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Bounds on the fair total domination number in trees and unicyclic graphs

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Abstract

For $k \geq 1$, a k-fair total dominating set (or just kFTD-set) in a graph G is a total dominating set S such that $|N(v) \cap S| = k$ for every vertex $v \in V - S$. The k-fair total domination number of G, denoted by $ftd_k(G)$, is the minimum cardinality of a kFTD-set. A fair total dominating set, abbreviated FTD-set, is a kFTD-set for some integer $k \geq 1$. The fair total domination number, denoted by $ftd(G)$, of G that is not the empty graph, is the minimum cardinality of an FTD-set in G. In this paper, we present upper bounds for the fair total domination number of trees and unicyclic graphs, and characterize trees and unicyclic graphs achieving equality for the upper bounds.

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1 Introduction

For notation and graph theory terminology not given here, we follow [13]. Specifically, let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from the context, we simply write $N(v)$ rather than $N_G(v)$. The *degree* of a vertex v, is $\deg(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. We denote the set of leaves and support vertices of a graph G by $L(G)$ and $S(G)$, respectively. A *strong support vertex* is a support vertex adjacent to at least two leaves, and a *weak support vertex* is a support vertex adjacent to precisely one leaf. A *double star* is a tree with precisely two vertices that are not leaves. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. The 2-*corona* $2 - cor(G)$ of a graph G is a graph obtained by joining any vertex of G to a leaf of a path P_2. The *distance* $d(u, v)$ between two vertices u and v in a graph G is the minimum number of edges of a path from u to v. The *diameter* $diam(G)$ of G, is $\max_{u,v \in V(G)} d(u, v)$. A path of length $diam(G)$ is called a *diametrical* path. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v, $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S in a graph with no isolated vertex is a *total dominating set* of G if every vertex in S is adjacent to a vertex in S. A subset $S \subseteq V(G)$ is a *double dominating set* of G, if every vertex in $V(G) - S$ has at least two neighbors in S and every vertex of S has a neighbor in S. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. The concept of double domination originally defined by Harary and Haynes [10] and further studied in, for example, [4, 11].

Caro et al. [1] studied the concept of fair domination in graphs. For $k \geq 1$, a *k-fair dominating set*, abbreviated kFD-set, in G is a dominating set S such that $|N(v) \cap D| = k$ for every vertex $v \in V - D$. The *k-fair domination number* of G, denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality $fd_k(G)$ is called a $fd_k(G)$-set. A *fair dominating set*, abbreviated FD-set, in G is a kFD-set for some integer $k \geq 1$. The *fair domination number*, denoted by $fd(G)$, of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G. An FD-set of G of cardinality $fd(G)$ is called a $fd(G)$-set. A *perfect dominating set* in a graph G is a dominating set S such that every vertex in $V(G) - S$ is adjacent to exactly one vertex in S. Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne et al. in [5], and Fellows et al. [8] with a different terminology which they called semiperfect domination. This concept was further studied in, for example, [2, 3, 6, 7, 12].

Maravilla et al. [15] introduced the concept of fair total domination in graphs. For an integer $k \geq 1$ and a graph G with no isolated vertex, a *k-fair total dominating set*, abbreviated kFTD-set, is a total dominating set $S \subseteq V(G)$ such that $|N(u) \cap S| = k$ for every $u \in V(G) - S$. The k-fair total domination number of G, denoted by $ftd_k(G)$, is the minimum cardinality of a kFTD-set. A kFTD-set of G of cardinality $ftd_k(G)$ is called a $ftd_k(G)$-set. A *fair total dominating set*, abbreviated FTD-set, in G is a kFTD-set for some integer $k \geq 1$. Thus, a fair total dominating set S of a graph G is a total dominating set S of G such that for every two distinct vertices u and v of $V(G) - S$, $|N(u) \cap S| = |N(v) \cap S|$; that is, S is both a fair dominating set and a total dominating set of G. The *fair total domination number* of G, denoted by $ftd(G)$, is the minimum cardinality of an FTD-set. A fair total dominating set of cardinality $ftd(G)$ is called a minimum fair total dominating set or a ftd-set of G.

In this paper, we present upper bounds for the fair total domination number of trees and unicyclic graphs, and characterize trees and unicyclic graphs achieving equality for the upper bounds. The following observation is easily verified.

Observation 1.1 *Any support vertex in a graph G with no isolated vertex belongs to every kFTD-set for each integer k.*

2 Trees

We begin with the following straightforward observation.

Observation 2.1 *If a tree T of order $n \geq 4$ is the 2-corona of a tree T', then $ftd_1(T) = 2n/3$. Furthermore, both $V(T) - L(T)$ and $S(T) \cup L(T)$ are $ftd_1(T)$-sets.*

Theorem 2.2 *If T is a tree of order $n \geq 3$, then $ftd_1(T) \leq 2n/3$, with equality if and only if T is the 2-corona of a tree.*

Proof. Let T be a tree of order $n \geq 3$. We use induction on n to show that $ftd_1(T) \leq 2n/3$. For the base step, if $3 \leq n \leq 6$, then it can be easily checked that $ftd_1(T) \leq 2n/3$. Assume that the result holds for all trees T' of order $n' < n$. Now consider the tree T of order $n \geq 7$. We root T at a leaf v_0 of a diametrical path $v_0 v_1 \dots v_d$, where $d = diam(T)$ such that $\deg(v_{d-1})$ is as large as possible. If $d = 2$, then T is a star, and clearly $ftd_1(T) = 2 < 2n/3$, since $n \geq 7$. If $d = 3$, then T is a double-star, and it can be seen that $ftd_1(T) = 2 < 2n/3$. Thus assume for the next that $d \geq 4$.

Assume that $\deg_T(v_{d-1}) \geq 3$. Let $T' = T - \{v_d\}$. By the induction hypothesis, $ftd_1(T') \leq 2n'/3 = 2(n-1)/3$. Let S' be a $ftd_1(T')$-set. By Observation 1.1, $v_{d-1} \in S'$. Then S' is a 1FTD-set in T, and so $ftd_1(T) < 2n/3$. Next assume that $\deg_T(v_{d-1}) = 2$. Assume that $\deg_T(v_{d-2}) = 2$. Let $T' = T - T_{v_{d-2}}$. By the induction hypothesis, $ftd_1(T') \leq 2n'/3 = 2(n-3)/3 = 2n/3 - 2$. Let S' be a $ftd_1(T')$-set. If $v_{d-3} \in S'$, then $\{v_{d-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in T and so $ftd_1(T) \leq 2n/3$ and if $v_{d-3} \notin S'$, then $\{v_{d-1}, v_d\} \cup S'$ is a 1FTD-set in T and so $ftd_1(T) \leq 2n/3$.

Thus assume that $\deg_T(v_{d-2}) \geq 3$. Assume that v_{d-2} is a support vertex. Let $T' = T - T_{v_{d-1}}$. By the induction hypothesis, $ftd_1(T') \leq 2n'/3 = 2(n-2)/3$. Let S' be a $ftd_1(T')$-set. By Observation 1.1, $v_{d-2} \in S'$. Then $\{v_{d-1}\} \cup S'$ is a 1FTD-set in T and so $ftd_1(T) \leq 2n/3$. Thus assume that v_{d-2} is not a support vertex of T. Let $x \neq v_{d-1}$ be a child of v_{d-2}. Clearly, x is a support vertex of T. By the choice of the path $v_0v_1 \ldots v_d$, $\deg_T(x) = 2$. Let y be the leaf adjacent to x, and $T' = T - \{v_d, v_{d-1}, y\}$. By the induction hypothesis, $ftd_1(T') \leq 2n'/3 = 2(n-3)/3$. Let S' be a $ftd_1(T')$-set. By Observation 1.1, $v_{d-2} \in S'$, since v_{d-2} is a support vertex of T'. Then $\{v_{d-1}, x\} \cup S'$ is a 1FTD-set in T, and thus $ftd_1(T) \leq 2n/3$.

