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Finding monarchs for excluded minor classes of matroids

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Abstract

We prove that the only 3-connected binary non-regular matroids with no minor isomorphic to a rank 5, 9-element binary matroid known as P_9^* are the rank 3 and 4 binary projective geometries, a 16-element rank 5 matroid, and two maximal 3-connected infinite families of matroids of rank $r \geq 5$ that we call monarchs: the well-known infinite family of binary spikes with $2r+1$ elements and a new infinite family with $4r-5$ elements. Both families are in matrix format. This is one of very few excluded-minor classes of matroids for which the members are so precisely determined. As a consquence, if M is a simple binary matroid of rank $r \geq 6$ with no P_9^* minor, then $|M| \leq \frac{r(r+1)}{2}$, with this bound being attained for $M \cong M(K_{r+1})$, where K_{r+1} is the rank r complete graph.

1 Introduction

In [4] we gave a reproof of the structural characterization of binary matroids with no $M(W_4)$-minor. This was a 1987 result by Oxley that appeared in [9]. He proved this result using Seymour's Splitter Theorem [11]. The main component of Oxley's proof was a complete identification of the class of binary matroids with no minor isomorphic to P_9 or P_9^*. Matrix representations for P_9 and P_9^* are shown below.

$$P_9 = \begin{array}{c} \begin{matrix} 1\cdots 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{array}\right] \end{array}, \quad P_9^* = \begin{array}{c} \begin{matrix} 1\cdots 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_5 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 \end{array}\right] \end{array}$$

Oxley showed that the excluded minor class for these two matroids contains one infinite family of 3-connected matroids known as the binary spikes Z_r. Matrix

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representations for Z_r and Z_r^* are shown below, where we use the name of the matroid to also stand for the matrix representing it:

$$
\begin{array}{c} \\ Z_r = \end{array}
\begin{array}{c}
\begin{array}{ccc|cccccc}
b_1 & \cdots & b_r & a_1 & a_2 & \cdots & a_{r-1} & a_r & c_r
\end{array}\\
\left[\begin{array}{c|cccccc}
 & 0 & 1 & \cdots & 1 & 1 & 1\\
 & 1 & 0 & \cdots & 1 & 1 & 1\\
I_r & 1 & 1 & \cdots & 1 & 1 & 1\\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
 & 1 & 1 & \cdots & 0 & 1 & 1\\
 & 1 & 1 & \cdots & 1 & 0 & 1
\end{array}\right],
\end{array}
$$

$$
\begin{array}{c} \\ Z_r^* = \end{array}
\begin{array}{c}
\begin{array}{ccc|ccccc}
b_1 & \cdots & b_{r+1} & a_1 & a_2 & \cdots & a_{r-1} & a_r
\end{array}\\
\left[\begin{array}{c|ccccc}
 & 0 & 1 & \cdots & 1 & 1\\
 & 1 & 0 & \cdots & 1 & 1\\
I_{r+1} & 1 & 1 & \cdots & \cdots & \\
 & \vdots & \vdots & \ddots & \vdots & \vdots\\
 & 1 & 1 & \cdots & 0 & 1\\
 & 1 & 1 & \cdots & 1 & 0\\
 & 1 & 1 & \cdots & 1 & 1
\end{array}\right].
\end{array}
$$

We call Z_r a rank r monarch. In general, let $\mathcal{M}$ be a class of matroids closed under minors. A 3-connected rank r matroid in $\mathcal{M}$ that has no further 3-connected extensions in $\mathcal{M}$ is called a *rank r monarch* for $\mathcal{M}$.

Following the terminology in [10], let $EX(M)$ denote the class of matroids with no minors isomorphic to M. Ding and Wu characterized the binary matroids in $EX(P_9)$ in terms of 3-sums in Theorem 1.2 of [2]. They proved that a binary 3-connected non-regular matroid M has no P_9-minor if and only if M is one of the 16 internally 4-connected non-regular minors of R_{16}^*; or M is a binary spike Z_r, Z_r^*, $Z_r\backslash b_r$, or $Z_r\backslash c_r$, for some $r \geq 4$; or M is formed by taking t disjoint triangles $T_1, T_2, \ldots, T_t$ of $M^*(K_{3,p})$, $M^*(K'_{3,p})$, $M^*(K''_{3,p})$, or $M^*(K'''_{3,p})$, where $p \geq 2$ and $1 \leq t \leq p$, and t copies of F_7^* and performing 3-sum operations consecutively. The last infinite family is formed by 3-summing copies of the Fano matroid to $M^*(K_{3,p})$, $M^*(K'_{3,p})$, $M^*(K''_{3,p})$, or $M^*(K'''_{3,p})$, where $p \geq 2$. This result extended Oxley's characterization of $EX(P_9, P_9^*)$ to $EX(P_9)$. Ding and Wu's result uses a chain theorem for internally 4-connected binary matroids [1]. We use a different proof technique, using the Strong Splitter Theorem in [6] to obtain the monarchs of $EX(P_9^*)$.

The next result is the main result of this paper.

Theorem 1.1 *A binary 3-connected non-regular matroid M has no P_9^*-minor if and only if M is isomorphic to F_7, $PG(3,2)$, R_{16}, Z_r for $r \geq 4$, Ω_r for $r \geq 5$, or one of their 3-connected deletion-minors.*

In Theorem 1.1, F_7 is the Fano plane, $P(3,2)$ is the rank 4 binary projective geometry, and Z_r is the binary spike, which can be represented by the matrix $[I_r|D]$, where D has $r+1$ columns. The first r columns of D have zeros on the diagonal and ones elsewhere and the last column is a column of ones as shown above and in [9]. A matrix representation for R_{16} is given below. Note that $R_{16} = R_{17} \backslash 17$, where R_{17} is the largest rank 5 matroid with no minor isomorphism to the prism $M^*(K_5 \backslash e)$ [5]. The matroid R_{17} is $AG(3,2) \times U_{1,1}$, which is the binary matroid obtained from the direct sum of $AG(3,2)$ and a coloop by completing the 3-point lines between every element in $AG(3,2)$ and the coloop. It was first described by Mayhew and Royle in [8].

$$
\begin{array}{c} \\ R_{16} = \end{array}
\begin{array}{c|ccccccccccc}
1\cdots 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\hline
 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
I_5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}
$$

The new rank r monarch in this larger class of binary matroids with no P_9^*-minor is the rank r binary matroid Ω_r for which a matrix representation is given in Figure 1. It has $4r - 5$ elements, and an intricate pattern. The fact that a pattern exists is quite surprising.

$b_1 \dots b_r$	a_1	a_2	a_3	a_4	a_5	c_5	d_5	c_6	d_6	...	c_{r-2}	d_{r-2}	c_{r-1}	d_{r-1}	c_r	d_r	e_r	f_r	$g_{r,1}$	$g_{r,2}$	$g_{r,3}$	...	$g_{r,r-5}$	$g_{r,r-4}$
	0	1	1	1	1	1	0	1	0		1	0	1	0	1	0	1	0	1	1	1		1	1
	1	0	1	1	1	1	0	1	0		1	0	1	0	1	0	1	0	1	1	1		1	1
	1	1	0	1	0	0	1	0	1		0	1	0	1	0	1	1	1	1	1	1		1	1
	1	1	1	1	0	0	1	0	1	...	0	1	0	1	0	1	0	1	1	1	1	...	1	1
	0	0	0	1	1	0	0	0	0		0	0	0	0	0	0	0	1	0	0	0		0	0
I_r	0	0	0	0	0	1	1	0	0		0	0	0	0	0	0	0	0	0	1	0		0	0
	0	0	0	0	0	0	0	1	1		0	0	0	0	0	0	0	0	0	0	1		0	0
	⋮								⋮	⋱			⋮	⋮								⋱		
											1		0	0	0	0	0	0	0	0	0		1	0
	0	0	0	0	0	0	0	0	0		0	0	1	1	0	0	0	0	0	0	0		0	1

Figure 1: The rank r matroid Ω_r with $4r - 5$ elements, for $r \geq 5$

Section 2 has the preliminaries and techniques used. The manner in which Ω_r is constructed is explained in Section 3, and the proof of Theorem 1.1 is in Section 4.

2 Preliminaries

The main technique used in the proof of Theorem 1.1 is the Strong Splitter Theorem [6]. How to use this result has been described in a fair amount of detail in [4], but the rank r monarch Ω_r has size $4r - 5$ and is much more complicated than the binary spikes Z_r of size $2r+1$. So more details are provided, especially on how single-element extensions and coextensions are computed.

