose $K_v - I$ into $(q-1)$ subgraphs isomorphic to $K_\ell - I$ (on vertex sets of the form Π_i, $i \in \{1, \ldots, q-1\}$), one subgraph isomorphic to $K_{\ell+r} - I$ (on vertex set Π_0), and subgraphs isomorphic to $K_{\ell,\ell}$ (on vertex sets of the form $\Pi_i \cup \Pi_j$, $i, j \in \{1, \ldots, q-1\}$) and $K_{\ell,\ell+r}$ (on vertex sets of the form $\Pi_i \cup \Pi_0$, $i \in \{1, \ldots, q-1\}$). By Lemmas 2.1 and 3.1 we can place equitably 2-colourable ℓ-cycle decompositions of $K_\ell - I$, $K_{\ell,\ell}$ and $K_{\ell,\ell+r}$ which respect the given colouring. The existence of an equitably 2-colourable ℓ-cycle decomposition of $K_{\ell+r} - I$ (by hypothesis) completes the decomposition. □

Corollary 4.2 *Let $\ell \geq 4$ be even. If $K_v - I$ admits an equitably 2-colourable ℓ-cycle decomposition for any ℓ-admissible even v satisfying $\ell \leq v < 2\ell$, then $K_v - I$ admits an equitably 2-colourable ℓ-cycle decomposition for any ℓ-admissible even v.*

PROOF: Let v be even and ℓ-admissible, and write $v = q\ell + r$, where $q \geq 0$ and $\ell \leq r < 2\ell$. There is an equitably 2-colourable ℓ-cycle decomposition of $K_r - I$ by hypothesis. The existence of such a decomposition of $K_v - I$ now follows by Theorem 4.1. □

5 Equitably 2-colourable even cycle systems

In this section, we show existence of equitably 2-colourable ℓ-cycle systems of $K_v - I$ for any $v \equiv 0$ or $2 \pmod{\ell}$, and provide a complete solution for the existence of an equitably 2-colourable ℓ-cycle system when ℓ is a power of 2, $\ell \in \{2q, 4q\}$ for q an odd prime power, or $\ell \leq 30$.

We begin with a construction in the case $v = \ell + 2$.

Lemma 5.1 *Let $\ell \geq 4$. There is an equitably 2-colourable ℓ-cycle decomposition of $K_{\ell+2} - I$.*

Proof: Let $V(K_{\ell+2}-I) = \mathbb{Z}_\ell \cup \{\infty_1, \infty_2\}$. Colour the elements in $\{0, 1, \ldots, \ell/2-1\}$ and ∞_1 blue and the ones in $\{\ell/2, \ldots, \ell-1\}$ and ∞_2 red.

First, take the ℓ-cycle $A = (0, 1, \ldots \ell - 1)$; we clearly have that A is fixed under the action of $\mathbb{Z}_\ell$, is equitably coloured, and $\partial A = \{1, -1\}$.

The remaining cycles arise from a starter cycle B formed as follows. Let P be a path containing edges having differences $2, 3, \ldots, (\ell/2 - 1)$. In particular, if $\ell \equiv 0 \pmod 4$, we let P be the path

$$P = 0, 2, (-1), 3, (-2), \ldots, (\ell/4), -(\ell/4 - 1)$$

and if $\ell \equiv 2 \pmod 4$, the path P is defined as

$$P = 0, 2, (-1), 3, (-2), \ldots, -(\ell - 6)/4, (\ell + 2)/4.$$

Now let B be the cycle C_P, as defined in Section 2.2.

Note that $\partial(B) = \mathbb{Z}_\ell \setminus \{0, 1, -1, \ell/2\}$. Thus, $\{A, B\}$ is a set of base cycles of type σ of a 2-pyramidal ℓ-cycle system of $K_{\ell+2}$. Since B and its translates each contain every vertex of $\mathbb{Z}_\ell \cup \{\infty_1, \infty_2\}$ except for two of the form x and $(x + \ell/2)$ (which have opposite colours), it is easy to see that this cycle system is equitably coloured. □

Theorem 5.2 *Let $\ell \geq 4$ be even and $v \equiv 0$ or $2 \pmod{\ell}$. There is an equitably 2-colourable ℓ-cycle decomposition of $K_v - I$.*

Proof: The result follows as a direct consequence of Theorem 4.1 together with the existence of equitably 2-colourable ℓ-cycle decompositions of $K_\ell - I$ and $K_{\ell+2} - I$ from Lemmas 2.1 and 5.1. □

For a prime power q, recalling that the $2q$-admissible integers v satisfy $v \equiv 0, 2 \pmod{2q}$, we have the following.

Corollary 5.3 *Let $\ell = 2q$, where q is a prime power. There is an equitably 2-colourable ℓ-cycle decomposition of $K_v - I$ if and only if v is ℓ-admissible.*

Note that in Corollary 5.3, q may be an odd or even prime power. Thus this result also includes the case that ℓ is a power of 2. In the case that $\ell = 4q$ for some odd prime power, q, we can also find necessary and sufficient conditions for the existence of an equitably 2-colourable ℓ-cycle decomposition.