We next prove the equality part. We prove by induction on the order n of a tree T with $ftd_1(T) = 2n/3$ to show that T is a 2-corona of a tree. For the base step, if $n = 3$, then $T = P_3$ which is 2-corona of the tree K_1. Assume that the result holds for all trees T' of order $n' < n$ with $ftd_1(T') = 2n'/3$. Now consider the tree T of order $n \geq 6$ with $ftd_1(T) = 2n/3$. Clearly, $2n \equiv 0 \pmod 3$. Suppose that T has a strong support vertex v, and v_1, v_2 are the leaves adjacent to v. Let $T_0 = T - v_1$. By the first part of the proof, $ftd_1(T_0) \leq 2n(T_0)/3 = 2(n-1)/3$. Let S be a $ftd_1(T_0)$-set. By Observation 1.1, $v \in S$ and thus S is a 1FTD-set in T, a contradiction. We deduce that every support vertex of T is adjacent to precisely one leaf.

We root T at a leaf v_0 of a diametrical path $v_0v_1 \ldots v_d$, where $d = diam(T)$ such that $\deg(v_{d-1})$ is as large as possible. As it was seen in the first part of the proof, if $2 \leq d \leq 3$, then T is a star or a double-star, and $ftd_1(T) < 2n/3$, a contradiction. Thus, $d \geq 4$. Observe that $\deg_T(v_{d-1}) = 2$, since T has no strong support vertex. We show that $\deg_T(v_{d-2}) = 2$. Suppose to the contrary, that $\deg_T(v_{d-2}) \geq 3$. Suppose that v_{d-2} is a support vertex. Let $T' = T - T_{v_{d-1}}$. By the first part of the proof, $ftd_1(T') \leq 2n'/3 = 2(n-2)/3$. Let S' be a $ftd_1(T')$-set. By Observation 1.1, $v_{d-2} \in S'$. Then $\{v_{d-1}\} \cup S'$ is a 1FTD-set in T and so $ftd_1(T) \leq 2(n-2)/3 + 1 = (2n-1)/3$, a contradiction. Thus assume that v_{d-2} is not a support vertex of T. Let $x \neq v_{d-1}$ be a child of v_{d-2}, and y be the leaf adjacent to x. Since y plays the same role as v_d, we find that $\deg_G(x) = 2$. Let $T' = T - \{v_d, v_{d-1}, y\}$. By the first part of the proof, $ftd_1(T') \leq 2n'/3 = 2(n-3)/3$. Suppose that $ftd_1(T') = 2n'/3 = 2n/3 - 2$. By the induction hypothesis, T' is the 2-corona of a tree. By Observation 2.1, $S(T') \cup L(T')$ is a $ftd_1(T')$-set. Then $S(T') \cup L(T') \cup \{v_{d-1}\}$ is a 1FTD-set in T, since $x, v_{d-2} \in S(T') \cup L(T')$. Then $ftd_1(T) \leq 2n/3 - 1$, a contradiction. Thus $ftd_1(T') < 2n'/3 = 2n/3 - 2$. Let S' be a $ftd_1(T')$-set. By Observation 1.1, $v_{d-2} \in S'$, since v_{d-2} is a support vertex of T'. Then $\{v_{d-1}, x\} \cup S'$ is a 1FTD-set in T and so $ftd_1(T) < 2n/3$, a contradiction. We conclude that $\deg_T(v_{d-2}) = 2$.

Let $T' = T - T_{v_{d-2}}$. By the first part of the proof, $ftd_1(T') \leq 2n'/3 = 2n/3 - 2$. Assume that $ftd_1(T') < 2n'/3 = 2n/3 - 2$. Let S' be a $ftd_1(T')$-set. If $v_{d-3} \in S'$, then $\{v_{d-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in T and so $ftd_1(T) < 2n/3$, a contradiction. Thus we assume that $v_{d-3} \notin S'$. Then $\{v_{d-1}, v_d\} \cup S'$ is a 1FTD-set in T and so $ftd_1(T) < 2n/3$, a contradiction. Thus $ftd_1(T') = 2n'/3 = 2n/3 - 2$. By the induction hypothesis T' is the 2-corona of a tree. Assume that $\deg_T(v_{d-3}) = 2$. Then v_{d-4} is a support vertex of T'. By Observation 2.1, $V(T') - L(T')$ is a $ftd_1(T')$-set. Thus $((V(T') - L(T')) - \{v_{d-4}\}) \cup \{v_{d-2}, v_{d-1}\}$ is a 1FTD-set in T and so $ftd_1(T) \leq$

$2n/3-1$, a contradiction. Thus $\deg_T(v_{d-3}) \geq 3$. Assume that v_{d-3} is a support vertex of T. Then v_{d-3} is a support vertex of T'. Let z be the leaf adjacent to v_{d-3}. By Observation 2.1, $S(T') \cup L(T')$ is a $ftd_1(T')$-set. Thus $S(T') \cup L(T') - \{z\} \cup \{v_{d-2}, v_{d-1}\}$ is a 1FTD-set in T and so $ftd_1(T) \leq 2n/3 - 1$, a contradiction. Thus v_{d-3} is not a support vertices of T'. Now, it is easy to check that T is the 2-corona of a tree, since T' is the 2-corona of a tree. The converse follows by Observation 2.1. ■

We next present a constructive characterization of trees T with $ftd_1(T) = (2n-1)/3$. For this purpose, we define a family of trees as follows: Let $\mathcal{T}$ be the class of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k = T$, of trees with $T_1 = P_5$, and if $k \geq 2$, then T_{i+1} is obtained from T_i by applying one of the following Operations $\mathcal{O}_1$ or $\mathcal{O}_2$, for $i = 1, 2, \ldots, k-1$.

Operation $\mathcal{O}_1$. Let v be a vertex of a tree T_i with $\deg(v) \geq 2$. Then T_{i+1} is obtained from T_i by adding a path P_3 and joining v to a leaf of P_3.

Operation $\mathcal{O}_2$. Let v be a support vertex of a tree T_i and let u be a leaf adjacent to v. Then T_{i+1} is obtained from T_i by adding a vertex u' and a path P_2, joining u to u' and joining v to a leaf of P_2.

The following is straightforward.

Observation 2.3 *Let $T \in \mathcal{T}$ be a tree of order n. Then*
(1) $2n \equiv 1$ (mod 3).
(2) $|L(T)| = (n+1)/3$.
(3) T has no strong support vertex. Furthermore, no pair of support vertices is adjacent.

Lemma 2.4 *If $T \in \mathcal{T}$, then every 1FTD-set in T contains every vertex of T of degree at least 2.*

Proof. Let $T \in \mathcal{T}$. Then T is obtained from a sequence $T_1, T_2, \ldots, T_k = T$, of trees with $T_1 = P_5$ and if $k \geq 2$, then T_{i+1} is obtained from T_i by one of the operations $\mathcal{O}_1$ or $\mathcal{O}_2$, for $i = 1, 2, \ldots, k-1$. We prove the result by an induction on k. For the base step of the induction, let $k = 1$, and so $T = P_5$. Clearly, every vertex of P_5 of degree at least two is contained in every 1FTD-set of T. Assume that the result holds for any k' with $2 \leq k' < k$. Now let $T = T_k$. Clearly, T is obtained from T_{k-1} by applying one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$. Let S be a 1FTD-set for T.

