AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **73(2)** (2019), Pages 372–384

The Gale-Ryser theorem modulo k

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Abstract

The Gale-Ryser theorem determines when there exists a $(0,1)$-matrix with prescribed row and column sum vectors R and S, respectively. We consider a mod k analogue of this theorem and give an algorithm for existence and construction of a matrix with prescribed R and S mod k. A necessary condition for existence is that the sum of the entries of R and the sum of the entries of S are congruent mod k. We show that if the size of the matrix is large enough, this condition is also sufficient.

1 Introduction

In this paper we continue our investigations begun in [3] concerning combinatorial properties of matrices over the integers modulo k.

The following theorem is well-known and is easy to prove by a simple recursive algorithm (see e.g. [1]).

Theorem 1.1 *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors. There exists a nonnegative integral matrix A with row sum vector R and column sum vector S if and only if*

$$r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n. \qquad (1)$$

Moreover, if (1) *holds then there exists such a matrix with at most $m+n-1$ nonzero entries.*

ISSN: 2202-3518

Theorem 1.1 characterizes when the set $\mathbb{Z}^+(R,S)$ of nonnegative integral matrices with row sum vector R and column sum vector S is nonempty. The classical Gale-Ryser theorem is a specialization of Theorem 1.1 whereby the entries of the matrix A are restricted to be zeros and ones.

Let $R = (r_1, r_2, \ldots, r_m)$ be a nonnegative integral vector with $\max\{r_i : 1 \leq i \leq m\} \leq n$ for some integer n, and let $R^* = (r_1^*, r_2^*, \ldots, r_n^*)$ be the *conjugate* of R considered as a partition of the integer τ defined to be $r_1 + r_2 + \cdots + r_m$. This conjugate is obtained by considering the *Ferrers diagram* of R, defined to be an $m \times n$ $(0,1)$-matrix in which row i has r_i 1's that have been left-justified $(1 \leq i \leq m)$. For each j with $1 \leq j \leq n$, r_j^* is the number of 1's in column j of the Ferrers diagram. We have $r_1^* \geq r_2^* \geq \cdots \geq r_n^* \geq 0$, $r_1^* + r_2^* + \cdots + r_n^* = \tau$, and $r_j^* = |\{i : r_i \geq j\}|$ for $1 \leq j \leq n$.

Now let $S = (s_1, s_2, \ldots, s_n)$ be another nonnegative integral vector, and let $\mathcal{A}(R,S)$ be the set of $(0,1)$-matrices in $\mathbb{Z}^+(R,S)$. The Gale [4] and Ryser [5] theorem (see also [1]) characterizes when $\mathcal{A}(R,S)$ is nonempty, that is, when $\mathbb{Z}^+(R,S)$ contains a $(0,1)$-matrix. This characterization is in terms of the notion of majorization which we now define.

Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two nonnegative integral vectors, and let $X' = (x_1', x_2', \ldots, x_n')$ and $Y = (y_1', y_2', \ldots, y_n')$ be, respectively, a reordering of the components of X and Y to get nonincreasing vectors, that is, $x_1' \geq x_2' \geq \cdots \geq x_n'$ and $y_1' \geq y_2' \geq \cdots \geq y_n'$. Then X is *majorized* by Y, written $X \preceq Y$ provided

$$\sum_{i=1}^{k} x_i' \leq \sum_{i=1}^{k} y_i' \text{ for all } k \text{ with equality when } k = n.$$

Theorem 1.2 *The set $\mathcal{A}(R,S)$ is nonempty if and only if S is majorized by R^*. When S is nonincreasing this is*

$$(\textit{Gale-Ryser conditions}) \qquad \sum_{i=1}^{j} s_i \leq \sum_{i=1}^{j} r_i^* \textit{ for all } j \textit{ with equality when } j = n. \quad (2)$$

In the results that follow, Theorem 1.2 is often used with nonincreasing vectors R and S in order to use conditions (2) as written here, but this is not required with the given definition of majorization. Note also that the conditions (2) imply that $r_i \leq n$ for all i so that R^* can be regarded as a vector with n components by including additional 0's. When (2) holds, the *Gale-Ryser algorithm* to construct a matrix in $\mathcal{A}(R,S)$ inserts s_n 1's in column n in those rows with the largest prescribed row sums (giving preference to the bottommost rows in case of ties) and then proceeds recursively.