A *triad* is a 3-element cocircuit whose removal does not disconnect the matroid. The next result is the Strong Splitter Theorem.

Theorem 2.1 *Suppose N is a 3-connected proper minor of a 3-connected matroid M such that, if N is a wheel or a whirl, then M has no larger minor isomorphic to a wheel or whirl, respectively. Let $j = r(M) - r(N)$. Then there is a sequence of 3-connected matroids $M_0, M_1, \dots, M_t$ such that $M_0 \cong N$, $M_t = M$, and M_{i-1} is a minor of M_i such that:*

(i) For $1 \le i \le j$, $r(M_i) - r(M_{i-1}) = 1$ and $|E(M_i) - E(M_{i-1})| \le 3$; and

(ii) For $j < i \le t$, $r(M_i) = r(M)$ and $|E(M_i) - E(M_{i-1})| = 1$.

Moreover, when $|E(M_i) - E(M_{i-1})| = 3$, for $1 \le i \le j$, $E(M_i) - E(M_{i-1})$ is a triad of M_i. □

Given a class $\mathcal{M}$ of matroids closed under minors, we may focus on the 3-connected members of $\mathcal{M}$ since matroids that are not 3-connected can be pieced together from 3-connected matroids using the operations of 1-sum and 2-sum [10](8.3.1). Let us denote a *simple* single-element extension of M by an element e as $M + e$ and a *cosimple* single-element coextension of M by an element f as $M \circ f$. Note that a simple extension of a 3-connected matroid is also 3-connected. Likewise for cosimple coextensions.

Suppose N is a 3-connected proper minor of a 3-connected matroid M such that, if N is a wheel or a whirl, then M has no larger minor isomorphic to a wheel or whirl, respectively. The Splitter Theorem states that there is a sequence of 3-connected matroids $M_0, M_1, \dots, M_t$ such that $M_0 = N$, $M_t \cong M$, and for $1 \le i \le t$ either $M_i = M_{i-1}+e$ or $M_i = M_{i-1}\circ f$. See [10](12.2.1). Thus to reach a matroid isomorphic to M, one may start with N and perform simple single-element extensions and cosimple single-element coextensions. The Splitter Theorem imposes no conditions to restrict how N can grow to (a matroid isomorphic to) M. Theorem 1.2 optimizes the Splitter Theorem by proving that after two simple single-element extensions a cosimple single-element coextension must be performed, and it puts additional restrictions on how the coextensions are obtained.

A minor-closed class $\mathcal{M}$ may have several rank r monarchs of varying sizes. For example the class in this paper has two rank r monarchs: Z_r for $r \ge 4$ and Ω_r for $r \ge 5$. Since Z_r is the rank r monarch for the class of binary matroids with no minors isomorphic to P_9 nor P_9^* (see the original proof in [9] and the new proof in [4]), we can use that result to begin with P_9 and exclude P_9^*. This new subclass has just one rank r monarch Ω_r. Our strategy is to take a large excluded minor class and break it down into smaller excluded-minor classes, repeatedly applying Theorem 1.2 to find rank r monarchs.

Theorem 1.2 implies that every 3-connected rank r monarch in $\mathcal{M}$ is a simple extension of a 3-connected rank r matroid M_r, where M_r is obtained from a 3-connected rank $r-1$ matroid M_{r-1} in the following ways:

(1) $M_r = M_{r-1} \circ f$;

(2) $M_r = M_{r-1} + e \circ f$, where f is added in series to an element in M_{r-1}; or

(3) $M_r = M_{r-1} + \{e_1, e_2\} \circ f$, where $\{e_1, e_2, f\}$ is a triad.

There is no reason to asssume *a priori* that M_r is unique for a specific excluded minor class. However, if M_r happens to be unique, we get a recursive way of defining it, and consequently a recursive way of defining the corresponding rank r monarch.

The focus then shifts to identifying the matroid M_r in the above description. We will call M_r the *rank r seed* corresponding to the rank r monarch and denote it by α_r. Thus Theorem 1.2 implies that every rank r monarch in $\mathcal{M}$ is the extension of a rank r seed α_r such that:

1. α_r is a cosimple single-element coextension of the rank $(r-1)$ matroid α_{r-1}; or

2. α_r is a cosimple single-element coextension of the simple single-element extensions of α_{r-1} formed by adding elements in series to existing elements; or

3. α_r is a cosimple single-element coextension of the simple double-element extensions of α_{r-1} formed by adding a triad made up of the two extensions elements and the coextension element.

It is possible for $\alpha_r = \Omega_r$, which would make the proof easier. However, if they are distinct, then finding the rank r seed α_r is more important than than finding the rank r monarch Ω_r, because within α_r lies the pattern of the infinite family. In [4] the monarch has just two more elements than the seed, so the distinction between seed and monarch was not really needed. In this paper the rank r monarch Ω_r is much larger than than the rank r seed α_r. Identifying rank r seeds and rank $4r$ monarchs when they differ by many elements requires the Strong Splitter Theorem.

In summary, the Strong Splitter Theorem implies that every 3-connected rank r monarch Ω_r in $\mathcal{M}$ is a simple extension of a 3-connected rank r seed α_r, where α_r is obtained from a 3-connected rank $r-1$ seed α_{r-1} in very specific ways because $|E(\alpha_r) - E(\alpha_{r-1})| = 3$. As described earlier, α_r is a cosimple single-element coextension of α_{r-1}, or a cosimple single-element coextension of a simple extension of α_{r-1} or a cosimple single-element coextension of a double element simple extension of α_{r-1}. There are additional restrictions in the second and third case.

Next, we have to describe our method for computing single-element extensions and coextensions. Let N be a $GF(q)$-representable n-element rank r matroid represented by the matrix $A = [I_r|D]$ over $GF(q)$. For $q = 2, 3, 4$ a 3-connected matroid over $GF(q)$ is uniquely representable, but for $q \geq 5$, there are inequivalent representations, so the method described below has to be modified considerably using the ideas in [3]. For this paper $q = 2$ and we do not need to consider inequivalent representations and the difficulties they create.

The columns of A may be viewed as a subset of the columns of the matrix that represents the projective geometry $PG(r-1, q)$. Let M be a simple single-element extension of N over $GF(q)$. Then $N = M\backslash e$ and M may be represented by $[I_r|D']$, where D' is the same as D, but with one additional column corresponding to the element e. The new column is distinct from the existing columns and has at least two non-zero elements. If the existing columns are labeled $\{1,\dots,r,\dots,n\}$, then the new column is labeled $(n+1)$. We can systematically construct all the non-isomorphic single-element extensions of N by adding the columns of $PG(r-1,q)$ that are missing in A one by one and keeping only the non-isomorphic single-element extensions. This procedure is similar to adding an edge to a rank r graph in all possible ways and keeping a list of the non-isomorphic edge-additions. Just like a rank r graph is a restriction of the complete graph K_{r+1}, a rank r (simple) binary matrix A is a restriction of the matrix representing $PG(r-1,q)$, so this method works even though binary matroids are exponentially larger objects than graphs.

Suppose M is a cosimple single-element coextension of N over $GF(q)$. Then $N = M/f$ and M may be represented by the matrix $[I_{r+1}|D'']$, where D'' is the same as D, but with one additional row. The new row is distinct from the existing rows and has at least two non-zero elements. The columns of $[I_{r+1}|D'']$ are labeled $\{1,\dots,r+1,r+2,\dots,n,n+1\}$. The coextension element f corresponds to column $r+1$. The coextension row is selected from $PG(n-r-1,q)$. We can visualize the new element f as appearing in the new dimension and lifting several points into the higher dimension. Observe that f forms a cocircuit with the elements corresponding to the non-zero entries in the new row. Note that in $[I_{r+1}|D'']$ the labels of columns beyond r are increased by 1 to accomodate the new column $r+1$. This method is similar to computing all possible non-isomorphic rank $(r+1)$ graphs obtained by splitting a vertex in a rank r graph. Again, vertex splits are easy to compute (and visualize), whereas doing a similar computation for a rank r binary matroid is much more complicated.

We refer to the simple single-element extensions of N as Type (i) matroids and the cosimple single-element coextensions of N as Type (ii) matroids. The structure of Type (i) and Type (ii) matroids are shown in Figure 2.