Assume that T is obtained from T_{k-1} by applying the Operation $\mathcal{O}_1$. Let $x_1x_2x_3$ be a path and x_1 be joined to $y \in V(T_{k-1})$, where $\deg_{T_{k-1}}(y) \geq 2$. By Observation 1.1, $x_2 \in S$. Observe that $\{x_1, x_3\} \cap S \neq \emptyset$. If $x_1 \notin S$, then $x_3 \in S$ and $y \notin S$. Then $S - \{x_2, x_3\}$ is a 1FTD-set for T_{k-1} that does not contain y, a contradiction to the induction hypothesis. Thus assume that $x_1 \in S$. Assume that $y \notin S$. Then $N_{T_{k-1}}(y) \cap S = \emptyset$. Clearly, y is not a support vertex. Let $y_1 \in N_{T_{k-1}}(y)$, and T' be the component of $T_{k-1} - y$ containing y_1. Then $(S - \{x_1, x_2, x_3\}) \cup V(T')$ is a 1FTD-set for T_{k-1} that does not contain y, a contradiction to the induction hypothesis. Thus $y \in S$. Clearly, $S - \{x_1, x_2, x_3\}$ is a 1FTD-set for T_{k-1}. By the induction hypothesis,

$S - \{x_1, x_2, x_3\}$ contains every vertex of T_{k-1} of degree at least two. Consequently, S contains every vertex of T_k of degree at least two.

Next assume that T_k is obtained from T_{k-1} by applying the Operation $\mathcal{O}_2$. Let u be a support vertex of the tree T_{k-1} and let v be the leaf of T_{k-1} adjacent to u. Let $P_2 : x_1x_2$ be a path and x_3 be a vertex that x_1 is joined to u, and x_3 is joined to v according to the Operation $\mathcal{O}_2$. By Observation 1.1, $x_1, v \in S$. Then $u \in S$, and so $S - \{x_1, v\}$ is a 1FTD-set for T_{k-1}. By the induction hypothesis, $S - \{x_1, v\}$ contains all vertices of T_{k-1} of degree at least two. Consequently, S contains every vertex of T_k of degree at least two. ■

Corollary 2.5 *If $T \in \mathcal{T}$ is a tree of order n, then*
(1) $V(T) - L(T)$ is the unique $ftd_1(T)$-set.
(2) $ftd_1(T) = (2n-1)/3$.

Theorem 2.6 *If T is a tree of order $n \geq 3$, then $ftd_1(T) = (2n-1)/3$ if and only if $T \in \mathcal{T}$.*

Proof. Let T be a tree of order $n \geq 3$ with $ftd_1(T) = (2n-1)/3$. Clearly, $2n \equiv 1 \pmod 3$. The proof is by induction on n. From $2n \equiv 1 \pmod 3$, we obtain that $n \geq 5$. For the base step of the induction, if $n = 5$, then it is easily seen that $T = P_5 \in \mathcal{T}$. Assume that the result holds for all trees T' of order $n' < n$ with $ftd_1(T') = (2n'-1)/3$. Now consider the tree T of order $n \geq 6$. We root T at a leaf v_0 of a diametrical path $v_0v_1 \dots v_d$, where $d = diam(T)$ such that $\deg_T(v_{d-1})$ is as large as possible. If $d = 2$ then T is a star, a contradiction, since $ftd_1(T) = 2 \neq (2n-1)/3$. If $d = 3$, then T is a double star, a contradiction, since $ftd_1(T) = 2 \neq (2n-1)/3$. Thus $d \geq 4$. Suppose that T has a strong support vertex x, and assume that x_1 and x_2 are two leaves adjacent to x. Let $T_0 = T - x_1$. By Theorem 2.2, $ftd_1(T_0) \leq 2n(T_0)/3 = 2(n-1)/3$. Let S be a $ftd_1(T_0)$-set. By Observation 1.1, $x \in S$ and thus S is a 1FTD-set in T, as well. This contradicts the fact that $ftd_1(T) = (2n-1)/3$. Thus we assume next that T has no strong support vertex. In particular, $\deg_T(v_{d-1}) = 2$. We consider the following cases.

Case 1. $\deg_T(v_{d-2}) \geq 3$. We show that v_{d-2} is not a support vertex of T. Suppose that v_{d-2} is a support vertex. Let x be the leaf adjacent to v_{d-2}, and $T' = T - T_{v_{d-1}}$. By Theorem 2.2, $ftd_1(T') \leq 2n'/3 = 2(n-2)/3$. Suppose that $ftd_1(T') = 2n'/3$. By Theorem 2.2, T is a 2-corona of a tree. Thus by Observation 2.1, $S(T') \cup L(T')$ is a $ftd_1(T')$-set. Then $S(T') \cup L(T') - \{x\} \cup \{v_{d-1}\}$ is a $ftd_1(T)$-set of cardinality at most $2(n-2)/3$, a contradiction. We deduce that $ftd_1(T') < 2n'/3$. Let S' be a $ftd_1(T')$-set. By Observation 1.1, $v_{d-2} \in S'$. Thus $\{v_{d-1}\} \cup S'$ is a 1FTD-set in T, and thus $ftd_1(T) < 2n'/3 + 1 = (2n-1)/3$, a contradiction. Thus assume that v_{d-2} is not a support vertex of T. Let $x \neq v_{d-1}$ be a child of v_{d-2}, and y be a child of x. Since y plays the same role as v_d, we find that $\deg_T(x) = 2$. Let $T' = T - \{v_d, v_{d-1}, y\}$. By Theorem 2.2, $ftd_1(T') \leq 2n'/3 = 2n/3 - 2$. Thus $ftd_1(T') \leq (2n'-1)/3 = (2n-1)/3 - 2$, since $2n \equiv 1 \pmod 3$. Suppose that $ftd_1(T') < (2n'-1)/3 = (2n-1)/3 - 2$. Let S' be a $ftd_1(T')$-set. By Observation

1.1, $v_{d-2} \in S'$, since v_{d-2} is a support vertex of T'. Then $\{v_{d-1}, x\} \cup S'$ is a 1FTD-set in T, and so $ftd_1(T) < (2n-1)/3$, a contradiction. Thus $ftd_1(T') = (2n'-1)/3$. By the induction hypothesis, $T' \in \mathcal{T}$. Now T is obtained from T' by Operation $\mathcal{O}_2$, and so $T \in \mathcal{T}$.

Case 2. $\deg_T(v_{d-2}) = 2$. We show that $\deg_T(v_{d-3}) \geq 3$. Suppose that $\deg_T(v_{d-3}) = 2$. Let $T' = T - T_{v_{d-3}}$. By Theorem 2.2, $ftd_1(T') \leq 2n'/3 = 2(n-4)/3 = (2n-2)/3 - 2$. Let S' be a $ftd_1(T')$-set. If $v_{d-4} \in S'$, then $\{v_{d-1}, v_d\} \cup S'$ is a 1FTD-set in T of cardinality at most $2(n-2)/3$, a contradiction. Thus $v_{d-4} \notin S'$. Then $\{v_{d-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in T of cardinality at most $2(n-2)/3$, a contradiction. We deduce that $\deg_T(v_{d-3}) \geq 3$. Let $T' = T - T_{v_{d-2}}$. By Theorem 2.2, $ftd_1(T') \leq 2n'/3 = 2(n-3)/3 = 2n/3 - 2$. Then $ftd_1(T') \leq (2n'-1)/3 = (2n-1)/3 - 2$, since $2n \equiv 1 \pmod 3$. Suppose that $ftd_1(T') < (2n'-1)/3 = (2n-1)/3 - 2$. Let S' be a $ftd_1(T')$-set. If $v_{d-3} \in S'$, then $\{v_{d-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in T, and so $ftd_1(T) < (2n-1)/3$, a contradiction. Thus $v_{d-3} \notin S'$. Then $\{v_{d-1}, v_d\} \cup S'$ is a 1FTD-set in T, and so $ftd_1(T) < (2n-1)/3$, a contradiction. We deduce that $ftd_1(T') = (2n'-1)/3$. By the induction hypothesis, $T' \in \mathcal{T}$. Now T is obtained from T' by Operation $\mathcal{O}_1$, and so $T \in \mathcal{T}$.