Let k be an integer with $k \geq 2$, and let $(\mathbb{Z}_k, +_k)$ be the additive group of integers modulo k. The set of elements of $\mathbb{Z}_k$ is taken to be $\{0, 1, \ldots, k-1\}$. The following mod k analogue of Theorem 1.1 was established in [2].

Theorem 1.3 *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be vectors with entries in $\mathbb{Z}_k$. Then there exists an $m \times n$ matrix with entries in $\mathbb{Z}_k$ with mod k row sum vector R and mod k column sum vector S if and only if we have the following congruence modulo k:*

$$r_1 + r_2 + \cdots + r_m \equiv s_1 + s_2 + \cdots + s_n \bmod k. \tag{3}$$

Moreover, if (3) *holds then there exists such a matrix with at most $m+n-1$ nonzero entries.*

Our goal here is to develop a mod k theorem having the same relationship to Theorem 1.3 as Theorem 1.2 has to Theorem 1.1, that is, a *mod k Gale-Ryser theorem.* Accordingly, let $\mathbb{Z}_k(R,S)$ denote the set of all matrices with entries in $\mathbb{Z}_k$ whose mod k row sum vector is R and whose mod k column sum vector is S, where R and S satisfy (3). Let $\mathcal{A}_k(R,S)$ denote the set of all $(0,1)$-matrices in $\mathbb{Z}_k(R,S)$.

If $k = 2$, that is, if we consider $\mathbb{Z}_2 = \{0,1\}$, then there is nothing new to investigate since $\mathbb{Z}_2$ has only the two elements 0 and 1, and so $\mathcal{A}_2(R,S)$ always equals $\mathbb{Z}_2(R,S)$. Thus we now assume that $k \geq 3$. The following examples indicate some of the subtleties that arise in our investigations. If U and V are integral vectors with the same number of components, then we write $U \equiv V \bmod k$ provided corresponding components of U and V are congruent modulo k.

Example 1.4 Let $k = 3$ and $R = S = (2,0)$. Then there is a matrix in $\mathbb{Z}_3(R,S)$, namely

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

but, as is easily checked, the Gale-Ryser conditions fail and there does not exist a matrix in $\mathcal{A}(R,S)$, nor a matrix in $\mathcal{A}_3(R,S)$.

Now let $R = S = (2,0,0)$. Then again the Gale-Ryser conditions fail and $\mathcal{A}(R,S)$ is empty. Define $R' = S' = (2,3,3)$ where $R' \equiv R \bmod 3$ and $S' \equiv S \bmod 3$. Then the Gale-Ryser conditions now hold and thus $\mathcal{A}_3(R,S) \neq \emptyset$; indeed the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is in $\mathcal{A}_3(R,S)$. This example generalizes to vectors $R = S = (2,0,\ldots,0)$ and $R' = S' = (2,3,\ldots,3)$ of arbitrary size at least 3. □

Example 1.5 Let $n = 3$, and let $R = (2,2,2)$ and $S = (1,1,1)$. Then there is a matrix in $\mathbb{Z}_3(R,S)$, namely

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix},$$

but there does not exist a $(0,1)$-matrix in $\mathbb{Z}_3(R,S)$; in fact, such a matrix would have to have exactly two 1's in each row and only one 1 in each column, an impossibility. Also, since the real sum of the proposed row sums does not equal the real sum of the proposed column sums, there does not exist a matrix in $\mathbb{Z}^+(R,S)$. □

More generally, let m and n be positive integers. Let $R = (2,2,\ldots,2)$ be an m-tuple of 2's, and let $S = (1,1,\ldots,1)$ be an n-tuple of 1's. In order that there exists a matrix A in $\mathcal{A}_3(R,S)$ we must have

$$2m \equiv n \bmod 3.$$

As Example 1.5 shows with $m = n = 3$, this does not suffice in general for there to be a matrix in $\mathcal{A}_3(R,S)$. But if $m = n = 6$, so that now each row of A can contain two or five 1's and each column can contain one or four 1's, there is a matrix in $\mathcal{A}_3(R,S)$, for instance, the matrix

$$\left[\begin{array}{c|c|c|c|c|c} & & 1 & 1 & & \\ \hline & & & 1 & 1 & \\ \hline & & & & 1 & 1 \\ \hline & & 1 & & & 1 \\ \hline 1 & & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

In Section 2 we obtain a mod k analogue of the Gale-Ryser algorithm which either uses the Gale-Ryser algorithm to construct a matrix in $\mathcal{A}_k(R,S)$ or concludes that no such matrix exists. In Section 3, we show that if m and n are large enough, the necessary condition (3) for the nonemptiness of $\mathcal{A}_k(R,S)$ is also sufficient. In Section 4, we make some final comments.