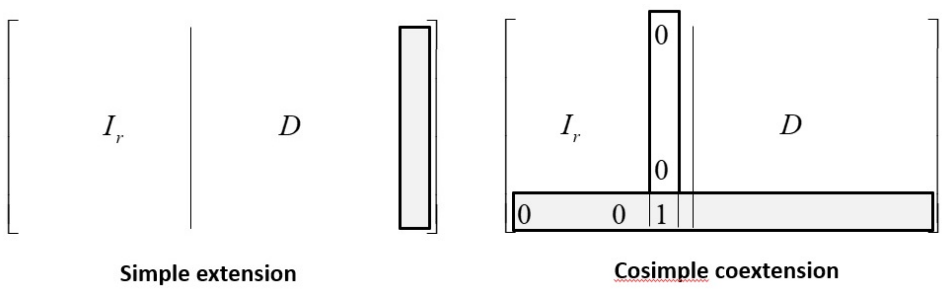

Figure 2: Structure of Type (i) and Type (ii) matroids

Once the simple single-element extensions (Type (i) matroids) and cosimple single-element coextensions (Type (ii) matroids) are determined, the number of permissable rows and columns give a bound on the choices for the cosimple single-element coextensions of the Type (i) matroids and the simple single-element extensions of the Type (ii) matroids, respectively. The structure of the cosimple single-element coextensions of a Type (i) matroid and the simple single-element extensions of a Type (ii) matroid are shown in Figure 3.

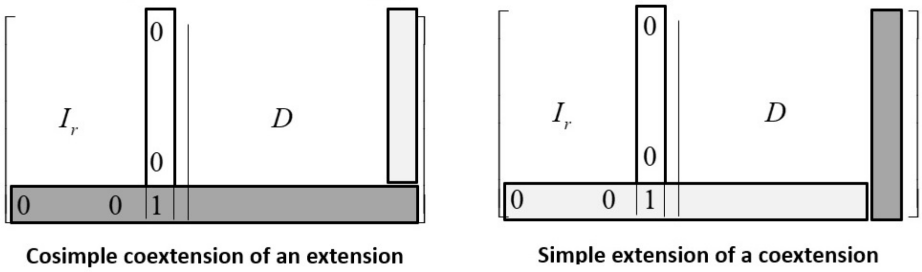

Figure 3: Structure of M, where $|E(M) - E(N)| = 2$

When computing the cosimple single-element coextension of a Type (i) matroid, there are three types of rows that may be inserted into the last row.

(I) Rows that can be added to the original matroid N to obtain a coextension, augmented by a 0 or 1 as the last entry;

(II) The identity rows augmented by a 1 in the last position; and

(III) Rows "in-series" to the right-hand side of the matrix with the last entry reversed.

When computing the simple single-element extensions of a Type (ii) matroid, there are three types of columns that may be inserted into the last column.

(I) Columns that can be added to the original matroid N to obtain an extension augmented by a 0 or 1 as the last entry;

(II) The identity columns augmented by a 1 in the last position; and

(III) Columns "in-parallel" to the right-hand side of matrix with the last entry reversed.

Note that this method can be applied to $GF(q)$-representable 3-connected matroids, for $q = 2, 3, 4$, but this paper is only on binary matroids, so we will talk only of zeros and ones.

Suppose N' is a simple double-element extension of N formed by adding columns e_1 and e_2 and M is a cosimple single-element coextension of N' by element f. By Theorem 1.2 $M \backslash e_1$ or $M \backslash e_2$ is 3-connected except when $\{e_1, e_2, f\}$ is a triad. Thus the only 3-connected coextension of N' we must check is the one formed by adding row $[00\dots011]$ to D. Moreover, no further calculations are necessary.

3 The rank r monarch Ω_r

Our goal in this paper is to find the binary matroids in $EX(P_9^*)$. The rank r seed matroid α_r is unique and it has $3r-5$ elements. A matrix representation is shown in Figure 4. The vertical and horizontal lines in the representation of α_r in Figure 4 shows how it is recursively constructed from α_5 shown below, which is the starting matroid for this family:

$$\alpha_5 = \begin{array}{c} \begin{array}{cccccc} b_1\cdots b_5 & a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ I_5 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}.$$

$$\begin{array}{c} \begin{array}{ccccccccccccccccc} b_1 & \ldots & b_r & a_1 & a_2 & a_3 & a_4 & a_5 & c_5 & d_5 & c_6 & d_6 & c_7 & d_7 & \ldots & c_{r-1} & d_{r-1} \end{array} \\ \left[\begin{array}{c|cccccccccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & & 1 & 0 \\ & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 0 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\ I_r & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & & 0 & 0 \\ & & & & & & \vdots & & & & & & \ddots & \vdots & \vdots \\ & & & & & & & & & & & & & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 \end{array}\right] \end{array}$$

Figure 4: The rank r seed α_r with $3r-5$-elements, for $r \geq 5$

The rank 6 seed matroid α_6 shown below is obtained from α_5 by adding two columns $c_5 = [11000]^T$ and $d_5 = [00110]^T$ and lifting by row [0000011]:

$$\alpha_6 = \begin{array}{c} \begin{array}{cccccccc} b_1\cdots b_6 & a_1 & a_2 & a_3 & a_4 & a_5 & c_5 & d_5 \end{array} \\ \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ I_6 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}.$$

In other words add c_5 and d_5 to form triangles with $\{b_1, b_2, c_5\}$ and $\{b_3, b_4, d_5\}$, respectively; then lift elements c_5 and d_5 into the next dimension to form a triad $\{c_5, d_5, b_6\}$ with the new lift element b_6. medskip In general, α_r is formed as follows: add parallel elements $\{c_5, c_6, \ldots, c_{r-1}\}$ so that each forms a triangle with basis points b_1 and b_2; add parallel elements $\{d_5, d_6, \ldots, d_{r-1}\}$ so that each forms a triangle with basis points b_3 and b_4; do a sequence of lifts by adding new basis elements $\{b_6, \ldots, b_r\}$ to form triads $\{b_i, c_{i-1}, d_{i-1}\}$ for $i = 6, \ldots, r$. Observe that

$$\alpha_r / b_r \backslash \{c_{r-1}, d_{r-1}\} = \alpha_{r-1}.$$

Once the construction of the rank r seed is understood, the construction of the rank r monarch follows by adding more columns. To obtain Ω_5 add the following five columns to α_5:

$$\begin{aligned} c_5 &= [11000]^T, \\ d_5 &= [00110]^T, \\ e_5 &= [11100]^T, \\ f_5 &= [00111]^T, \\ g_{5,1} &= [111100]^T \end{aligned}$$

to get:

$$\Omega_5 = \begin{array}{c} \begin{array}{cccccccccccc} b_1 \cdots b_5 & a_1 & a_2 & a_3 & a_4 & a_5 & c_5 & d_6 & e_5 & f_5 & g_{5,1} \end{array} \\ \left[\begin{array}{c|cccccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ I_5 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \end{array}.$$

To obtain Ω_6 add the following six columns to α_6:

$$\begin{aligned} c_6 &= [110000]^T, \\ d_6 &= [001100]^T, \\ e_6 &= [111000]^T, \\ f_6 &= [001110]^T, \\ g_{6,1} &= [111100]^T, \\ g_{6,2} &= [111101]^T \end{aligned}$$

to get:

$$\Omega_6 = \begin{array}{c} \begin{array}{cccccccccccccc} b_1 \cdots b_6 & a_1 & a_2 & a_3 & a_4 & a_5 & c_5 & d_5 & c_6 & d_6 & e_6 & f_6 & g_{6,1} & g_{6,2} \end{array} \\ \left[\begin{array}{c|ccccccccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ I_6 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \end{array}.$$

In general, to obtain Ω_r from α_r add the following r columns:

$$\begin{aligned} c_r &= [110000, \ldots, 00]^T, \\ d_r &= [001100, \ldots, 00]^T, \\ e_r &= [111000, \ldots, 00]^T, \\ f_r &= [001110\ldots 00]^T, \\ g_{r,1} &= [11110000, \ldots, 000]^T, \\ g_{2,2} &= [11110100, \ldots, 000]^T, \\ g_{2,3} &= [11110010, \ldots, 000]^T, \ldots, \\ g_{2,r-5} &= [1111000, \ldots, 010]^T, \\ g_{r,r-4} &= [1111000, \ldots, 001]^T. \end{aligned}$$