The converse follows by Corollary 2.5. ∎

Lemma 2.7 (Chellali [4]) *If T is a nontrivial tree of order n, with ℓ leaves and s support vertices, then $\gamma_{\times 2}(T) \geq (2n + \ell - s + 2)/3$.*

Proposition 2.8 *In a tree T, every ftd_1-set is a ftd-set.*

Proof. If S is a kFTD-set for T for some $k \geq 2$, then $|N(x) \cap S| = k \geq 2$ for all $x \in V(T) - S$. Thus every vertex of S has a neighbor in S, implying that S is a double dominating set, and thus $|S| \geq \gamma_{\times 2}(T)$. By Lemma 2.7, we have $\gamma_{\times 2}(T) \geq (2n+2)/3$. By Theorem 2.2, $ftd_1(T) \leq 2n/3$. Thus, $ftd(T) < ftd_k(T)$ for each $k \geq 2$. ∎

We are now ready to state the main theorems of this section.

Theorem 2.9 *If T is a tree of order $n \geq 3$, then $ftd(T) \leq 2n/3$, with equality if and only if T is the 2-corona of a tree.*

Theorem 2.10 *If T is a tree of order $n \geq 3$, then $ftd(T) = (2n-1)/3$ if and only if $T \in \mathcal{T}$.*

We propose characterization of trees T of order $n \geq 3$ with $ftd(T) = (2n-2)/3$ as a problem.

3 Unicyclic graphs

The following is easily verified.

Observation 3.1 *For $n \geq 3$, $ftd(C_n) = \gamma_t(C_n)$ unless $n \equiv 3 \pmod 4$ and $n \geq 5$ in which case $ftd(C_n) = \gamma_t(C_n) + 1$.*

For a unicyclic graph G with the cycle C, any vertex of degree 2 on C is called the *special vertex* of G. We prove that $ftd_1(G) \leq (2n+1)/3$ for any unicyclic graphs G of order n, and then present a constructive characterization of unicyclic graphs G of order n with $ftd_1(G) = (2n + 1)/3$. For this purpose, we define a family of unicyclic graphs as follows. Let $\mathcal{C}_1$ be the class of all graphs G that can be obtained from the 2-corona of a cycle C_k ($k \geq 3$) by removing precisely one support vertex v and the leaf adjacent to v. Let $\mathcal{G}$ be the class of all unicyclic graphs G that can be obtained from a sequence $G_1, G_2, \ldots, G_k = G$, of unicyclic graphs, where $G_1 \in \mathcal{C}_1$, and if $k \geq 2$, then G_{i+1} is obtained from G_i by one of the following Operations $\mathcal{O}_1$ or $\mathcal{O}_2$, for $i = 1, 2, \ldots, k - 1$.

Operation $\mathcal{O}_1$. Let v be a vertex of a unicyclic graph G_i with $\deg_{G_i}(v) \geq 2$ such that v is not a special vertex. Then G_{i+1} is obtained from G_i by adding a path P_3 and joining v to a leaf of P_3.

Operation $\mathcal{O}_2$. Let v be a support vertex of a unicyclyc graph G_i and let u be a leaf adjacent to v. Then G_{i+1} is obtained from G_i by adding a vertex u' and a path P_2, joining u to u' and joining v to a leaf of P_2.

The following observation is straightforward.

Observation 3.2 *(1) Each graph $G \in \mathcal{G}$ has precisely one special vertex.*
(2) If $G \in \mathcal{G}$ is a unicyclic graph of order n, then $|L(G)| = (n - 1)/3$.
(3) If C is the cycle of a graph $G \in \mathcal{G}$, then no vertex of C is a support vertex of G.

Lemma 3.3 *If $G \in \mathcal{G}$, then every 1FTD-set in G contains every vertex of G of degree at least 2.*

Proof. Let $G \in \mathcal{G}$. Then G is obtained from a sequence $G_1, G_2, \ldots, G_k = G$, of unicyclic graphs, where $G_1 \in \mathcal{C}_1$, and if $k \geq 2$, then G_{i+1} is obtained from G_i by one of the operations $\mathcal{O}_1$ or $\mathcal{O}_2$, for $i = 1, 2, \ldots, k-1$. Let C be the cycle of G. We prove the result by induction on k. For the base step of the induction, let $k = 1$. Clearly, $V(G_1) - L(G_1)$ is contained in every 1FTD-set of G. Assume that the result holds for each k' with $2 \leq k' < k$. Now let $G = G_k$. Clearly, G is obtained from G_{k-1} by applying one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$. Let S be a 1FTD-set for G.

Assume that G is obtained from G_{k-1} by applying Operation $\mathcal{O}_1$. Let $x_1x_2x_3$ be a path and x_1 be joined to $y \in V(G_{k-1})$, where $\deg_{G_{k-1}}(y) \geq 2$ and y is not a special vertex of G_{k-1}. By Observation 1.1, $x_2 \in S$. Observe that $\{x_3, x_1\} \cap S \neq \emptyset$. If $x_1 \notin S$ then $x_3 \in S$ and $y \notin S$. Then $S - \{x_2, x_3\}$ is a 1FTD-set for G_{k-1} that does not contain y, a contradiction to the induction hypothesis. Thus assume that $x_1 \in S$. Assume that $y \notin S$. Then $N_{G_{k-1}}(y) \cap S = \emptyset$. Clearly, y is not a support vertex. Note that $G_{k-1} - y$ has a component G' with $V(G') \cap V(C) = \emptyset$. Let $y_1 \in N_{G_{k-1}}(y) \cap V(G')$. Then $(S - \{x_1, x_2, x_3\}) \cup V(G')$ is a 1FTD-set for G_{k-1} that does not contain y, a contradiction to the induction hypothesis. Thus $y \in S$. Clearly, $S - \{x_1, x_2, x_3\}$ is a 1FTD-set for G_{k-1}. By the induction hypothesis $S - \{x_1, x_2, x_3\}$ contains every vertex of G_{k-1} of degree at least two. Consequently, S contains every vertex of G_k of degree at least two.

Next assume that G is obtained from G_{k-1} by applying Operation $\mathcal{O}_2$. Let u be a support vertex of a unicyclic graph G_{k-1} and let v be the leaf adjacent to u. Let x_1x_2 be a path and x_1 be joined to u, and let x_3 be a vertex that is joined to v according to the Operation $\mathcal{O}_2$. By Observation 1.1, $x_1, v \in S$. Thus $u \in S$, and so $S - \{x_1\}$ is a 1FTD-set for G_{k-1}. By the induction hypothesis, S contains all vertices of G_{k-1} of degree at least two. Consequently, S contains every vertex of G_k of degree at least two. ■

As a consequence of Observation 3.2 (2) and Lemma 3.3, we obtain the following.

Corollary 3.4 *If $G \in \mathcal{G}$ is a unicyclic graph of order n, then $V(G) - L(G)$ is the unique $ftd_1(G)$-set.*

We recall the following result of [14].

Theorem 3.5 ([14]) *For $n \geq 3$, $\gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

Theorem 3.6 *If G is a unicyclic graph of order $n \geq 4$, then $ftd_1(G) \leq (2n+1)/3$, with equality if and only if $G = C_7$ or $G \in \mathcal{G}$.*

Proof. Let G be a unicyclic graph of order $n \geq 4$. We first use induction on n to show that $ftd_1(G) \leq (2n+1)/3$. For the base step of the induction note that if $n = 4$, then $G = C_4$ or G is obtained from C_3 by adding a leaf to a vertex of C_3, and we can see that $ftd_1(G) = 2 \leq (2n+1)/3$. Assume that the result holds for all unicyclic graphs G' of order $n' < n$. Now consider the unicyclic graph G of order $n \geq 5$. Let $C = u_1, u_2, .., u_k, u_1$ be the cycle of G. If $G = C$, then by Observation 3.1, $ftd_1(G) \leq \gamma_t(G) + 1$, and so by Theorem 3.5 $ftd_1(G) \leq (2n(G)+1)/3$ if $n \neq 5, 6$. However, for $n = 5, 6$, we have $ftd_1(G) \leq \gamma_t(G) \leq (2n(G)+1)/3$. Thus assume that $G \neq C$. Let v_d be a vertex of G such that $d(v_d, C)$ is as large as possible and $\deg(v_{d-1})$ is as large as possible, where v_{d-1} is the neighbor of v_d on the shortest path from v_d to C. Let $v_0v_1 \dots v_d$ be the shortest path from v_d to C, where v_0 is the common vertex of this path with C.