2 An Algorithm

Examples 1.4 and 1.5 motivate the following discussion.

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be vectors with entries in $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$. By Theorem 1.3 a necessary condition for $\mathcal{A}_k(R,S)$ to be nonempty is that (3) holds. The following lemma concerning the nonemptiness of $\mathcal{A}_k(R,S)$ is now obvious.

Lemma 2.1 *The set $\mathcal{A}_k(R,S)$ is nonempty if and only if there exist nonnegative integral vectors $R' = (r'_1, r'_2, \ldots, r'_m)$ and $S' = (s'_1, s'_2, \ldots, s'_n)$ where $r'_i \equiv r_i \bmod k$ $(1 \le i \le m)$ and $s'_j \equiv s_j \bmod k$ $(1 \le j \le n)$, such that $\mathcal{A}(R',S')$ is nonempty, that is, if and only if there exist vectors R' and S', obtained from R and S by adding multiples of k to their components, which satisfy the Gale-Ryser conditions.*

So the question of nonemptiness of $\mathcal{A}_k(R,S)$ reduces to:

(*) *Given* R *and* S *such that* (3) *holds, when does there exist* R' *and* S' *with* $R' \equiv R \bmod k$ *and* $S' \equiv S \bmod k$ *such that* R' *and* S' *satisfy the Gale-Ryser conditions* (2)?

The following algorithm either constructs a matrix in $\mathcal{A}_k(R,S)$ or gives the conclusion that no such matrix exists.

$GR_k(R,S)$**: Algorithm for the Existence of a Matrix in** $\mathcal{A}_k(R,S)$

The algorithm takes as input a positive integer $k \geq 3$, and integral vectors $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$, with $n \geq r_1 \geq r_2 \geq \cdots \geq r_m$ and $m \geq s_1 \geq s_2 \geq \cdots \geq s_n$, and with entries in $\{0, 1, \ldots, k-1\}$ such that

$$r_1 + r_2 + \cdots + r_m \equiv s_1 + s_2 + \cdots + s_n \bmod k. \tag{4}$$

The algorithm either ends in FAILURE or it outputs integral vectors R' and S' with $R' \equiv R \bmod k$ and $S' \equiv S \bmod k$ along with a matrix in $\mathcal{A}(R', S')$, which is also in $\mathcal{A}_k(R,S)$. Assume without loss of generality that the real sums of the components of R and S satisfy $s_1 + s_2 + \cdots + s_n \leq r_1 + r_2 + \cdots + r_m$.

To start, copy R and S into new integral vectors $R' = (r'_1, \ldots, r'_m)$ and $S' = (s'_1, \ldots, s'_n)$. We will update these as the algorithm progresses.

(i) If $r'_1 + r'_2 + \cdots + r'_m = s'_1 + s'_2 + \cdots + s'_n$, then go to step (ii). Otherwise, $r'_1 + r'_2 + \cdots + r'_m > s'_1 + s'_2 + \cdots + s'_n$ and, by assumption (4), the difference is a multiple of k. If $s'_n + k > m$, we stop and declare FAILURE. Otherwise we increase the smallest entry of S' (that is, s'_n) by k and sort the new entries giving $S'' = (s''_1 = s'_n + k, s''_2 = s'_1, \ldots, s''_n = s'_{n-1})$. Repeat this step, treating S'' as the new S' until FAILURE or directed to step (ii).

(ii) If $S' \preceq R'^*$, then we reorder the entries of R' and S' so that $R' \equiv R \bmod k$ and $S' \equiv S \bmod k$ and use the Gale-Ryser algorithm to construct a matrix A in $\mathcal{A}(R', S')$. We then output R', S', and the matrix A and stop. If $s'_n + k > m$ or $r'_m + k > n$, then we stop and declare FAILURE. Otherwise, we increase the smallest entries in R' and S' (that is, r'_m and s'_n) by k and sort the new entries, giving nonincreasing vectors $R'' = (r''_1 = r'_m + k, r''_2 = r'_1, \ldots, r''_m = r'_{m-1})$ and $S'' = (s''_1 = s'_n + k, s''_2 = s'_1, \ldots, s''_n = s'_{n-1})$ which preserves $r''_1 + r''_2 + \cdots + r''_m = s''_1 + s''_2 + \cdots + s''_n$. Repeat this step, treating R'' and S'' as the new R' and S' until failure or the algorithm outputs a valid matrix.