Column $g_{r,1}$ has ones in the first four positions and zeros elsewhere. The rest of the columns $g_{r,t}$, where $2 \leq t \leq r-4$, have five ones (See Figure 1). Observe that

$$\Omega_r / b_r \backslash \{c_r, d_r, g_{r,r-4}\} = \Omega_{r-1}.$$

Proposition 3.1 *The matroid α_r has no P_9^*-minor.*

Proof: Observe that P_9^* has odd size circuits and α_5 has no odd size circuits. If P_9^* were a deletion-minor of α_5, then the odd size circuits in P_9^* would remain in α_5. Since α_r is obtained from α_5 by adding only triangles and triads, P_9^* is not a minor of α_5. □

Proposition 3.2 *The matroid Ω_r has no P_9^*-minor.*

Proof: The proof is by induction on $r \geq 6$. Since $\Omega_r / b_r \backslash \{c_r, d_r, g_{r,r-4}\} = \Omega_{r-1}$, and by induction Ω_{r-1} has no P_9^*-minor, any possible P_9^*-minor in Ω_r must have columns $c_r, d_r, g_{r,r-4}$ and row b_r. Therefore,

$$\Omega_r / \{b_7, \ldots b_{r-1}\} \backslash \{c_7, d_7, \ldots, c_{r-1}, d_{r-1}, e_r, f_r, g_{r,1}, \ldots, g_{r,r-5}\}$$

gives the following matrix:

$$\left[\begin{array}{c|cccccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ I_6 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

will have a P_9^*-minor, which is not true. □

The matroid α_r has two 3-connected non-isomorphic binary single-element extensions in $EX(P_9^*)$:

1. $\alpha_{r,1}$ formed by adding columns c_r, d_r or $g_{r,1}$; and
2. $\alpha_{r,2}$ formed by adding any one of the remaining columns e_r, f_r, $g_{r,2}, \dots g_{r,r-4}$.

This will follow from the induction argument in the proof of Theorem 1.1 (as we will see later) using the automorphism in Ω_r that takes

$$(b_i, b_j, c_{i-1}, d_{j-1}, g_{7,i-4}, g_{7,j-4})$$

to

$$(b_j, b_i, d_{j-1}, c_{i-1}, g_{7,j-4}, g_{7,i-4})$$

for $b_j > b_i \geq 6$, and leaves the remaining columns unchanged. To obtain the pattern in Ω_r shown in Figure 1, $\alpha_{r,1}$ is formed by adding column c_r and $\alpha_{r,2}$ is formed by adding column e_r. The matroid $\alpha_{r,1}$ has two non-isomorphic single-element extensions $\alpha_{r-1,1,1}$ formed by adding d_r and $\alpha_{r-1,1,2}$ formed by adding e_r. The matroid $\alpha_{r,2}$ has two non-isomorphic single-element extensions $\alpha_{r-1,1,1}$ and $\alpha_{r-1,2,2}$. Note that $\alpha_{r-1,2,2}$ is formed by adding f_r to $\alpha_{r,2}$. The notable matroid that gives rise to the rank $(r+1)$ seed matroid is $\alpha_{r,1,1}$.

4 Proof of Theorem 1.1.

The proof of Theorem 1.1 is by induction on $r \geq 5$. The base case is somewhat longer than in a typical induction proof. But it is quite straightforward in the sense that only careful computation of small rank matroids is required and that is also greatly reduced by repeated application of Theorem 1.2.

Proof: Let M be a 3-connected binary non-regular matroid. If M has no P_9 or P_9^*-minor, then M is isomorphic to F_7 or a deletion-minor of Z_r, for $r \geq 4$ [9]. Therefore assume that M has a P_9-minor, but no P_9^*-minor. The proof is by induction on the rank. The base case $r \leq 6$ is in the Appendix.

Assume a binary non-regular 3-connected matroid with rank at most $(r-1)$ is in $EX(P_9^*)$ if and only if it is a member of the known classes of matroids. In other words, the rank $(r-2)$ seed α_{r-2} has no cosimple coextensions in $EX(P_9^*)$; its simple single-element extensions $\alpha_{r-2,1}$ and $\alpha_{r-2,2}$ have no cosimple coextensions; their simple single-element extensions $\alpha_{r-2,1,1}$, $\alpha_{r-2,1,2}$, and $\alpha_{r-2,2,2}$ have as their only cosimple single-element coextension the rank $(r-1)$ seed α_{r-1}; and finally these matroids extend to the rank $(r-1)$ monarch Ω_{r-1}. (See Figure 5 and note that the position of the monarchs are not drawn to scale since they are too large.)

We must prove that the rank $(r-1)$ seed α_{r-1} has no cosimple coextensions in $EX(P_9^*)$; its simple single-element extensions $\alpha_{r-1,1}$ and $\alpha_{r-1,2}$ have no cosimple coextensions; their simple single-element extensions $\alpha_{r-1,1,1}$, $\alpha_{r-1,1,2}$, and $\alpha_{r-1,2,2}$ have as their only cosimple single-element coextension the rank r seed α_r; and finally these matroids extend to the rank r monarch Ω_r.

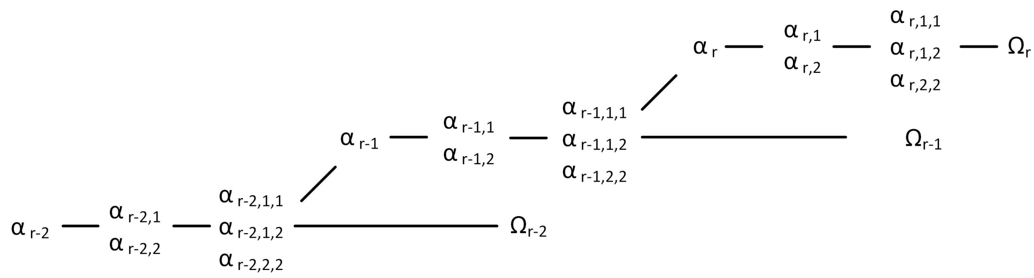

Figure 5: Growth pattern of the seed and monarch

We summarize this in short by saying we will show that the rank $(r-1)$ seed α_{r-1} gives rise to the rank r seed α_r, and α_r extends to the rank r monarch Ω_r and prove the assertions in the form of two claims, both using the induction hypothesis.

Claim A. The rank $(r-1)$ seed α_{r-1} gives rise to the rank r seed α_r.

Proof. By the Strong Splitter Theorem M must be a cosimple single-element coextension of:

(i) α_{r-1};

(ii) $\alpha_{r-1,1}$ or $\alpha_{r-1,2}$ by adding rows in series; or

(iii) $\alpha_{r-1,1,1}$, $\alpha_{r-1,1,2}$, or $\alpha_{r-1,2,2}$ by adding row $[00\ldots011]$);

where

$$\begin{aligned}
\alpha_{r-1,1} &= \alpha_{r-1} + c_{r-1},\\
\alpha_{r-1,2} &= \alpha_{r-1} + e_{r-1},\\
\alpha_{r-1,1,1} &= \alpha_{r-1} + \{c_{r-1}, d_{r-1}\},\\
\alpha_{r-1,1,2} &= \alpha_{r-1} + \{c_{r-1}, e_{r-1}\},\\
\alpha_{r-1,2,2} &= \alpha_{r-1} + \{e_{r-1}, f_{r-1}\}.
\end{aligned}$$

If row $[00\ldots011]$ is added to $\alpha_{r-1,1,1}$, we get α_r, which is the rank r seed matroid. We will show that the other matroids do not have cosimple single-element coextensions in $EX(P_9^*)$.

Case i. By the induction hypothesis α_{r-1} is formed by adding row $[00\ldots011]$ to $\alpha_{r-2,1,1}$, and therefore has no further single-element coextension in $EX(P_9^*)$.

Case ii(a). Suppose, if possible, M is a cosimple single-element coextension of $\alpha_{r-1,1} = \alpha_{r-1} + c_{r-1}$. Only Type II and Type III rows may be added to $\alpha_{r-1,1}$. Type II rows are the identity rows with a one in the last entry and Type III rows are the rows of $\alpha_{r-1,1}$ (shown in Figure 6) with the last entry switched (*i.e.* put 0 if the last entry is 1 and 1 if it is 0). The superscripts indicate if the last entry is a 1 or a 0.