Assume that $d \geq 3$. Assume that $\deg_G(v_{d-1}) \geq 3$. Let $G' = G - \{v_d\}$. By the induction hypothesis, $ftd_1(G') \leq (2n'+1)/3 = (2n-1)/3$. Let S' be a $ftd_1(G')$-set. By Observation 1.1, $v_{d-1} \in S'$. Clearly, S' is a 1FTD-set in G, and so $ftd_1(G) < (2n+1)/3$. Thus assume that $\deg_G(v_{d-1}) = 2$. Assume that $\deg_G(v_{d-2}) = 2$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}\}$. By the induction hypothesis, $ftd_1(G') \leq (2n'+1)/3 = (2n-5)/3$. Let S' be a $ftd_1(G')$-set. If $v_{d-3} \in S'$, then $\{v_{d-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in G, and so $ftd_1(G) \leq (2n+1)/3$, and if $v_{d-3} \notin S'$, then $\{v_{v-1}, v_d\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2n+1)/3$. Thus assume that $\deg_G(v_{d-2}) \geq 3$. Assume that v_{d-2} is a support vertex. Let $G' = G - \{v_{d-1}, v_d\}$. By the induction hypothesis, $ftd_1(G') \leq (2n'+1)/3 = (2n-3)/3$. Let S' be a $ftd_1(G')$-set. By Observation 1.1,

$v_{d-2} \in S'$. Then $\{v_{d-1}\} \cup S'$ is a 1FTD-set in G and so $ftd_1(T) \leq (2n+1)/3$. Thus assume that v_{d-2} is not a support vertex of G. Let $x \neq v_{d-1}, v_{d-3}$ be a support vertex of G such that $x \in N(v_{d-2})$. By the choice of the path $v_0v_1 \ldots v_d$, (the part "$\deg(v_{d-1})$ is as large as possible") $\deg_G(x) = 2$. Let y be the leaf adjacent to x, and $G' = G - \{v_d, v_{d-1}, y\}$. By the induction hypothesis $ftd_1(G') \leq (2n'+1)/3 = (2n-5)/3$. Let S' be a $ftd_1(G')$-set. By Observation 1.1, $v_{d-2} \in S'$, since v_{d-2} is a support vertex of G'. Then $\{v_{d-1}, x\} \cup S'$ is a 1FTD-set in G, and so $ftd_1(G) \leq (2n+1)/3$.

Next assume that $d = 2$. Assume that $\deg(u_i) \geq 3$ for every i with $1 \leq i \leq k$. Let $D = S(G) - V(C)$. Clearly, $v_{d-1} \in D$. Then $n \geq 2k + |D|$. Clearly, $V(C) \cup D$ is a 1FTD-set in G of cardinality $k + |D| \leq (2n+1)/3$. Thus assume that $\deg_G(u_j) = 2$ for some $j \in \{1, 2, \ldots, k\}$. Assume that u_j and u_{j+1} are two consecutive vertices on C such that $\deg_G(u_j) = 2$ and $\deg_G(u_{j+1}) \geq 3$. Then $T = G - u_{j-1}u_j$ is a tree. Let S' be a $ftd_1(T)$-set. By Theorem 2.2, $ftd_1(T') \leq 2n/3$. Clearly, T is not a 2-corona of a tree, and so by Theorem 2.2, $ftd_1(T) < 2n/3$. Observe that u_{j+1} is either a strong support vertex of T or is adjacent to at least one support vertex of T. Thus by Observation 2.3, $T \notin \mathcal{T}$. Then by Theorem 2.6, $ftd_1(T) < (2n-1)/3$ and so $ftd_1(T) \leq (2n-2)/3$. By Observation 1.1, $u_{j+1} \in S'$. If $\mid S' \cap \{u_j, u_{j-1}\} \mid \in \{0, 2\}$, then S' is a 1FTD-set for G of cardinality at most $2n/3$. Assume that $\mid S' \cap \{u_j, u_{j-1}\} \mid = 1$. If $u_{j-1} \in S'$, then $S' \cup \{u_j\}$ is a 1FTD-set for G of cardinality at most $(2n+1)/3$, and so $ftd_1(G) \leq (2n+1)/3$. Thus assume that $u_j \in S'$. Then u_{j+1} is not adjacent to a support vertex of T and so u_{j+1} is a strong support vertex of T. Let $z \neq u_j$ be a leaf adjacent to u_{j+1}. Then $S - \{u_j\} \cup \{z\}$ is a 1FTD-set for G of cardinality at most $(2n+1)/3$, and so $ftd_1(G) \leq (2n+1)/3$.

Now assume that $d = 1$. If $\deg_G(u_i) \geq 3$ for each i with $1 \leq i \leq k$, then $V(C)$ is a 1FTD-set in G of cardinality at most $n/2$, and so $ftd_1(G) \leq \frac{n}{2} < (2n+1)/3$. Assume $\deg(u_i) = 2$ for some $i \in \{1, 2, \ldots, k\}$. Let u_j and u_{j+1} be two consecutive vertices on C such that $\deg_G(u_j) = 2$ and $\deg_G(u_{j+1}) \geq 3$. Then $T = G - u_{j-1}u_j$ is a tree. Let S' be a $ftd_1(T)$-set. By Theorem 2.2, $ftd_1(T) \leq 2n/3$. Clearly, T is not a 2-corona of a tree, since u_{j+1} is a strong support vertex of T. By Theorem 2.2, $ftd_1(T) < 2n/3$. Then by Observation 2.3, $T \notin \mathcal{T}$. By Theorem 2.6, $ftd_1(T) < (2n-1)/3$ and so $ftd_1(T) \leq (2n-2)/3$. By Observation 1.1, $u_{j+1} \in S'$. If $\mid S' \cap \{u_j, u_{j-1}\} \mid \in \{0, 2\}$, then S' is a 1FTD-set for G of cardinality at most $(2n-2)/3$, and so $ftd_1(G) \leq (2n+1)/3$. Thus assume that $\mid S' \cap \{u_j, u_{j-1}\} \mid = 1$. Assume that $u_{j-1} \in S'$. Then $S' \cup \{u_j\}$ is a 1FTD-set for G of cardinality at most $(2n+1)/3$, and so $ftd_1(G) \leq (2n+1)/3$. Next assume that $u_j \in S'$. Let $z \neq u_j$ be a leaf adjacent to u_{j+1}. Then $S' - \{u_j\} \cup \{z\}$ is a 1FTD-set for G of cardinality at most $(2n+1)/3$, and so $ftd_1(G) \leq (2n+1)/3$.

We next prove the equality part. We prove by induction on the order n of a unicyclic graph $G \neq C_7$ with $ftd_1(G) = (2n+1)/3$ to show that $G \in \mathcal{G}$. If $4 \leq n \leq 7$, then by a directly checking of all possible unicyclic graphs, we find that $G \in \mathcal{G}$. Assume that the result holds for all unicyclic graph $G' \neq C_7$ of order $n' < n$ with $ftd_1(G') = (2n'+1)/3$. Now consider a unicyclic graph $G \neq C_7$ of order n with $ftd_1(G) = (2n+1)/3$. Clearly, $2n+1 \equiv 0 \pmod 3$. Suppose that G has a strong support vertex v, and assume that v_1 and v_2 are two leaves adjacent to v. Let $G' = G - v_1$. By the first part of the proof, $ftd_1(G') \leq (2n(G')+1)/3 = (2n-1)/3$.

