AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **75(2)** (2019), Pages 256–258

Erratum to the article ‘A lexicographic product for signed graphs’

Maurizio Brunetti

Dipartimento di Matematica e Applicazioni
Università ‘Federico II’ di Napoli
Italy
mbrunett@unina.it

Matteo Cavaleri   Alfredo Donno

Università degli Studi Niccolò Cusano di Roma
Italy
matteo.cavaleri@unicusano.it   alfredo.donno@unicusano.it

Let $\Gamma = (G, \sigma)$ and $\Lambda = (H, \tau)$ be two signed graphs. A lexicographic product of signed graphs $\Gamma \overline{*} \Lambda$ has been firstly introduced in [3], and some of its spectral properties are discussed in [2]. In [1] we proposed an alternative signature $\sigma * \tau$ on the (unsigned) lexicographic product $G * H$ claiming that such new signature, contrarily to the one appeared in [3], maps balanced graphs onto balanced graphs.

Dimitri Lajou, who is currently working at University of Bordeaux, kindly contacted us, pointing out that this claim is not true in general, providing the counter-example depicted in Fig. 1: $(P_2, +)$ and (P_3, τ) are both balanced but $(P_2, +) * (P_3, \tau)$ is not. In fact, the latter contains unbalanced cycles, like the triangle with vertices (u_0, v_0), (u_0, v_1) and (u_1, v_2).

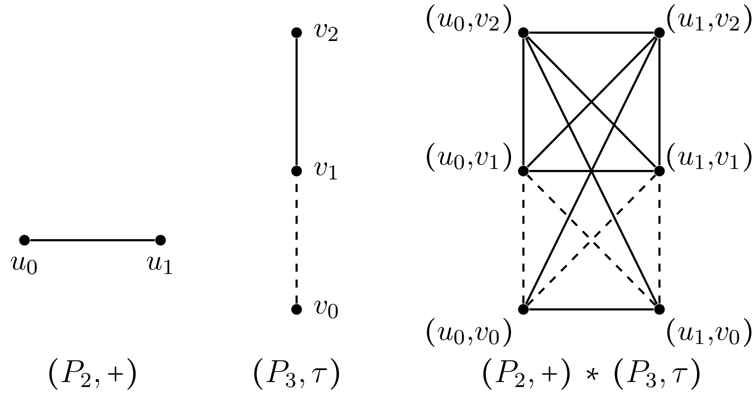

Fig. 1: The unbalanced graph $(P_2, +) * (P_3, \tau)$. Dashed lines represent negative edges.

ISSN: 2202-3518

Our incorrect claim was a consequence of a mistake in the proof of [1, Theorem 2.3] concerning the signature of sloping edges, hence the statements of Theorem 2.3, our claim that such Theorem admits a purely matrix-theoretical proof, and Corollary 2.4 in [1] are not correct.

In order to precisely locate the error, we begin by recalling the meaning of notation and terminology used there.

For $\Gamma = (G, \sigma_\Gamma)$ and $U \subset V(G)$, let $\Gamma^U = (G, \sigma_{\Gamma^U})$ be the signed graph obtained from Γ by negating the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be switching equivalent to Γ. Sometimes the subscript of σ_Γ is omitted when it is clear which signed graph the signature refers to. Finally, $(G,+)$ denotes a signed graph whose edges are all positively signed.

Our (wrong) Theorem 2.3 in [1] says that given two signed graphs $\Gamma = (G, \sigma)$ and $\Lambda = (H, \tau)$, and given U and Y subsets of $V(G)$ and $V(H)$ respectively, the two signed graphs $\Gamma * \Lambda$ and $\Gamma^U * \Lambda^Y$ are switching equivalent.