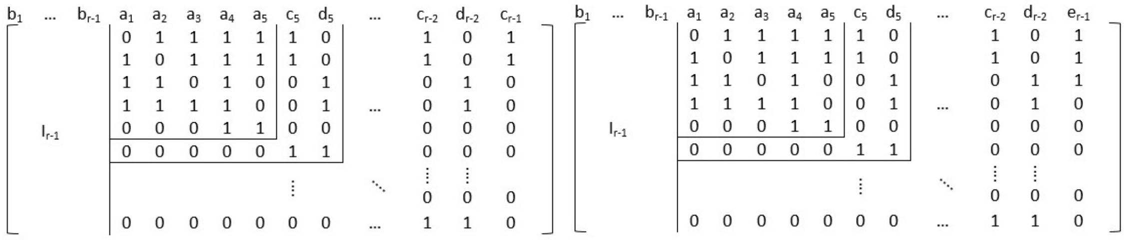

Figure 6: $\alpha_{r-1,1} = \alpha_{r-1} + c_{r-1}$ and $\alpha_{r-1,2} = \alpha_{r-1} + e_{r-1}$

Type II rows are:

$$\begin{aligned} a_1^1 &= [100\ldots0000\mathbf{1}], \\ a_2^1 &= [010\ldots0000\mathbf{1}], \ldots, \\ c_{r-2}^1 &= [000\ldots0010\mathbf{1}], \\ d_{r-2}^1 &= [000\ldots0001\mathbf{1}]. \end{aligned}$$

Type III rows are:

$$\begin{aligned} b_1^0 &= [0111110\ldots1010\mathbf{0}], \\ b_2^0 &= [1011110\ldots1010\mathbf{0}], \\ b_3^1 &= [1101001\ldots0101\mathbf{1}], \\ b_4^1 &= [1111001\ldots0101\mathbf{1}], \\ b_5^1 &= [0001100\ldots0000\mathbf{1}], \\ b_6^1 &= [0000011\ldots0000\mathbf{1}], \ldots, \\ b_{r-2}^1 &= [0000000\ldots1100\mathbf{1}], \\ b_{r-1}^1 &= [0000000\ldots0011\mathbf{1}]. \end{aligned}$$

However, observe that $\alpha_{r-1,1}/b_{r-1}\backslash c_{r-2} \cong \alpha_{r-2,1,1}$. Note the isomorphism instead of equality. This happens because in $\alpha_{r-1,1}\backslash c_{r-2}/b_{r-1}$ the last two columns d_{r-2} and c_{r-1} are switched. Otherwise it would be exactly equal to $\alpha_{r-2,1,1}$. By the induction hypothesis, $\alpha_{r-2,1,1}$ has exactly one cosimple single-element coextension in the class (the one formed by row $x = [000\ldots011]$). The isomorphism instead of equality is of no consequence since the last two entries are both ones.

Thus we have Type I rows with a zero or one in the third last entry (the position of the deleted column c_{r-2}), Type II rows which are the identity rows with a one in the third last entry, and Type III rows with the third last entry switched. Specifically, Type I rows are:

$$\begin{aligned} x^0 &= [000\ldots0\mathbf{0}11], \\ x^1 &= [000\ldots0\mathbf{1}11]. \end{aligned}$$

Type II rows are:

$$\begin{aligned} a_1^1 &= [100\ldots00\mathbf{1}00], \\ a_2^1 &= [010\ldots00\mathbf{1}00], \ldots, \\ c_{r-3}^1 &= [000\ldots01\mathbf{1}00], \\ d_{r-3}^1 &= [000\ldots00\mathbf{1}10], \\ d_{r-2}^1 &= [000\ldots00\mathbf{1}01]. \end{aligned}$$

Type III rows are:

$$\begin{aligned} b_1^0 &= [0111111\ldots10\mathbf{0}01], \\ b_2^0 &= [1011111\ldots10\mathbf{0}01], \\ b_3^1 &= [1101000\ldots01\mathbf{1}10], \\ b_4^1 &= [1111000\ldots01\mathbf{1}10], \\ b_5^1 &= [0001100\ldots00\mathbf{1}00], \\ b_6^1 &= [0000011\ldots00\mathbf{1}00], \\ b_{r-2}^1 &= [0000000\ldots11\mathbf{1}00]. \end{aligned}$$

The only common rows are $[000\ldots00011]$, $[000\ldots00111]$ and $[000\ldots0101]$. Therefore the only matrices that must be checked explicitly for a P_9^* minor are the ones shown below formed with these three rows. They have the following three rank 7 minors, respectively, obtained by contracting $\{b_6,\ldots,b_{r-2}\}$ and deleting

$$\{c_5, d_5, c_6, d_6, \ldots, c_{r-3}, d_{r-3}\}.$$

$$M_1 = \left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&1 \\ & 1&0&1&1&1&1&0&1 \\ I_7 & 1&1&0&1&0&0&1&0 \\ & 1&1&1&1&0&0&1&0 \\ & 0&0&0&1&1&0&0&0 \\ & 0&0&0&0&0&1&1&0 \\ & 0&0&0&0&0&0&1&1 \end{array}\right], \quad M_2 = \left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&1 \\ & 1&0&1&1&1&1&0&1 \\ I_7 & 1&1&0&1&0&0&1&0 \\ & 1&1&1&1&0&0&1&0 \\ & 0&0&0&1&1&0&0&0 \\ & 0&0&0&0&0&1&1&0 \\ & 0&0&0&0&0&1&1&1 \end{array}\right],$$

$$M_3 = \left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&1 \\ & 1&0&1&1&1&1&0&1 \\ I_7 & 1&1&0&1&0&0&1&0 \\ & 1&1&1&1&0&0&1&0 \\ & 0&0&0&1&1&0&0&0 \\ & 0&0&0&0&0&1&1&0 \\ & 0&0&0&0&0&1&0&1 \end{array}\right].$$

Each of M_1, M_2, and M_3 has a P_9^*-minor. Thus M cannot be a cosimple single-element coextension of $\alpha_{r-1,1}$.

Case ii(b). Suppose, if possible, M is a cosimple single-element coextension of $\alpha_{r-1,2} = \alpha_{r-1} + e_{r-1}$ (shown in Figure 6). The argument is similar to that of Case ii(a). Only Type II and Type III rows may be added to α_{r-1}. Type II rows are:

$$\begin{aligned} a_1^1 &= [100\ldots0000\mathbf{1}], \\ a_2^1 &= [010\ldots0000\mathbf{1}], \ldots, \\ c_{r-2}^1 &= [000\ldots0010\mathbf{1}], \\ d_{r-2}^1 &= [000\ldots0001\mathbf{1}]. \end{aligned}$$

Type III rows are:

$$\begin{aligned} b_1^0 &= [0111110\ldots1010\mathbf{0}], \\ b_2^0 &= [1011110\ldots1010\mathbf{0}], \\ b_3^0 &= [1101001\ldots0101\mathbf{0}], \\ b_4^1 &= [1111001\ldots0101\mathbf{1}], \\ b_5^1 &= [0001100\ldots0000\mathbf{1}], \\ b_6^1 &= [0000011\ldots0000\mathbf{1}], \ldots, \\ b_{r-2}^1 &= [0000000\ldots1100\mathbf{1}], \\ b_{r-1}^1 &= [0000000\ldots0011\mathbf{1}]. \end{aligned}$$

However, observe that $\alpha_{r-1,2}/b_{r-1}\backslash d_{r-2} = \alpha_{r-2,1,2}$. There are no Type I rows to be added to $\alpha_{r-1,1}$ by the induction hypothesis. Type II rows that may be added to α_{r-1} are:

$$\begin{aligned} a_1^1 &= [100\ldots000\mathbf{1}0], \\ a_2^1 &= [010\ldots000\mathbf{1}0], \\ c_{r-2}^1 &= [000\ldots001\mathbf{1}0], \\ e_{r-1}^1 &= [000\ldots0000\mathbf{1}1]. \end{aligned}$$

Type III rows are:

$$\begin{aligned} b_1^0 &= [0111110\ldots101\mathbf{1}1], \\ b_2^0 &= [1011110\ldots101\mathbf{1}1], \\ b_3^1 &= [1101001\ldots010\mathbf{0}0], \\ b_4^1 &= [1111001\ldots010\mathbf{0}0], \\ b_5^1 &= [0001100\ldots000\mathbf{1}0], \\ b_6^1 &= [0000011\ldots000\mathbf{1}0], \ldots, \\ b_{r-2}^1 &= [0000000\ldots1100\mathbf{1}0]. \end{aligned}$$

The only common row is $[000\ldots00011]$. Therefore the only matrix that must be checked explicitly for a P_9^* minor is the matrix with row $[000\ldots00011]$. This matrix has the following rank 7 minor M_4 obtained by contracting $\{b_6, \ldots, b_{r-2}\}$ and deleting $\{c_5, d_5, \ldots, c_{r-3}, d_{r-3}\}$:

$$M_4 = \left[\begin{array}{c|cccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ I_7 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

.