Theorem 5.4 *Let $\ell = 4q$, where q is an odd prime power. There is an equitably 2-colourable ℓ-cycle decomposition of $K_v - I$ if and only if v is ℓ-admissible.*

PROOF: By Corollary 4.2, it suffices to consider the case $v \in [4q, 8q)$; the only $4q$-admissible values in this range are $v = 4q, 4q+2, 6q, 6q+2$. The cases $v = 4q$ and $v = 4q+2$ are covered by Theorem 5.2. In the remaining two cases, we construct a cyclic $4q$-cycle decomposition of $K_v - I$, and show that it is equitably 2-colourable.

Case 1. $v = 6q + 2$

Consider the following two $4q$-cycles, A and B:

$$A = [0, 1, -1, 2, -2, \dots, (q-1)/2, -(q-1)/2]_{-(3q+1)/2}, \text{ and}$$
$$B = [0, q, -3, q+1, -4, q+2, -5, \dots, -(q+1), 2q-1]_{3q+1}.$$

Note that the lists of partial differences of these cycles are

$$\partial A = \pm\{1, 2, \dots, q-1, q+1\} \text{ and } \partial B = \pm\{q, q+2, \dots, 3q\};$$

hence A and B form a set of base cycles for a cyclic $4q$-cycle system for $K_{6q+2} - I$. It is easy to check that this cycle system can be equitably 2-coloured by colouring the vertices in $[0, 3q]$ red and those in $[3q+1, 6q-1]$ blue.

Case 2. $v = 6q$

We will build a set of base cycles using $(q-1)/2$ cycles with orbit of length 3, and one with orbit of length $3q$.

Set $(q-1)/2 = 2s$ if $q \equiv 1 \pmod 4$, and $(q-1)/2 = 2s+1$ if $q \equiv 3 \pmod 4$. Thus, we need to find a collection of cycles whose partial differences cover the set $\pm\{1, 2, \dots, 12s+2\}$ if $q \equiv 1 \pmod 4$, and $\pm\{1, 2, \dots, 12s+8\}$ if $q \equiv 3 \pmod 4$.

We first build 2s cycles $A_{i,1}$, $A_{i,2}$ for $i = 0, \dots, s-1$ as follows:

$$A_{i,1} = [0, 6i+1]_{-3}, \quad A_{i,2} = [0, 6i+2]_{-3}.$$

It is clear that

$$\bigcup_{i=0}^{s-1} (\partial A_{i,1} \cup \partial A_{i,2}) = \pm(\{1, 2, \dots, 6s-1\} \setminus \{3, 6, 9, \dots, 6s-3\}).$$

Now, if $q \equiv 1 \pmod 4$, take

$$\begin{aligned} B &= [0, 3, -3, 6, -6, \dots, 3s, -3s, 3s+1, -(3s+2), 3s+2, -(3s+3), \\ &\qquad 3s+3, \dots, -(6s+1), 6s+1]_{12s+3}. \end{aligned}$$

It is easy to check that

$$\partial B = \pm\{3, 6, 9, \dots, 6s-3, 6s, 6s+1, 6s+2, \dots, 12s+2\}.$$

If $q \equiv 3 \pmod 4$, we also take the extra cycle $A^* = [0, 12s+7]_{-3}$, with $\partial A^* = \pm\{12s+7, 12s+8\}$; then we choose the last cycle B as follows:

$$\begin{aligned} B &= [0, 3, -3, 6, -6, \dots, 3s, -3s, 3s+1, -(3s+1), 3s+2, -(3s+2), 3s+3, \\ &\qquad -(3s+4), (3s+4), -(3s+5) \dots, -(6s+3), 6s+3]_{12s+9}. \end{aligned}$$

We have that

$$\partial B = \pm\{3, 6, 9, \dots, 6s-3, 6s, 6s+1, 6s+2, \dots, 12s+6\}.$$

(If $q = 3$, then note that $B = [0, 1, -1, 2, -2, 3]_9$ and $\partial B = \pm\{1, 2, 3, 4, 5, 6\}$.)

It is easily checked that these cycle systems can be equitably 2-coloured by colouring the vertices with an even label red and those with odd label blue. Indeed, the cycle B is made from the concatenation of two paths of length $\ell/2$, namely P_1 and P_2, where $V(P_2) = V(P_1) + 3q$ (with computations done modulo $6q$). Since q is odd, vertices $i \in \mathbb{Z}_{6q}$ and $i + 3q$ have opposite parity and are thus coloured differently; it follows that B and its translates are equitably coloured. The other cycles in the decomposition are all built by concatenating an even number of paths of length two. In this concatenation, initial vertices of consecutive paths have opposite parity, as do the internal vertices of consecutive paths; thus each cycle formed in this way has an equal number of vertices of each colour. □

Example 5.5 Let $q = 7$; a set of base cycles for a cyclic 28-cycle system of $K_{44} - I$ is the following.

$$A = [0, 1, -1, 2, -2, 3, -3]_{-11}, \text{ and}$$
$$B = [0, 7, -3, 8, -4, 9, -5, 10, -6, 11, -7, 12, -8, 13]_{22}.$$

The resulting cycle system can be equitably coloured by colouring vertices in $[0, 21]$ red and those in $[22, 43]$ blue.