Before verifying this algorithm, we give an example.

Example 2.2 We take $k = 4$. Let $m = n = 5$, and let $R = (3,3,3,0,0)$ and $S = (1,1,1,1,1)$, where $5 = s_1 + s_2 + s_3 + s_4 + s_5 \equiv r_1 + r_2 + r_3 + r_4 + r_5 = 9 \bmod 4$. Since $5 < 9$, in step (i) we increase s_5 by 4 which, after sorting, gives $R' = (3,3,3,0,0)$ and $S' = (5,1,1,1,1)$ with $3+3+3+0+0 = 5+1+1+1+1$. We now go to step (ii). Since $S' \not\preceq R'^*$, we increase both r'_5 and s'_5 by 4 and sort to give the new $R' = (4,3,3,3,0)$ and new $S' = (5,5,1,1,1)$. Since we still have $S' \not\preceq R'^*$, we increase both the new r'_5 and new s'_5 by 4 and sort to now give $R' = (4, 4, 3, 3, 3)$ and $S' = (5, 5, 5, 1, 1)$. Now we have $S' \preceq R'^*$ and so $\mathcal{A}(R', S') \neq \emptyset$. In fact, we have

$$A' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \in \mathcal{A}(R', S').$$

Reversing our sorting, this gives

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \in \mathcal{A}_4(R, S).$$

□

We now verify the correctness of the algorithm.

Theorem 2.3 *The class $\mathcal{A}_k(R, S)$ is nonempty if and only if the $GR_k(R, S)$ algorithm terminates with vectors R' and S' satisfying $S' \preceq R'^*$ and a matrix in $\mathcal{A}_k(R, S)$.*

Proof. If the algorithm outputs $A \in \mathcal{A}_k(R, S)$, then the class is clearly nonempty.

Conversely, we need to show that if there is a matrix $A \in \mathcal{A}_k(R, S)$, then the $GR_k(R, S)$ algorithm does not end in FAILURE. For the sake of contradiction, we assume $\mathcal{A}_k(R, S) \neq \emptyset$ and the $GR_k(R, S)$ algorithm ends in FAILURE.

Suppose the algorithm stops in step (i) with FAILURE. Since the algorithm always adds k to the smallest component of the current column sum vector, it follows that if $S'' = (s''_1, s''_2, \ldots, s''_n)$ is any vector obtained from S by successively increasing components by k to get $s''_1 + s''_2 + \cdots + s''_n = r_1 + r_2 + \cdots + r_m$, then for at least one i we have $s''_i > m$. Thus there cannot exist vectors R' and S', obtained from R and S, respectively, by adding positive multiples of k to components, such that $\mathcal{A}(R', S') \neq \emptyset$. This implies that $\mathcal{A}_k(R, S) = \emptyset$, a contradiction.

Now suppose the algorithm outputs FAILURE in step (ii), but there exists $A \in \mathcal{A}(\hat{R}, \hat{S})$ for some $\hat{R} = (\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_m) \equiv R \bmod k$ and $\hat{S} = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n) \equiv S \bmod k$ where $\hat{S} \preceq \hat{R}^*$ by Theorem 1.2. At some point in step (ii) we considered R' and S' with $r'_1 + r'_2 + \cdots + r'_m = \hat{r}_1 + \hat{r}_2 + \cdots + \hat{r}_m$. Both R' and $\hat{R}$ are obtained from R by adding multiples of k to the entries of R. Since we obtain R' by recursively adding k to the smallest components of R, we have $\hat{R}^* \preceq R'^*$. Similarly, $S' \preceq \hat{S}$. Thus we have $S' \preceq \hat{S} \preceq \hat{R}^* \preceq R'^*$, and the algorithm would have returned R' and S', a contradiction. □

In the next section we show that if m and n are large enough in terms of k, then $\mathcal{A}_k(R, S) \neq \emptyset$ provided only that the obvious congruence equation (3) holds.