Since M_4 has a P_9^*-minor, M cannot be a cosimple single-element coextension of $\alpha_{r-1,2}$.

Case iii. Suppose, if possible, M is a cosimple single-element coextension of $\alpha_{r-1,1,2}$ or $\alpha_{r-1,2,2}$. Then M is formed by adding row $[000\ldots011]$ to $\alpha_{r-1,1,2} = \alpha_{r-1} + \{c_{r-1}, e_{r-1}\}$ or $\alpha_{r-1,2,2} = \alpha_{r-1} + \{e_{r-1}, f_{r-1}\}$. The matrices formed in this manner have as minors M_5 and M_6 shown below obtained by contracting $\{b_6, \ldots, b_{r-2}\}$ and deleting $\{c_5, d_5, \ldots c_{r-3}, d_{r-3}\}$:

$$M_5 = \left[\begin{array}{c|ccccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ I_7 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right],$$

$$M_6 = \left[\begin{array}{c|ccccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ I_7 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Observe that M_5 and M_6 have a P_9^*-minor.

Claim B. The rank r seed α_r extends to the rank r monarch Ω_r

Proof. We will prove that the only columns that can be added to α_r are

$$c_r, d_r, e_r, f_r, g_{r,1}, \ldots, g_{r,r-4}.$$

Adding all these columns give Ω_r. To begin with we will show that α_r has two single-element extensions $\alpha_{r,1}$ and $\alpha_{r,2}$. Observe that

$$\alpha_r / b_r \backslash \{c_r, d_r\} = \alpha_{r-1}.$$

By the induction hypothesis the only columns that can be added to α_{r-1} are

$$e_{r-1}, f_{r-1}, g_{r,1}, \ldots, g_{r,r-5}.$$

Thus there are three types of columns that can be added to α_r. Type I columns are those that can be added to α_{r-1}, namely, $e_{r-1}, f_{r-1}, g_{r,1}, \ldots, g_{r,r-5}$, with a zero or one in the last entry. Type II and III columns are the columns of α_{r-1} with the last entry switched. Specifically, Type I columns are:

$$\begin{aligned}
e^0_{r-1} &= [1110000\ldots00\mathbf{0}]^T,\\
e^1_{r-1} &= [1110000\ldots10\mathbf{1}]^T,\\
f^0_{r-1} &= [0011100\ldots00\mathbf{0}]^T,\\
f^1_{r-1} &= [0011100\ldots00\mathbf{1}]^T,\\
g^0_{r-1,1} &= [1111000\ldots00\mathbf{0}]^T,\\
g^1_{r-1,1} &= [1111000\ldots00\mathbf{1}]^T,\\
g^0_{r-1,2} &= [1111010\ldots00\mathbf{0}]^T,\\
g^1_{r-1,2} &= [1111010\ldots00\mathbf{1}]^T, \ldots,\\
g^0_{r-1,r-5} &= [1111000\ldots01\mathbf{0}]^T,\\
g^1_{r-1,r-5} &= [1111000\ldots01\mathbf{1}]^T.
\end{aligned}$$

Type II columns are:

$$\begin{aligned}
b^1_1 &= [100\ldots00\mathbf{1}]^T,\\
b^1_2 &= [010\ldots00\mathbf{1}]^T, \ldots,\\
b^1_{r-1} &= [000\ldots01\mathbf{1}]^T
\end{aligned}$$

and Type III columns are:

$$\begin{aligned}
a^0_1 &= [0111000\ldots00\mathbf{1}]^T,\\
a^0_2 &= [1011000\ldots00\mathbf{1}]^T,\\
a^1_3 &= [1101000\ldots00\mathbf{1}]^T,\\
a^1_4 &= [1111100\ldots00\mathbf{1}]^T,\\
a^1_5 &= [1100100\ldots00\mathbf{1}]^T,\\
c^1_5 &= [1100010\ldots00\mathbf{1}]^T,\\
d^1_5 &= [0011010\ldots00\mathbf{1}]^T,\\
c^1_6 &= [1100001\ldots00\mathbf{1}]^T,\\
d^1_6 &= [0011001\ldots00\mathbf{1}]^T, \ldots,\\
c^1_{r-2} &= [1100000\ldots01\mathbf{1}]^T,\\
d^1_{r-2} &= [0011000\ldots01\mathbf{1}]^T,\\
c^0_{r-1} &= [1100000\ldots00\mathbf{0}]^T,\\
d^0_{r-1} &= [0011000\ldots00\mathbf{0}]^T.
\end{aligned}$$

Observe that, $c^0_{r-1} = c_r$, $d^0_{r-1} = d_r$ $e^0_{r-1} = e_r$, $f^0_{r-1} = f_r$, $g^0_{r-1,1} = g_{r,1}, g^0_{r-1,2} = g_{r,2}, g^0_{r-1,r-5} = g_{r,r-5}$, and $g^1_{r-1,1} = g_{r,r-4}$. We will show that the matrices obtained by adding the other columns have a P_9^*-minor.

Consider Type I columns e^1_{r-1} and f^1_{r-1}. Writing out the matrices it is easy to see that

$$(\alpha_r + e^1_{r-1})/\{b_6, \dots b_{r-1}\}\backslash\{c_5, d_5, \dots c_{r-2}, d_{r-2}\} = \alpha_6 + e^1_6$$

and

$$(\alpha_r + f^1_{r-1})/\{b_6, \dots b_{r-1}\}\backslash\{c_5, d_5, \dots c_{r-2}, d_{r-2}\} = \alpha_6 + f^1_6$$

which are shown below:

$$\alpha_6 + e^1_6 = \left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&1\\ & 1&0&1&1&1&1&0&1\\ I_6 & 1&1&0&1&0&0&1&1\\ & 1&1&1&1&0&0&1&0\\ & 0&0&0&1&1&0&0&0\\ & 0&0&0&0&0&1&1&1 \end{array}\right],$$

$$\alpha_6 + f^1_6 = \left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&0\\ & 1&0&1&1&1&1&0&0\\ I_6 & 1&1&0&1&0&0&1&1\\ & 1&1&1&1&0&0&1&1\\ & 0&0&0&1&1&0&0&1\\ & 0&0&0&0&0&1&1&1 \end{array}\right].$$

Both the above matroids have a P_9^*-minor.

Consider the Type I columns $g^1_{r-1,2}, g^1_{r-1,3}, \dots g^1_{r-1,r-5}$. For $2 \le k \le r-5$, the matrix $\alpha_r + g^1_{r-1,k}$ has as minor $\alpha_7 + g^1_{6,1}$ obtained by contracting $b_6, \dots, b_{r-1}$ except b_{k+4} and deleting $c_5, d_5, \dots, c_{r-2}, d_{r-2}$ except c_{k+3} and d_{k+3}. The matrix $\alpha_7 + g^1_{6,1}$ is shown below and it has a P_9^*-minor:

$$\alpha_7 + g^1_{6,1} = \left[\begin{array}{c|cccccccccc} & 0&1&1&1&1&1&0&1&0&1\\ & 1&0&1&1&1&1&0&1&0&1\\ I_7 & 1&1&0&1&0&0&1&0&1&1\\ & 1&1&1&1&0&0&1&0&1&1\\ & 0&0&0&1&1&0&0&0&0&0\\ & 0&0&0&0&0&1&1&0&0&1\\ & 0&0&0&0&0&0&0&1&1&1 \end{array}\right].$$

Consider the Type II columns $b^1_1, \dots b^1_{r-1}$. Writing out the matrices we see that for $1 \le k \le 5$

$$(\alpha_r + b^1_k)/\{b_6, b_7, \dots b_{r-1}\}\backslash\{c_5, d_5, \dots c_{r-2}, d_{r-2}\} = \alpha_6 + b^1_k.$$