Let S be a $ftd_1(G')$-set. By Observation 1.1, $v \in S$ and thus S is a $1FTD$-set in G, as well. This contradicts the fact that $ftd_1(G) = (2n+1)/3$. Thus we assume for the next that G has no strong support vertex. Let $C = u_1, u_2, .., u_k, u_1$ be the cycle of G. By Observation 3.1, $G \neq C$. Let v_d be a vertex of G such that $d(v_d, C)$ is as large as possible, $\deg(v_{d-1})$ is as large as possible, and $\deg_G(v_0)$ is as large as possible, where $v_0 v_1 \dots v_d$ is the shortest path from v_d to C, where $v_0 \in C$ is the common vertex of this path with C.

Suppose that $d = 1$. Assume that $\deg_G(u_i) \geq 3$ for each i with $1 \leq i \leq k$. Then $V(C)$ is a 1FTD-set G of cardinality at most $n/2$, a contradiction. Thus $\deg_G(u_j) = 2$ for some j with $1 \leq j \leq k$. Let $D_0 = \{u_i | \deg_G(u_i) = 2\}$ and $D_1 = \{u_i | u_i \text{ is a support vertex of } V(C)\}$. We show that if $\deg_G(u_j) = 2$, then $\deg_G(u_{j+1}) = 3$ and $\deg_G(u_{j-1}) = 3$. Suppose that $\deg_G(u_j) = \deg_G(u_{j+1}) = 2$ for some $1 \leq j \leq k$. Among such vertices choose u_j and u_{j+1} such that $\deg_G(u_{j-1}) = 3$. Let $T = G - u_j$. By Theorem 2.2, $ftd_1(T) \leq 2n(T)/3$. Assume that $ftd_1(T) = 2n(T)/3$. By Theorem 2.2, $V(T) - L(T)$ is a $ftd_1(T)$-set (Note that $\deg_T(u_{j-1}) = 2$ and $\deg_T(u_{j+1}) = 1$). Then $V(T) - L(T)$ is a 1FTD-set in G of cardinality at most $(2n-2)/3$, a contradiction. Thus $ftd_1(T) < 2n(T)/3$. Let S be a $ftd_1(T)$-set. By Observation 1.1, $u_{j-1}, u_{j+2} \in S$. If $u_{j+1} \notin S$, then S is a 1FTD-set in G of cardinality at most $(2n-2)/3$, a contradiction. Thus $u_{j+1} \in S$. Then $S \cup \{u_j\}$ is a 1FTD-set in G, and so $ftd_1(G) < (2n+1)/3$, a contradiction. Thus if $\deg_G(u_j) = 2$ then $\deg_G(u_{j+1}) = 3$ and $\deg_G(u_{j-1}) = 3$. Thus $| D_0 | \leq | D_1 |$, and so $V(C)$ is a 1FTD-set in G of cardinality at most $2n/3$, a contradiction. Thus, assume that $d \geq 2$. Clearly, $\deg_G(v_{d-1}) = 2$, since G has no strong support vertex. We consider the following cases.

Case 1. $d \geq 4$. Assume that $\deg_G(v_{d-2}) \geq 3$. Suppose that v_{d-2} is a support vertex. Let x be the leaf adjacent to v_{d-2}, and $G' = G - \{v_{d-1}, v_d\}$. By the first part of the proof, $ftd_1(G') \leq (2n(G')+1)/3 = (2n-3)/3$. Let S' be a $ftd_1(G')$-set. By Observation 1.1, $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FTD-set in G and so $ftd_1(G) < (2n+1)/3$, a contradiction. Thus assume that v_{d-2} is not a support vertex of G. Let $x \neq v_{d-1}, v_{d-3}$ be a support vertex of G such that $x \in N(v_{d-2})$. By the choice of the path $v_0 v_1 \dots v_d$, (the part "$\deg(v_{d-1})$ is as large as possible"), we have $\deg_G(x) = 2$. Let y be the leaf adjacent to x. Let $G' = G - \{v_d, v_{d-1}, y\}$. By the first part of the proof, $ftd_1(G') \leq (2n(G')+1)/3 = (2n+1)/3 - 2$. If $ftd_1(G') < (2n(G')+1)/3 = (2n+1)/3 - 2$ and S' is a $ftd_1(G')$-set, then by Observation 1.1, $v_{d-2} \in S'$, since v_{d-2} is a support vertex of G'. Then $\{v_{d-1}, x\} \cup S'$ is a 1FTD-set in G and so $ftd_1(T) < (2n+1)/3$, a contradiction. Thus $ftd_1(G') = (2n(G')-1)/3$. By the induction hypothesis, $G' \in \mathcal{G}$. Thus G is obtained from G' by Operation $\mathcal{O}_2$ and so $G \in \mathcal{G}$.

Assume next that $\deg_G(v_{d-2}) = 2$. Suppose that $\deg_G(v_{d-3}) = 2$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}, v_{d-3}\}$. By the first part of the proof, $ftd_1(G') \leq (2n(G')+1)/3 = (2n-7)/3$. Let S' be a $ftd_1(G')$-set. If $v_{d-4} \in S'$, then $\{v_{d-1}, v_d\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2n-1)/3$, a contradiction. Thus $v_{d-4} \notin S'$, and so $\{v_{d-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2n-1)/3$, a contradiction. We deduce that $\deg_G(v_{d-3}) \geq 3$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}\}$. By the first part of the proof, $ftd_1(G') \leq (2n(G')+1)/3 = (2n-5)/3$. Suppose that $ftd_1(G') < (2n(G')+1)/3 = (2n-5)/3$. Let S' be a $ftd_1(G')$-set. If $v_{d-3} \in S'$, then $\{v_{d-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in G, and so $ftd_1(G) < (2n+1)/3$, a contradiction. Thus $v_{d-3} \notin S'$. Then $\{v_{d-1}, v_d\} \cup S'$ is a 1FTD-set in G, and so $ftd_1(G) < (2n-1)/3$, a contradiction. Thus $ftd_1(G') = (2n(G')+1)/3$. By the induction hypothesis, $G' \in \mathcal{G}$. Clearly, v_{d-3} is not a special vertex of G', since $d \geq 4$. Thus G is obtained from G' by Operation $\mathcal{O}_1$ and so $G \in \mathcal{G}$.

Case 2. $d = 3$. Observe that $\deg_G(v_2) = 2$, since G has no strong support vertex.

Assume that $\deg_G(v_1) \geq 3$. Suppose that v_1 is a support vertex. Let $G' = G - \{v_2, v_3\}$. By the first part of the proof, $ftd_1(G') \leq (2n(G')+1)/3 = (2n-3)/3$. Let S' be a $ftd_1(G')$-set. By Observation 1.1, $v_1 \in S'$. Then $S' \cup \{v_2\}$ is a 1FTD-set in G, and so $ftd_1(G) < (2n+1)/3$, a contradiction. Thus assume that v_1 is not a support vertex of G. Let $x \neq v_2, v_0$ be a support vertex of G such that $x \in N(v_1)$. By the choice of the path $v_0v_1 \ldots v_d$, (the part "$\deg(v_{d-1})$ is as large as possible") $\deg_G(x) = 2$. Let y be the leaf adjacent to x. Let $G' = G - \{v_3, v_2, y\}$. By the first part of the proof, $ftd_1(G') \leq (2n(G')+1)/3 = (2n+1)/3 - 2$. If $ftd_1(G') < (2n(G')+1)/3 = (2n+1)/3 - 2$ and S' is a $ftd_1(G')$-set, then by Observation 1.1, $v_1 \in S'$, since v_1 is a support vertex of G'. Then $\{v_2, x\} \cup S'$ is a 1FTD-set in G and so $ftd_1(T) < (2n+1)/3$, a contradiction. Thus $ftd_1(G') = (2n(G')-1)/3$. By the induction hypothesis $G' \in \mathcal{G}$. Then G is obtained from G' by Operation $\mathcal{O}_2$, and so $G \in \mathcal{G}$.