3 An Existence Theorem

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$. Theorem 2.3 provides an algorithm to determine when a class $\mathcal{A}_k(R, S)$ is nonempty. If $k \geq \max\{m, n\}$, then $\mathcal{A}_k(R, S) \neq \emptyset$ if and only if $\mathcal{A}(R, S) \neq \emptyset$, and thus the Gale-Ryser conditions (2) give a necessary and sufficient condition for $\mathcal{A}_k(R, S)$ to be nonempty. If $k = 2$, then, for any m and n, $\mathcal{A}_k(R, S) \neq \emptyset$ if $\sum_{i=1}^m r_i \equiv \sum_{j=1}^n s_j \bmod 2$ by Theorem 1.3. When $k \geq 3$, we now show that if m and n are large enough as a function of k (a linear bound), then the obvious necessary condition (3) guarantees that $\mathcal{A}_k(R, S) \neq \emptyset$.

Theorem 3.1 *Let k be an integer with $k \geq 2$, and let m and n be integers with $m, n \geq 3k - 1$. Assume that $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ are vectors with $r_i, s_j \in \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then the following are equivalent:*

(i) $\mathbb{Z}_k(R, S) \neq \emptyset$.

(ii) $\mathcal{A}_k(R, S) \neq \emptyset$.

(iii) $\sum_{i=1}^m r_i \equiv \sum_{j=1}^n s_j \bmod\ k$.

Proof. We know that in general (ii) implies (i), and (i) and (iii) are equivalent. Thus we need only show that if $m, n \geq 3k - 1$, then (iii) implies (ii). So we assume that (iii) holds. We refer to the $GR_k(R, S)$ algorithm.

Claim 1: Step (i) of $GR_k(R, S)$ does not end in failure.

Now let $|R| = \sum_{i=1}^m r_i$ and $|S| = \sum_{j=1}^n s_j$, a real sum in both instances, so that $|R| \equiv |S| \bmod k$. Without loss of generality we assume that $|R| \geq |S|$ so that $|R| = |S| + tk$ for some nonnegative integer t. We may recursively add k to the smallest integer in S arriving at a vector, continued to be labelled $S = (s_1, s_2, \ldots, s_n)$, such that $|R| = |S|$. This new vector S depends on $|R|$ and not on the individual components of R. We now assume that the components of R and S have been rearranged so that as real numbers they are nonincreasing. Since the entries of R have not changed, we continue to have that $r_i \in \{0, 1, \ldots, k-1\}$ for all i and $|R| \leq m(k-1)$. If the algorithm fails in Step (i), then at some point we obtain a nonincreasing vector $S' = (s'_1, s'_2, \ldots, s'_n)$ such that $s'_n > m - k$ and thus $|S'| > n(m-k) = mn - nk$. If $m \geq n$, then using the assumption that $n \geq 3k - 1$, we get

$$m(k-1) \geq |R| > |S'| > mn - nk \geq mn - mk = m(n-k) \geq m(2k-1),$$

a contradiction. If $n > m$, then using that $m \geq 3k - 1$, we have

$$m(k-1) \geq |R| > |S'| > mn - nk = n(m-k) \geq m(2k-1),$$

which is also a contradiction. Thus in either case, in the $GR_k(R, S)$ algorithm we succeed to advance to Step (ii). This completes the verification of Claim 1.

We now relabel the new vector obtained from Step (i) using $S = (s_1, s_2, \ldots, s_n)$ as before, and assume that $R = (r_1, r_2, \ldots, r_m)$ and S are arranged to be nonincreasing. Thus now $|R| = |S|$. If $S \preceq R^*$, then the Gale-Ryser theorem produces a matrix in $\mathcal{A}(R, S)$ and thus a matrix in $\mathcal{A}_k(R, S)$. Thus we now assume that $S \not\preceq R^*$.