These matrices are shown below:

$$\alpha_6+b_1^1=\left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&1\\ & 1&0&1&1&1&1&0&0\\ I_6 & 1&1&0&1&0&0&1&0\\ & 1&1&1&1&0&0&1&0\\ & 0&0&0&1&1&0&0&0\\ & 0&0&0&0&0&1&1&1\end{array}\right],\quad \alpha_6+b_2^1=\left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&0\\ & 1&0&1&1&1&1&0&1\\ I_6 & 1&1&0&1&0&0&1&0\\ & 1&1&1&1&0&0&1&0\\ & 0&0&0&1&1&0&0&0\\ & 0&0&0&0&0&1&1&1\end{array}\right],$$

$$\alpha_6+b_3^1=\left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&0\\ & 1&0&1&1&1&1&0&0\\ I_6 & 1&1&0&1&0&0&1&1\\ & 1&1&1&1&0&0&1&0\\ & 0&0&0&1&1&0&0&0\\ & 0&0&0&0&0&1&1&1\end{array}\right],\quad \alpha_6+b_4^1=\left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&0\\ & 1&0&1&1&1&1&0&0\\ I_6 & 1&1&0&1&0&0&1&0\\ & 1&1&1&1&0&0&1&1\\ & 0&0&0&1&1&0&0&0\\ & 0&0&0&0&0&1&1&1\end{array}\right],$$

$$\alpha_6+b_5^1=\left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&0\\ & 1&0&1&1&1&1&0&0\\ I_6 & 1&1&0&1&0&0&1&0\\ & 1&1&1&1&0&0&1&0\\ & 0&0&0&1&1&0&0&1\\ & 0&0&0&0&0&1&1&1\end{array}\right],$$

They have a P_9^*-minor. For $6 \leq k \leq r-1$, the matroid $\alpha_r + b_k$ has minor $\alpha_7 + b_6^1$ obtained by contracting all columns $\{b_6, \ldots, b_{r-1}\}$ except b_k and deleting all columns $\{c_5, d_5, \ldots, c_{r-2}, d_{r-2}\}$ except c_{r-2} and d_{r-2}. The matrix $\alpha_7 + b_6^1$ is shown below and it has a P_9^*-minor:

$$\alpha_7+b_6^1=\left[\begin{array}{c|cccccccccc} & 0&1&1&1&1&1&0&1&0&0\\ & 1&0&1&1&1&1&0&1&0&0\\ I_7 & 1&1&0&1&0&0&1&0&1&0\\ & 1&1&1&1&0&0&1&0&1&0\\ & 0&0&0&1&1&0&0&0&0&0\\ & 0&0&0&0&0&1&1&0&0&1\\ & 0&0&0&0&0&0&0&1&1&1\end{array}\right].$$

Consider the Type III columns $a_1^1, a_2^1, a_3^1, a_4^1, a_5^1, c_5^1, d_5^1, c_6^1, d_6^1, \ldots, c_{r-1}^0, d_{r-1}^0$. Writing out the matrices we see that for $1 \leq k \leq 5$:

$$\alpha_r + a_k^1/\{b_6, b_7, \ldots, b_{r-1}\}\backslash\{c_5, d_5, \ldots c_{r-2}, d_{r-2}\} = \alpha_6 + a_k^1$$

These matrices are shown below and each has a P_9^*-minor:

$$\alpha_6+a_1^1=\left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&0\\ & 1&0&1&1&1&1&0&1\\ I_6 & 1&1&0&1&0&0&1&1\\ & 1&1&1&1&0&0&1&1\\ & 0&0&0&1&1&0&0&0\\ & 0&0&0&0&0&1&1&1\end{array}\right],\quad \alpha_6+a_2^1=\left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&1\\ & 1&0&1&1&1&1&0&0\\ I_6 & 1&1&0&1&0&0&1&1\\ & 1&1&1&1&0&0&1&1\\ & 0&0&0&1&1&0&0&0\\ & 0&0&0&0&0&1&1&1\end{array}\right],$$

$$\alpha_6+a_3^1=\left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&1\\ & 1&0&1&1&1&1&0&1\\ I_6 & 1&1&0&1&0&0&1&0\\ & 1&1&1&1&0&0&1&1\\ & 0&0&0&1&1&0&0&0\\ & 0&0&0&0&0&1&1&1 \end{array}\right],\quad \alpha_6+a_4^1=\left[\begin{array}{c|cccccccc} & 0&1&1&1&1&1&0&1\\ & 1&0&1&1&1&1&0&1\\ I_6 & 1&1&0&1&0&0&1&1\\ & 1&1&1&1&0&0&1&1\\ & 0&0&0&1&1&0&0&1\\ & 0&0&0&0&0&1&1&1 \end{array}\right].$$

For $6 \leq k \leq r-2$. the matrix $\alpha_r + c_k^1$ has as minor $\alpha_7 + c_5^1$ obtained by contacting columns $\{b_6, \ldots, b_{r-1}\}$ except b_{k+1} and deleting columns $\{c_5, d_5, \ldots, c_{r-2}, d_{r-2}\}$ except c_k and d_k. Similarly, $\alpha_r + d_k^1$ has as minor $\alpha_7 + d_5^1$. The matrices $\alpha_7 + c_5^1$ and $\alpha_7 + d_5^1$ are shown below and each has a P_9^*-minor:

$$\alpha_7+c_5^1=\left[\begin{array}{c|cccccccccc} & 0&1&1&1&1&1&0&1&0&1\\ & 1&0&1&1&1&1&0&1&0&1\\ I_7 & 1&1&0&1&0&0&1&0&1&0\\ & 1&1&1&1&0&0&1&0&1&0\\ & 0&0&0&1&1&0&0&0&0&0\\ & 0&0&0&0&0&1&1&0&0&1\\ & 0&0&0&0&0&0&0&1&1&1 \end{array}\right],$$

$$\alpha_7+d_5^1=\left[\begin{array}{c|cccccccccc} & 0&1&1&1&1&1&0&1&0&0\\ & 1&0&1&1&1&1&0&1&0&0\\ I_7 & 1&1&0&1&0&0&1&0&1&1\\ & 1&1&1&1&0&0&1&0&1&1\\ & 0&0&0&1&1&0&0&0&0&0\\ & 0&0&0&0&0&1&1&0&0&1\\ & 0&0&0&0&0&0&0&1&1&1 \end{array}\right].$$

Therefore, α_r extends to Ω_r.

To finish the proof it is easy to see using the above induction argument that α_r has two non-isomorphic single-element extensions $\alpha_{r,1}$ formed by adding columns c_r, d_r or $g_{r,1}$, and the remaining columns give $\alpha_{r,2}$. Since the pattern in columns $g_{r,t}$, where $2 \leq t \leq r-4$ begins with α_7 we only need to check that:

$$\alpha_7 + g_{7,2} \cong \alpha_7 + g_{7,3}.$$

This is true due to the mapping from $\alpha_7 + g_{7,2}$ to $\alpha_7 + g_{7,3}$ that takes:

$$(b_1, b_2, b_3, b_4, b_5, \mathbf{b_6}, \mathbf{b_7}, a_1, a_2, a_3, a_4, a_5, \mathbf{c_5}, \mathbf{d_5}, \mathbf{c_6}, \mathbf{d_6}, \mathbf{g_{7,2}})$$

to

$$(b_1, b_2, b_3, b_4, b_5, \mathbf{b_7}, \mathbf{b_6}, a_1, a_2, a_3, a_4, a_5, \mathbf{c_6}, \mathbf{d_6}, \mathbf{c_5}, \mathbf{d_5}, \mathbf{g_{7,3}}).$$

Similarly, $\alpha_{r,1}$ has two non-isomorphic single-element extensions, the notable one that gives rise to the rank-(r+1) seed matroid is $\alpha_{r,1,1}$ formed by adding c_r and d_r to α_r, and $\alpha_{r,2}$ also has two non-isomorphic single-element extensions. □

The next result follows immediately since the size of the rank r non-regular infinite families Z_r and Ω_r are, respectively, $2r+1$ and $4r-5$ and the complete graph K_{r+1} is the largest rank r regular member with no minor isomorphic to P_9^*.

Corollary 4.1 *Let M be a simple binary matroid of rank $r \geq 6$ with no P_9^* minor. Then $|M| \leq \frac{r(r+1)}{2}$, with this bound being attained for $M \cong M(K_{r+1})$.* □

The above result may be added to a short list of similar results. See for example Table 1 in [7] that has a list of size functions for some classes of binary matroids.