Example 5.6 Let $q = 7$; a set of base cycles for a cyclic 28-cycle system of $K_{42} - I$ is as follows.

$$A_{0,1} = [0, 1]_{-3}, \quad A_{0,2} = [0, 2]_{-3},$$
$$A^* = [0, 19]_{-3}, \text{ and}$$
$$B = [0, 3, -3, 4, -4, 5, -5, 6, -7, 7, -8, 8, -9, 9]_{21}.$$

The cycle system can be equitably 2-coloured by colouring the even vertices red, and the odd ones blue.

Our final result is to prove the existence of an equitably 2-colourable ℓ-cycle decomposition of $K_v - I$ whenever $4 \le \ell \le 30$ is even and v is ℓ-admissible. Before we prove this, we first present some small cases.

Lemma 5.7 *There is an equitably 2-colourable cyclic ℓ-cycle decomposition of $K_v - I$ for $(\ell, v) \in \{(24, 32), (30, 42), (30, 50)\}$.*

Proof: For $(\ell, v) = (24, 32)$, the following cycles form a set of base cycles of a cyclic 24-cycle system of $K_{32} - I$:

$$A = [0, 1, 31]_{-4} \text{ and } B = [0, 4, 31, 5, 29, 6, 28, 7, 27, 8, 26, 9]_{16}.$$

Indeed, note that $\partial A = \pm\{1,2,3\}$ and $\partial B = \pm\{4,5,\dots,15\}$. Colouring the vertices in the interval $[0,15]$ blue and those in $[16,31]$ red gives an equitable 2-colouring.

For $(\ell, v) = (30, 42)$, base cycles of a cyclic 30-cycle system of $K_{42} - I$ are given by

$$A = [0,1,41,2,40]_{35}$$

and

$$B = [0,6,41,7,40,8,39,9,38,11,37,12,36,13,35]_{21};$$

note that $\partial A = \pm\{1,2,3,4,5\}$ and $\partial B = \pm\{6,7,\dots,20\}$. An equitable 2-colouring is given by colouring the vertices in the interval $[0,20]$ red and those in $[21,39]$ blue.

For $(\ell, v) = (30, 50)$, the following three cycles form a set of base cycles for a 30-cycle decomposition of $K_{50} - I$:

$$A = [0,6,48]_{35},$$
$$B = [0,1,49,2,48,3]_{10}, \text{ and}$$
$$C = [0,9,49,10,48,12,47,13,46,15,45,16,44,17,43]_{25}.$$

Here, we have that

$$\partial A = \pm\{6,8,13\},$$
$$\partial B = \pm\{1,2,3,4,5,7\}, \text{ and}$$
$$\partial C = \pm\{9,10,11,12\} \cup \{14,15,\dots,24\}.$$

This decomposition may be equitably 2-coloured by colouring vertices with an even label red and those with an odd label blue. □

Lemma 5.8 *There is an equitably 2-colourable 2-pyramidal 24-cycle decomposition of $K_{42} - I$.*

Proof: We define a set of base cycles for a 2-pyramidal 24-cycle decomposition of type σ on vertex set $\mathbb{Z}_{40} - I$. In the cycles below, we will thus consider differences in $\mathbb{Z}_{40}$.

First, define cycles

$$A_1 = [0,1,39]_5, \quad A_2 = [0,3,36]_5, \quad \text{and } A_3 = [0,4,33]_5.$$

We have that $\partial A_1 = \pm\{1,2,6\}$, $\partial A_2 = \pm\{3,7,9\}$ and $\partial A_3 = \pm\{4,11,12\}$.

Now, define the path

$$P = 0,5,37,7,34,8,33,9,32,10,31.$$

Letting $B = C_P$, we have that $\partial B = \pm\{5,8,10\} \cup \{13,14,\dots,19\}$.

It is now easy to see that A_1, A_2, A_3 and B form the required set of base cycles.

The 24-cycle decomposition can be equitably coloured by colouring vertices in the interval $[0,19]$ as well as ∞_1 red, and vertices in $[20,39]$ along with ∞_2 blue. □

Theorem 5.9 *Let* $4 \le \ell \le 30$ *be even. There is an equitably* 2*-colourable* ℓ*-cycle decomposition of* $K_v - I$ *if and only if* v *is* ℓ*-admissible.*

Proof: Corollary 5.3 and Theorem 5.4 cover all ℓ-values except $\ell \in \{24, 30\}$. As a consequence of Corollary 4.2 and Theorem 5.2, we need only consider values of $v \in [\ell, 2\ell) \setminus \{\ell, \ell+2\}$. Existence of an equitably 2-colourable cycle decomposition in each of these cases is shown in Lemmas 5.7 and 5.8. □

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