The proof consisted of two steps. We first showed that

$$\Gamma * \Lambda \sim \Gamma^U * \Lambda. \tag{1}$$

Thereafter, we claimed that

$$\Gamma^U * \Lambda \sim \Gamma^U * \Lambda^Y. \tag{2}$$

Now, (1) is in fact correct since, in all cases, $\Gamma^U * \Lambda$ is equal to $(\Gamma * \Lambda)^{U \times V(H)}$. Indeed, by (2.1) in [1], we see that

$$\sigma_{\Gamma_U} * \tau\,((u,v)(u',v')) \quad \text{and} \quad (\sigma * \tau)_{(\Gamma*\Lambda)^{U\times V(H)}}\,((u,v)(u',v'))$$

are both equal to

$$\begin{array}{rl} \sigma(u,u') & \text{if } v \nsim v' \text{ and } (u,u') \in (U\times U) \cup ((V(G)\setminus U)\times(V(G)\setminus U)); \\ -\sigma(u,u') & \text{if } v \nsim v' \text{ and } (u,u') \notin (U\times U) \cup ((V(G)\setminus U)\times(V(G)\setminus U)); \\ \sigma(u,u')\tau(v,v') & \text{if } v \sim v' \text{ and } (u,u') \in (U\times U) \cup ((V(G)\setminus U)\times(V(G)\setminus U)); \\ -\sigma(u,u')\tau(v,v') & \text{if } v \sim v' \text{ and } (u,u') \notin (U\times U) \cup ((V(G)\setminus U)\times(V(G)\setminus U)); \\ \tau(v,v') & \text{if } u = u' \text{ and } v \sim v'. \end{array}$$

On the contrary (2) is false in general: $(P_3,+)$ and (P_3,τ) in Fig. 1 are switching equivalent, yet $(P_2,+) * (P_3,+)$ and $(P_2,+) * (P_3,\tau)$ are not. In [1] we said that (2) comes from the equality $\Gamma^U * \Lambda^Y = (\Gamma^U * \Lambda)^{V(G)\times Y}$, but there are cases where such equality fails to hold. In fact, suppose that U contains two adjacent vertices u and u', and $V(H)$ contains two non-adjacent vertices v and v'. If we take $Y = \{v\}$, we get

$$\sigma_{\Gamma_U} * \tau_{\Lambda_Y}\,((u,v)(u',v')) = \sigma(u,u'),$$

whereas

$$(\sigma_{\Gamma_U} * \tau)_{(\Gamma^U*\Lambda)^{V(G)\times Y}}\,((u,v)(u',v')) = -\sigma(u,u').$$

For the mentioned alternative matrix-theoretical proof of Theorem 2.3 we had in mind, we used a wrong state matrix based on the false equality (2).

Fortunately, Theorem 2.3 and Corollary 2.4 do not affect the remaining results in [1, Section 2] concerning signed regularity and the spectral computations of [1, Section 3].

The problem of finding a suitable signature $\sigma \circledast \tau$ on $E(G * H)$ such that

1. $\sigma \circledast \tau((u,v)(u',v)) = \sigma(uu')$ for all $uu' \in E(G)$ and $v \in V(H)$;
2. $\sigma \circledast \tau((u,v)(u,v')) = \tau(vv')$ for all $u \in V(G)$ and $vv' \in E(H)$;
3. the signatures $\sigma \circledast \tau$ and $\sigma' \circledast \tau'$ are switching equivalent whenever σ is switching equivalent to σ' and τ is switching equivalent to τ';

remains wide open.

References

[1] M. Brunetti, M. Cavaleri and A. Donno, A lexicographic product for signed graphs, *Australas. J. Combin.* **74** (2019), 332–343.

[2] K. A. Germina, S. Hameed K and T. Zaslavsky, On products and line graphs of signed graphs, their eigenvalues and energy, *Linear Algebra Appl.* **435** (2011), 2432–2450.

[3] S. Hameed K and K. A. Germina, On composition of signed graphs, *Discuss. Math. Graph Theory* **32** (2012), 507–516.

(Received 28 June 2019; revised 2 Sep 2019)