Claim 2: $|R|$ satisfies

$$|R| < (k-1)s_1. \tag{5}$$

Let t be the smallest integer such that the tth majorization inequality fails:

$$\sum_{i=1}^{t-1} s_i \leq \sum_{i=1}^{t-1} r_i^* \text{ and } \sum_{i=1}^{t} r_i^* < \sum_{i=1}^{t} s_i, \text{ so that } r_t^* < s_t.$$

(If $t = 1$, then we are only asserting that $r_1^* < s_1$, that is, that the number of nonzero r_i is strictly less than s_1.) Since the components of R are from $\{0, 1, \ldots, k-1\}$ and those of both R and S are nonincreasing (R^* is always nonincreasing by definition), we then have

$$\begin{aligned} |R| = \sum_{i=1}^{k-1} r_i^* = \sum_{i=1}^{t} r_i^* + \sum_{i=t+1}^{k-1} r_i^* &< \sum_{i=1}^{t} s_i + (k-1-t) r_{t+1}^* \\ &\leq \sum_{i=1}^{t} s_i + (k-1-t) r_t^* \\ &< \sum_{i=1}^{t} s_i + (k-1-t) s_t \\ &\leq t s_1 + (k-1-t) s_1 = (k-1) s_1. \end{aligned}$$

This completes the verification of Claim 2.

Claim 3: The largest component s_1 of S satisfies $s_1 < 2k - 1$.

If $s_1 \leq k$, then the claim certainly holds, so we assume that $s_1 > k$. Since $s_1 > k$ and $s_n + k \geq s_1$, we also have

$$|S| = \sum_{i=1}^{n} s_i \geq s_1 + (n-1)s_n \geq s_1 + (n-1)(s_1 - k) = ns_1 - kn + k.$$

Combining this with (5), we get

$$ns_1 - kn + k \leq |S| = |R| < s_1(k-1). \tag{6}$$

Using (6) and the assumption that $n \geq 3k - 1$, we get

$$s_1 < \frac{kn - k}{n - k + 1} = k + \frac{k(k-2)}{n-k+1} \leq k + \frac{k(k-2)}{2k} < \frac{3k}{2} \leq 2k - 1.$$

This completes the verification of Claim 3.

The remainder of the proof depends on which of m and n is larger. We only consider in detail the case when n is as least as large as m.

First assume that $m \leq n$. Since $n \geq 3k-1 \geq 2k-1 \geq r_1+k$ and $m \geq 3k-1 = (2k-1)+k > s_1+k$, we can now add k to each component of R and to the m smallest components of S to obtain, after reordering, nonincreasing vectors $R' = (r'_1, r'_2, \ldots, r'_m)$ and $S' = (s'_1, s'_2, \ldots, s'_n)$ with $|R'| = |S'|$.

Claim 4: $S' \preceq R'^*$, and hence, after reordering the rows and columns to match the original R and S, there exists a matrix $A \in \mathcal{A}(R', S')$ where $A \in \mathcal{A}_k(R, S)$.

Since the components of S originally were also in $\{0, 1, \ldots, k-1\}$, the recursive process of adding k to the smallest component implies that the components of S' lie in an interval $l, l+1, \ldots, l+k$ (for some $l \geq 0$) of $k+1$ consecutive integers (thus $s'_n + k \geq s'_1$). The sums in the majorization assertion satisfy:

$$\sum_{i=1}^{t} r_i'^* = \begin{cases} mt, & \text{if } t \leq k \\ mk + \sum_{i=1}^{t-k} r_i^*, & \text{if } k < t < 2k-1 \\ |R'|, & \text{if } 2k-1 \leq t \end{cases} \tag{7}$$

and

$$\sum_{i=1}^{t} s'_i \leq tk + \sum_{i=1}^{t} s_i. \tag{8}$$

The majorization assertion in the claim is verified in three parts.

(a) First, if $t \geq 2k-1$, then

$$\sum_{i=1}^{t} s'_i \leq |S'| = |R'| = \sum_{i=1}^{t} r_i'^*.$$

(b) Next, suppose that $t \leq k$. Then, combining (7) and the fact that every entry in S' is less than m,

$$\sum_{i=1}^{t} s'_i \leq ts_1 \leq tm = \sum_{i=1}^{t} r_i'^*.$$

(c) Finally suppose that $k < t < 2k-1$. Then we do some calculation. We have

$$\begin{aligned} m &\geq 3k-2 \\ m-2t &\geq 3k-2-2t \\ m-k-t &> m-2t \geq 3k-2-2t \\ m-t &> 4k-2-2t = 2(2k-1-t), \text{ and so,} \\ \frac{m-t}{2k-1-t} &> 2. \end{aligned}$$

Combining this inequality with claims 2 and 3, we get

$$|R| < s_1(k-1) < 2k(k-1) < \frac{m-t}{2k-1-t} k(k-1),$$

and so

$$\begin{aligned} |R|\frac{2k-1-t}{k-1} &< k(m-t) \\ |R|\left(\frac{k-1}{k-1}-\frac{t-k}{k-1}\right) &< km-tk \\ |R|+tk &< km+\frac{t-k}{k-1}|R|. \end{aligned} \tag{9}$$