Appendix

In many instances we have to check matrices for a P_9^*-minor. The presence or absence of the minor has been determined by the matroid software programs Oid and Macek. While Macek only gives a yes/no answer, Oid gives the columns that must be deleted and contracted, making it easy to verify by hand.

Suppose M has rank $r \leq 6$. Since P_9^* is a rank 5 matroid, $EX(P_9^*)$ contains $PG(3,2)$. The matroid P_9 has three non-isomorphic simple single-element extensions, D_1, D_2, and D_3, and eight non-isomorphic cosimple single-element coextensions, of which just one matroid E_7 has no P_9^*-minor. See Appendix Tables 1 and 2 of [5]. The matroid E_7 is shown below. It is α_5, the rank 5 seed matroid. Most importantly, note that since α_5 is formed by adding to P_9 just one row [00011], α_5 has no further cosimple coextensions in $EX(P_9^*)$.

$$E_7 = \alpha_5 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

We need to show that:

(i) α_5 is the rank 5 seed matroid and it extends to the rank 5 monarch Ω_5;

(ii) α_5 gives rise to the rank 6 seed α_6; and

(iii) There is an additional rank 5, 16-element matroid R_{16} that results in no larger rank matroids in $EX(P_9^*)$.

By the Strong Splitter Theorem M must be a cosimple single-element coextension of P_9, or of its single-element extensions D_1, D_2, or D_3 formed with a row in series to an existing row, or of its double-element extensions X_1, X_2, X_3 formed with row [0000011]. The matroids D_1, D_2, and D_3 and X_1, X_2, X_3 are shown below:

$$D_1 = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right], D_2 = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right],$$

$$D_3 = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 0 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right],$$

$$X_1 = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{array}\right], X_2 = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right],$$

$$X_3 = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{array}\right].$$

In every instance the resulting matroid has an α_5-minor or P_9^*-minor. Observe from Table 1 shown below that α_5 has three simple single-element extensions with no P_9^*-minor (extensions 2, 3, and 5).

Extension Columns	**Name**	**P_9^*-minor**
[00011] [00101] [11101]	Ext 1	Yes
[00110] [11000] [11110]	Ext 2 ($\alpha_{\mathbf{5,1}}$)	No
[00111] [11100]	Ext 3 ($\alpha_{\mathbf{5,2}}$)	No
[01001] [01010] [01100] [01111] [10001] [10010] [10100] [10111]	Ext 4	Yes
[01011] [01101] [10011] [10101]	Ext 5 ($\alpha_{\mathbf{5,3}}$)	No
[11011]	Ext 6	Yes

Table 1: Simple single-element extensions of α_5

Let $\alpha_{5,1} = (\alpha_5, ext2)$, $\alpha_{5,2} = (\alpha_5, ext3)$ and $\alpha_{5,3} = (\alpha_5, ext5)$. Matrix representations for $\alpha_{5,1}$, $\alpha_{5,2}$, and $\alpha_{5,3}$ are shown below:

$$\alpha_{5,1} = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right], \alpha_{5,2} = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right],$$

$$\alpha_{5,3} = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

Further, observe that $\alpha_{5,1}$ is obtained by adding columns $a = [00110]^T$, $b = [11000]^T$, and $c = [11110]^T$; $\alpha_{5,2}$ is obtained by adding column $d = [00111]^T$ and

$e = [11100]^T$; and $\alpha_{5,3}$ is obtained by adding column $f = [01011]^T$, $g = [01101]^T$, $h = [10011]^T$, and $i = [10101]^T$.

We can check that $\alpha_{5,3}$ (formed by adding column f to α_5) has only one simple single-element extension in $EX(P_9^*)$ and it is obtained by adding any one of columns g, h, i, d, or e. Up to isomorphism all five columns give the same single-element extension. Let us call this matroid $\alpha_{5,3,1}$ obtained by adding, say, column g:

$$\alpha_{5,3,1} = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

.

Similarly, adding to $\alpha_{5,3,1}$ any one of columns h, i, d, e, say h, gives $\alpha_{5,3,1,1}$, and so on; we get $\alpha_{5,3,1,1}$ by adding i; $\alpha_{5,3,1,1,1}$ by adding d; and finally $\alpha_{5,3,1,1,1,1} = R_{16}$ by adding e. This gives all the rank 5 members in $EX(P_9^*)$ with an $\alpha_{5,3}$-minor.

It remains to show that there are no higher rank matroids with an $\alpha_{5,3}$-minor. Suppose M is a rank 6 cosimple single-element matroid in $EX(P_9^*)$ with an $\alpha_{5,3}$-minor. By the Strong Splitter Theorem, M is a cosimple single-element coextension of $\alpha_{5,3}$ by Type II and III rows or $\alpha_{5,3,1}$ with row [0000011]. In every instance M has a P_9^*-minor.

Thus we may assume M has an $\alpha_{5,1}$-minor or $\alpha_{5,2}$-minor. Renaming columns to fit the pattern in Ω_5, Table 1 shows that $\alpha_{5,1}$ is formed by adding columns $c_5 = [11000]^T$, $d_5 = [00110]^T$, or $g_{5,1} = [11110]^T$ and $\alpha_{5,2}$ is formed by adding columns $e_5 = [11100]^T$ or $f_5 = [00111]^T$. Adding all these columns to α_5 gives Ω_5. Using the same method explained above for $\alpha_{5,3}$ we can show that every cosimple single-element coextension of $\alpha_{5,1}$ and $\alpha_{5,2}$ also has a P_9^*-minor.

Lastly, $\alpha_{5,1}$ (with c_5) has two simple single-element extensions in $EX(P_9^*)$, namely, $\alpha_{5,1,1}$, formed by adding d_5 or $g_{5,1}$, and $\alpha_{5,1,2}$ formed by adding e_5 or f_5. The matroid $\alpha_{5,2}$ (with e_5) also has two single-element extensions, $\alpha_{5,2,1}$ formed by adding c_5, d_5 or $g_{5,1}$ and $\alpha_{5,2,2}$ formed by adding f_5:

$$\alpha_{5,1,1} = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right], \alpha_{5,1,2} = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right],$$

$$\alpha_{5,2,2} = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right].$$

Further, note that $\alpha_{5,2,1} = \alpha_{5,1,2}$.

By the Strong Splitter Theorem we must only check one single-element coextension of $\alpha_{5,1,1}$, $\alpha_{5,1,2}$, and $\alpha_{5,2,2}$, namely the one formed by adding row [0000011].

Observe that $\alpha_{5,1,1}$ with row [0000011] is precisely α_6, whereas each of $\alpha_{5,1,2}$ and $\alpha_{5,2,2}$ with row [0000011] has a P_9^*-minor. This completes the base case for the induction argument.

The base case is summarized in Figure 7, where the numbers below the figure represent the size of the matroids. The rank 5 seed α_5 has size 10 and the monarch Ω_5 has size 15. The rank 6 seed α_6 has size 13 and the monarch Ω_6 has size 19. The figure also shows how R_{16} manifests as an extension of α_5 via its third single-element extension $\alpha_{5,3}$, which has no coextensions in $EX(P_9^*)$.

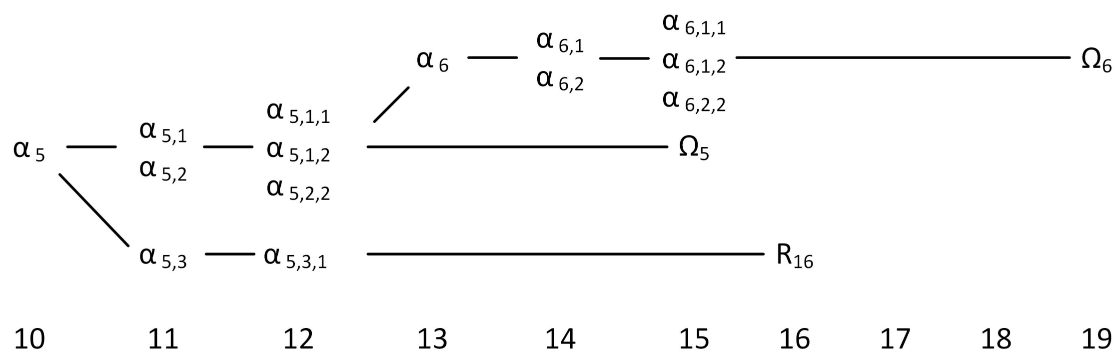

Figure 7: Base case of the induction argument

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