Next assume that $\deg_G(v_1) = 2$. We show that $\deg(v_0) \geq 4$. Suppose that $\deg(v_0) = 3$. Let $G' = G - \{v_1, v_2, v_3\}$. Assume that $ftd_1(G') = (2n(G')+1)/3$. By the induction hypothesis $G' \in \mathcal{G}$. By Observation 3.2(1), v_0 is the unique special vertex of G', since $\deg_{G'}(v_0) = 2$. We show that $\deg_{G'}(x) = 3$, for each $x \in \{u_1, \ldots, u_k\} - \{v_0\}$. Assume that $\deg_{G'}(u_j) \geq 4$ for some $u_j \in \{u_1, \ldots, u_k\} - \{v_0\}$. If there is a vertex $w \in V(G) - C$ such that $d(w, C) = d(w, u_j) = 3$, then w plays the same role of v_d, and thus $\deg(u_j) = 3$, a contradiction. Thus there is no vertex $w \in V(G) - C$ such that $d(w, C) = d(w, u_j) = 3$. Then any vertex of $N(u_j) - C$ is a leaf or a weak support vertex. Assume that $N(u_j) - C$ contains t_1 leaves and t_2 support vertices, where $t_1 + t_2 \geq 2$. By Observation 3.2(3), $t_1 = 0$, since $G' \in \mathcal{G}$. Thus $t_2 \geq 2$. Let z_1 and z_2 be two weak support vertices in $N(u_j) - C$. Let z'_1 and z'_2 be the leaves adjacent to z_1 and z_2, respectively. (We switch for a while to G.) Let $G'' = G - \{z_1, z'_1, z'_2\}$. By the first part of the proof, $ftd_1(G'') \leq (2n(G'')+1)/3$. Suppose that $ftd_1(G'') = (2n(G')+1)/3$. By the induction hypothesis, $G'' \in \mathcal{G}$. Clearly, $\deg_{G''}(u_j) \geq 3$, since v_0 is the unique special vertex of G', a contradiction (by Observation 3.2(1)). Thus $ftd_1(G'') < (2n(G')+1)/3 = (2n-5)/3$. Let S'' be a $ftd_1(G'')$-set. By Observation 1.1, $u_j \in S''$. Then $S'' \cup \{z_1, z_2\}$ is a 1FTD-set of G, and so $ftd_1(G) < (2n(G)+1)/3$, a contradiction. We deduce that $\deg_{G'}(x) = 3$ for each $x \in \{u_1, \ldots, u_k\} - \{v_0\}$. Note that by Observation 3.2, u_i is not a support vertex for each i with $1 \leq i \leq k$ in G', since $G' \in \mathcal{G}$. (We switch for a while to G.) Let $F = \cup_{i=1}^{k}(N_G(u_i)) - \{u_1, \ldots, u_k\}$. Clearly, $|F| = k$, since $\deg_{G'}(u_i) = 3$ for each $u_i \in \{u_1, \ldots, u_k\} - \{v_0\}$ and $\deg_G(v_0) = 3$. Let $F = \{u'_1, u'_2, \ldots, u'_k\}$. Clearly, $\deg_G(u'_i) \geq 2$, for each i with $1 \leq i \leq k$, since u_i is not a support vertex for $1 \leq i \leq k$ in G'. Clearly, u'_i is not a strong support vertex of G for $1 \leq i \leq k$. If u'_i is adjacent to a support vertex $u''_i \in V(G) - C$, for some integer i, then since the leaf of u''_i plays the role of v_3, we obtain that $\deg(u'_i) = 2$. Since $\deg_G(u'_i) \geq 2$, for each i with $1 \leq i \leq k$, we find that $\deg_G(u'_i) = 2$, for each i with $1 \leq i \leq k$. Let $F' = \cup_{i=1}^{r} N_G(u'_i) - \{u_1, \ldots, u_k\}$. Clearly, $|F'| = k$, since $\deg_G(u'_i) = 2$, for each $u'_i \in \{u'_1, \ldots, u'_k\}$. Clearly, $F \cup F'$ is a 1FTD-set in G of cardinality at most $2n/3$, a contradiction. We deduce that $ftd_1(G') < (2n(G') + 1)/3$. Let S' be a $ftd_1(G')$-set. If $v_0 \in S'$, then $S' \cup \{v_1, v_2\}$ is a 1FTD-set in G, and so $ftd_1(G) < (2n + 1)/3$, a contradiction. Thus assume that $v_0 \notin S'$. Then $S' \cup \{v_2, v_3\}$ is a 1FTD-set in G, and so $ftd_1(G) < (2n + 1)/3$, a contradiction. Thus $\deg(v_0) \geq 4$. Let $G' = G - \{v_1, v_2, v_3\}$. By the first part of the proof, $ftd_1(G') \leq (2n(G') + 1)/3$. Assume that $ftd_1(G') < (2n(G') + 1)/3$. Let S' be a $ftd_1(G')$-set. If $v_0 \in S'$, then $S = S' \cup \{v_1, v_2\}$ is a 1FTD-set for G and so $ftd_1(G) < (2n + 1)/3$, a contradiction. Thus assume that $v_0 \notin S'$. Then $S = S' \cup \{v_2, v_3\}$ is a 1FTD-set for G and so $ftd_1(G) < (2n + 1)/3$, a contradiction. Hence, $ftd_1(G') = (2n(G') + 1)/3$. By the induction hypothesis, $G' \in \mathcal{G}$. Since $\deg(v_0) \geq 4$, v_0 is not a special vertex of G'. Thus G is obtained from G' by Operation $\mathcal{O}_1$ and so $G \in \mathcal{G}$.

Case 3. $d = 2$. We show that $\deg_G(v_0) = 3$. Suppose that $\deg_G(v_0) \geq 4$. Assume that v_0 is a support vertex. Let $G' = G - \{v_1, v_2\}$. By the first part of the proof, $ftd_1(G') \leq (2n(G') + 1)/3 = (2n - 3)/3$. Let S' be a $ftd_1(G')$-set. By Observation 1.1, $v_0 \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FTD-set in G, and so $ftd_1(G) < (2n + 1)/3$, a contradiction. Thus assume that v_0 is not a support vertex of G. Let $x \neq v_1$ be a support vertex of G such that $x \in N(v_0) - V(C)$. By the choice of the path $v_0v_1 \ldots v_d$, (the part "$\deg(v_{d-1})$ is as large as possible"), $\deg_G(x) = 2$. Let y be the leaf adjacent to x, and $G' = G - \{v_2, v_1, y\}$. By the first part of the proof, $ftd_1(G') \leq (2n(G') + 1)/3 = (2n + 1)/3 - 2$. Let $ftd_1(G') < (2n(G') + 1)/3 = (2n + 1)/3 - 2$. Let S' be a $ftd_1(G')$-set. By Observation 1.1, $v_0 \in S'$, since v_0 is a support vertex of G'. Then $\{v_1, x\} \cup S'$ is a 1FTD-set in G, and so $ftd_1(T) < (2n + 1)/3$, a contradiction. Thus $ftd_1(G') = (2n(G') - 1)/3$. By the induction hypothesis, $G' \in \mathcal{G}$, a contradiction by Observation 3.2 (3), since v_0 is a support vertex of G'. Thus $\deg_G(v_0) = 3$. Observe that G has no strong support vertex. If u_i is adjacent to a support vertex u'_i of $N(u_i) - C$ for some i, then the leaf of u'_i plays the role of v_2, and thus $\deg(u_i) = 3$. Thus we may assume that $\deg_G(u_i) \leq 3$ for each i with $i = 1, 2, \ldots, k$. Assume that $\deg_G(u_i) = 3$ for each i with $1 \leq i \leq k$. Let D_1 be the set of support vertices of C and D_2 be the set of non-support vertices of C. Let $D'_2 = N(D_2) - C$. Then $S = V(C) \cup D'_2$ is a 1FTD-set in G of cardinality at most $2n/3$, a contradiction. Thus $\deg_G(u_j) = 2$ for some j with $1 \leq j \leq k$.