Since each entry of R is at most $k-1$ and thus R^* has at most $k-1$ nonzero components, the average of the components of R^* is at least $\frac{|R|}{k-1}$ and thus the sum of the first $t-k$ (therefore the largest) components of R^* is at least $\frac{t-k}{k-1}|R|$. Hence

$$\sum_{i=1}^{t} r_i'^* \geq km+\frac{t-k}{k-1}|R|. \tag{10}$$

From (8),

$$\sum_{i=1}^{t} s_i' \leq tk+\sum_{i=1}^{t} s_i \leq tk+|S| = tk+|R|. \tag{11}$$

Combining (9), (10), and (11), we get

$$\sum_{i=1}^{t} s_i' \leq tk+|R| \leq mk+\frac{t-k}{k-1}|R| \leq \sum_{i=1}^{t} r_i^*.$$

Thus the majorization inequality holds if $k<t<2k-1$ and this completes the verification of Claim 4.

In the case when $m>n$, we add k to all the components of S and to the n smallest components of R. We get an inequality similar to (7) of the form

$$\sum_{i=1}^{t} r_i'^* = \begin{cases} nt+\alpha, & \text{if } t\leq k \\ nk+\sum_{i=1}^{t-k} r_i^*+\beta, & \text{if } k<t<2k-1 \\ |R'|, & \text{if } 2k-1\leq t \end{cases},$$

where α and β are nonnegative quantities coming from the largest $m-n$ components of R that do not get changed and so become the smallest components of R', and (8) becomes an equality (which does not change the argument). The remainder of the verification of the majorization inequalities is very similar to the case when $n\geq m$, and we omit the details. □

Remark 3.2 As can be seen in Example 1.5, some version of the hypothesis in Theorem 3.1 requiring n and m to be large relative to k is necessary for the conclusion to hold. While our proof uses $3k-1$ as a lower bound, this bound is not tight in general. It is very likely that the constant 3 in the linear bound in terms of k can be replaced with 2. If $k=3$, it is not difficult to show that $m,n\geq 4$ suffice. The next example shows that in general a lower bound of $2k-3$ does not guarantee the existence of a matrix in $\mathcal{A}_k(R,S)$. □

Example 3.3 Let k be an integer with $k \geq 2$, and let $n = 2k - 3$. The vectors $R = (r_1, r_2, \ldots, r_n)$ and $S = (s_1, s_2, \ldots, s_n)$ with $s_i = k - 1 > r_i = k - 2$ for $1 \leq i \leq k$ and $s_i = r_i = k - 1$ for $k + 1 \leq i \leq n$. Then

$$r_1 + r_2 + \cdots + r_n \equiv s_1 + s_2 + \cdots + s_n \bmod k$$

and so by Theorem 1.3 there is an $n \times n$ matrix with entries in $\mathbb{Z}_k$ with mod k row sum vector R and mod k column sum vector S. However, the algorithm $GR_k(R, S)$ fails in step (i) because the vectors R and S have different real sums, and adding k to the smallest entry in R gives $2k - 2 > n$. □

4 Coda

A different approach for a mod k analogue of the Gale-Ryser theorem is also possible.

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be vectors with $r_i, s_j \in \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, such that $\sum_{i=1}^m r_i \equiv \sum_{j=1}^n s_j \bmod k$ but $\mathcal{A}_k(R, S) = \emptyset$. Suppose that we extend R to an m'-vector R' by including an additional $m' - m \geq 0$ components equal to zero and extend S to an n'-vector S' by including an additional $n' - n \geq 0$ components equal to zero. Then it follows that $\mathcal{A}(R, S) \neq \emptyset$ if and only if $\mathcal{A}(R', S') \neq \emptyset$ but, while $\mathcal{A}_k(R, S) \neq \emptyset$ trivially implies that $\mathcal{A}_k(R', S') \neq \emptyset$, we may have $\mathcal{A}_k(R, S) = \emptyset$ but $\mathcal{A}_k(R', S') \neq \emptyset$.