Claim 1.: If $\deg_G(u_j) = 2$ for some j with $1 \leq j \leq k$, then $\deg_G(u_{j+1}) = 3$ and $\deg_G(u_{j-1}) = 3$

Proof of Claim 1. Assume that $\deg_G(u_j) = \deg_G(u_{j+1}) = 2$ for some j with $1 \leq j \leq k$, and among such vertices choose u_j such that $\deg_G(u_{j-1}) = 3$. Let $T = G - u_j$. By Theorem 2.2, $ftd_1(T) \leq 2n(T)/3$. Assume that $ftd_1(T) = 2n(T)/3$. By Theorem 2.2, $V(T) - L(T)$ is a $ftd_1(T)$-set (Note that $\deg_T(u_{j-1}) = 2$ and $\deg_T(u_{j+1}) = 1$). Then $V(T) - L(T)$ is a 1FTD-set in G of cardinality at most $(2n - 2)/3$, a contradiction. Thus we assume that $ftd_1(T) < 2n(T)/3$. Let S be a $ftd_1(T)$-set. If $|\{u_{j-1}, u_{j+1}\} \cap S| = 1$, then S is a 1FTD-set in G of cardinality at most $(2n-2)/3$, a contradiction. Thus $|\{u_{j-1}, u_{j+1}\} \cap S| \in \{0, 2\}$. If $|\{u_{j-1}, u_{j+1}\} \cap S| = 0$, then $S \cup \{u_{j+1}\}$ a 1FTD-set in G and so $ftd_1(T) < (2n+1)/3$, a contradiction. Thus $|\{u_{j-1}, u_{j+1}\} \cap S| = 2$. Now $S \cup \{u_j\}$ a 1FTD-set in G and so $ftd_1(G) < (2n+1)/3$, a contradiction. □

Claim 2.: If $\deg_G(u_{j_1}) = \deg_G(u_{j_2}) = 2$ for some j_1 and j_2 with $j_1 < j_2$, then there is an integer j' with $j_1 \leq j' \leq j_2$ such that $u_{j'}$ is a support vertex of G.

Proof of Claim 2. Assume that $\deg_G(u_{j_1}) = \deg_G(u_{j_2}) = 2$ for some j_1 and j_2 with $j_1 < j_2$. By Claim 1, $j_1 \leq j_2 - 2$. Among such vertices choose u_{j_1} and u_{j_2} such that there is no vertex u_i with $\deg(u_i) = 2$ and $j_1 < i < j_2$. Suppose to the contrary, that u_i is not a support vertex of G for each i with $j_1 < i < j_2$. By Claim 1, $\deg_G(u_{j_1-1}) = \deg_G(u_{j_2+1}) = 3$. Let $T = G - u_{j_1}u_{j_1+1} - u_{j_2}u_{j_2+1}$, T' be the component of T such that $u_{j_2} \in V(T')$, and T'' be the component of T such that $u_{j_1} \in V(T'')$. By Theorem 2.2, $ftd_1(T'') \leq 2n(T'')/3$. Assume that $ftd_1(T'') = 2n(T'')/3$. By Theorem 2.2, $S = V(T'') - L(T'')$ is a $ftd_1(T'')$-set (Note that $\deg_{T''}(u_{j_1}) = 1$ and $\deg_{T''}(u_{j_2+1}) = 2$). Then $S \cup (V(T') - V(C))$ is a 1FTD-set in G of cardinality at most $(2n-2)/3$, a contradiction. Thus $ftd_1(T'') < 2n(T'')/3$. Let S be a $ftd_1(T'')$-set. Suppose that $u_{j_1} \in S$. If $u_{j_2+1} \notin S$, then $S \cup (V(T') - L(T'))$ is a 1FTD-set in G, and so $ftd_1(G) < (2n+1)/3$, a contradiction, and if $u_{j_2+1} \in S$, then $S \cup (V(T') - L(T')) \cup \{u_{j_2}\}$ is a 1FTD-set in G, and so $ftd_1(G) < (2n+1)/3$, a contradiction. Thus, $u_{j_1} \notin S$. If $u_{j_2+1} \in S$, then $S \cup (V(T') - V(C))$ is a 1FTD-set in G, and so $ftd_1(G) < (2n+1)/3$, a contradiction. Thus, $u_{j_2+1} \notin S$. Then $S \cup (V(T') - L(T')) \cup \{u_{j_1}\}$ is a 1FTD-set in G, and so $ftd_1(G) < (2n+1)/3$, a contradiction. □

Let $D_0 = \{u_i | \deg_G(u_i) = 2\}$, $D_1 = \{u_i | u_i$ is a support vertex of V(C)$\}$, $D_2 = \{u_i | u_i$ is a not support vertex of $V(C)$ such that $\deg_G(u_i) = 3\}$ and $D_2' = N(D_2) - V(C)$. If $|D_0| \leq |D_1|$, then $V(C) \cup D_2'$ is a 1FTD-set G of cardinality at most $2n/3$, a contradiction. Thus $|D_1| < |D_0|$. Then by Claims 1 and 2 we obtain that $|D_0| = 1$ and $|D_1| = 0$. Thus G is obtained from 2-corona of a cycle C by removal of a support vertex and its leaf. Consequently, $G \in \mathcal{G}$.

For the converse, if $G \neq C_7$, then by Corollary 3.4, $V(G) - L(G)$ is the unique $ftd_1(G)$-set. Now Observation 3.2 implies that $ftd_1(G) = (2n+1)/3$. The result for C_7 is obvious. ■

Theorem 3.7 *If G is a unicyclic graph of order $n \geq 4$, then $\gamma_{\times 2}(G) \geq 2n/3$.*

Proof. Let G be a unicyclic graph of order n, and let S be a $\gamma_{\times 2}(G)$-set. Assume that $C = u_1u_2\ldots u_ku_1$ be the cycle of G. If $\{u_1, u_2, \ldots, u_k\} \subseteq S$ then S is a double dominating set of the tree $T = G - u_1u_2$, and thus by a result of Chellali [4], $|S| \geq \gamma_{\times 2}(T) \geq (2n+2)/3$. Thus $\gamma_{\times 2}(G) \geq (2n+2)/3$. Next assume that $u_j \notin S$ for some $1 \leq j \leq k$. Let $T_1', T_2', \ldots, T_r'$ be $r \geq 1$ components of $G - u_j$. Clearly, $S \cap T_i'$ is a double dominating set of the tree T_i' for each $1 \leq i \leq r$. Then by a result of Chellali [4], $|S| \geq (2(n-1)+2r)/3 \geq 2n/3$ and so $\gamma_{\times 2}(G) \geq 2n/3$. ■

Corollary 3.8 *In a unicyclic graph of order* $n \geq 4$*, every* ftd_1*-set is a* ftd*-set.*

Proof. If S is a kFTD-set for a unicyclic graph G for some $k \geq 2$, then $|N(x) \cap S| = k \geq 2$ for all $x \in V - S$. Thus every vertex of S has a neighbor in S, implying that S is a double dominating set, and thus $|S| \geq \gamma_{\times 2}(G)$. By Theorem 3.7, $|S| \geq 2n/3$. Assume that $2n \equiv 1, 2 (\text{mod } 3)$. Then $|S| \geq (2n+1)/3$. By Theorem 3.6, $ftd_1(G) \leq (2n+1)/3$ and so $ftd_1(G) \leq ftd_k(G)$, for each $k \geq 2$. Next assume that $2n \equiv 0 (\text{mod } 3)$. Then by Theorem 3.6, $ftd_1(G) < (2n+1)/3$ and so $ftd_1(G) \leq 2n/3$ and $ftd_1(G) \leq ftd_k(G)$ for each $k \geq 2$. ■

We are now ready to state the main theorem of this section.

Theorem 3.9 *If* G *is a unicyclic graph of order* $n \geq 4$*, then* $ftd(G) \leq (2n+1)/3$*, with equality if and only if* $G = C_7$ *or* $G \in \mathcal{G}$*.*

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