Example 4.1 With $k = 3$, let $m = n = 3$, and let $R = (2, 2, 2)$ and $S = (1, 1, 1)$. As pointed out in Example 1.5, there does not exist a $(0, 1)$-matrix in $\mathcal{A}_3(R, S)$. With $R' = (2, 2, 2, 0)$ and $S' = S = (1, 1, 1)$, the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

in $\mathcal{A}_3(R', S')$. □

A more general possibility is the following. Let $k \geq 2$ and $l \geq 2$ be integers, and let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors. What are necessary and sufficient conditions that there exist a nonnegative integral matrix (respectively, a $(0, 1)$-matrix) A such that the mod k row sums of A equal R and the mod l column sums of A equal S? Let $\mathbb{Z}^+_{k,l}(R, S)$ be the set of all nonnegative integral matrices with mod k row sum vector equal to R and mod l column sum vector equal to S. Let $\mathcal{A}_{k,l}(R, S) \subseteq \mathbb{Z}^+_{k,l}(R, S)$ be the set of all (0,1)-matrices with mod k row sum vector equal to R and mod l column sum vector equal to S.

Example 4.2 Let $k = 3$ and $l = 2$, and let $R = (1, 1, 1, 2)$ and $S = (1, 1, 0, 0, 1)$. Then

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has mod 3 row sum vector R and mod 2 column sum vector S. Thus $A \in \mathcal{A}_{3,2}(R, S)$. □

Perhaps a more approachable but still interesting possibility is to take one of k and l equal to ∞ and interpret $\mathbb{Z}^+_\infty$ as the nonnegative integers using real arithmetic. Then $\mathbb{Z}^+_{\infty,\infty}(R, S)$ is what we have previously called $\mathbb{Z}^+(R, S)$, and $\mathcal{A}_{\infty,\infty}(R, S)$ is $A(R, S)$. We can also consider $\mathbb{Z}^+_{k,\infty}(R, S)$ and $\mathcal{A}_{k,\infty}(R, S)$.

Let $R = (r_1, r_2, \ldots, r_m)$ be a vector with $r_i \in \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ for $1 \le i \le m$, and let $S = (s_1, s_2, \ldots, s_n)$ be a vector with $s_j \in \mathbb{Z}^+$. Assume that $s_1 + s_2 + \cdots + s_n \ge r_1 + r_2 + \cdots + r_m$. Suppose there exists a matrix $A \in \mathcal{A}_{k,\infty}(R, S)$, and let $\tau = (s_1 + s_2 + \cdots + s_n) - (r_1 + r_2 + \cdots + r_m)$. Then $\tau \in \mathbb{Z}$ and $\tau \equiv 0 \bmod k$, say $\tau = pk$ where $p \ge 0$. This leads to the following simple algorithm to determine the nonemptiness of $\mathcal{A}_{k,\infty}(R, S)$:

- Start with R and its Ferrers diagram F (including empty rows if some components of R equal 0).
- Iteratively insert k 1's in the row of F with the smallest sum until pk 1's have been inserted. The result is the Ferrers diagram $\widetilde{F}$ of a vector $\widetilde{R}$ whose conjugate is $\widetilde{R}^*$.
- Then $\mathcal{A}_{k,\infty}(R, S) \ne \emptyset$ if and only if $S \preceq \widetilde{R}^*$.

The justification is: $\mathcal{A}_{k,\infty}(R, S) \ne \emptyset$ if and only if *some* way of iteratively inserting k 1's in the rows of F gives a vector $\widehat{R}$ whose conjugate $\widehat{R}^*$ satisfies $S \preceq \widehat{R}^*$. By always inserting k 1's in the row with the smallest sum guarantees that $\widehat{R}^* \preceq \widetilde{R}^*$.

Example 4.3 Let $k = 3$ and let $R = (2, 2, 1, 0, 0)$ and $S = (3, 3, 3, 2, 2, 1)$, and consider $\mathcal{A}_{3,\infty}(R, S)$. Then $\tau = 9 = 3 \cdot 3$ and $R^* = (3, 2, 0, 0, 0)$. Using the above algorithm, we get $\widetilde{R} = (2, 2, 4, 3, 3)$ and $\widetilde{R}^* = (5, 5, 3, 1, 0, 0)$. We have that $S \preceq \widetilde{R}^*$ and so there exists a matrix in $\mathcal{A}(\widetilde{R}, S)$. An example of a matrix in $\mathcal{A}_{3,\infty}(R, S)$ is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

□

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(Received 23 Apr 2018; revised 17 Dec 2018)