AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **73(2)** (2019), Pages 291–312

The domination equivalence classes of paths

I. Beaton J.I. Brown*

Department of Mathematics and Statistics
Dalhousie University
Halifax, NS B3H 3J5
Canada
iain.ac.beaton@gmail.com jason.brown@dal.ca

Abstract

A dominating set S of a graph G of order n is a subset of the vertices of G such that every vertex is either in S or is adjacent to a vertex of S. The domination polynomial is defined by $D(G,x) = \sum d(G,i)x^i$ where $d(G,i)$ is the number of dominating sets in G with cardinality i. Two graphs G and H are considered $\mathcal{D}$-equivalent if $D(G,x) = D(H,x)$. Extending previous results, we determine the equivalence classes of all paths.

1 Introduction

Let $G = (V,E)$ be a graph. A set S of the vertex set V of graph G is a *dominating set* if for each $v \in V(G)$, either $v \in S$ or there exists $u \in S$ which is adjacent to v. The *domination number* of G, denoted $\gamma(G)$, is the cardinality of the smallest dominating set of G. There is a long history of interest in domination, in both pure and applied settings [10, 11].

As for many graph properties, one can more thoroughly investigate domination via a generating function. Let $\mathcal{D}(G,i)$ be the collection of dominating sets of a graph G with cardinality i and let $d(G,i) = |\mathcal{D}(G,i)|$. Then the *domination polynomial* $D(G,x)$ of G is defined as

$$D(G,i) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i.$$

A natural question to ask is to what extent can a graph polynomial describe the underlying graph (for example, a survey of what is known with regards to chromatic polynomials can be found in [9, ch. 3]). We say that two graphs G and H are

* Corresponding author.

ISSN: 2202-3518

domination equivalent or simply $\mathcal{D}$-*equivalent* (written $G \sim_{\mathcal{D}} H$) if they have the same domination polynomial. As in [1], we let $[G]$ denote the $\mathcal{D}$-equivalence class determined by G, that is $[G] = \{H | H \sim_{\mathcal{D}} G\}$ (we identify isomorphic graphs). A graph G is said to be *dominating unique* or simply $\mathcal{D}$-*unique* if $[G]$ contains only G.

Two problems arise. Which graphs are $\mathcal{D}$-*unique*, that is, are completely determined by their domination polynomials? More generally, can we determine the $\mathcal{D}$-equivalence class of a graph? Both problems appear difficult, but there are some partial results known. In [2] Akbari and Oboudi showed all cycles are $\mathcal{D}$-unique. Anthony and Picollelli classified all complete r-partite graphs which are $\mathcal{D}$-unique in [7], while Alikhani and Peng [5] showed most cubic graphs of order 10 (including the Petersen graph) are $\mathcal{D}$-unique. Kotek, Preen, and Simon [13] defined and characterized irrelevant edges (those which can be removed without changing the domination polynomial of a graph), and showed various trees (in particular paths [1]) barbell graphs [12], and other graphs, are not $\mathcal{D}$-unique.

Paths form an interesting class. In [1] Akbari, Alikhani and Peng considered the $\mathcal{D}$-equivalence classes of paths, and showed that $[P_n] = \{P_n, \widetilde{P_n}\}$ for $n \equiv 0 \pmod 3$ where $\widetilde{P_n}$ is the graph obtained by adding an edge between the two stems in P_n. In this paper we extend this result and determine the $\mathcal{D}$-equivalence class for path P_n for *all* n. Our plan is as follows: we first prove some results concerning the top-most coefficients of domination polynomials for relevant classes of graphs, and some conditions that hold for graphs whose domination polynomials satisfy the same three-term recurrence that paths follow. We then determine the equivalence class for a path P_n ($n \geq 5$) by showing first that any $\mathcal{D}$-equivalent graph must arise as the disjoint union of some cycles together with P_n or the graph formed from it by joining its two stems. Finally, we rule out the presence of cycles.

We shall rely on a few standard definitions. The *order* of a graph is its number of vertices; P_n and C_n denote the *path* and *cycle* of order n, respectively. The set of vertices $N(v) = \{u | uv \in E(G)\}$ is called the *open neighbourhood* of v; similarly, $N[v] = N(v) \cup \{v\}$ is called the *closed neighbourhood* of v (if S is a subset of vertices, then the closed neighbourhood of S is defined to be $N[S] = \cup_{v \in S} N[v]$). A vertex of degree 1 is a *leaf*, its neighbour is a *stem*, and the edge between them is called a *pendant edge*.

2 Coefficients of Domination Polynomials

We start with an examination of what the domination polynomial encodes about graphs in general, and about paths in particular. Some coefficients of domination polynomials are known for general graphs. In [6], Alikhani and Peng determined $d(G, n-1)$ and $d(G, n-2)$ in terms of certain properties within the graph. In this section we will present formulae for $d(G, n-3)$ and $d(G, n-4)$ for large classes of graphs, as well as derive some properties of the coefficients specifically for the domination polynomials of paths. All of these will be essential in the following section where we determine the equivalence class for paths.

Theorem 2.1 [6] *Let G be a graph of order n with t vertices of degree one and r isolated vertices. If $D(G,x) = \sum_{i=1}^{n} d(G,i)x^i$ is its domination polynomial then the following hold:*

(i) $d(G, n-1) = n - r$.

(ii) $d(G, n-2) = \binom{n}{2} - t$ *if G has no isolated vertices and no K_2 components.*

When counting the number of dominating sets with cardinality close to n, it is sometimes simpler to count the number of subsets which are not dominating. A subset $S \subseteq V(G)$ is not dominating if there exists a vertex v in G such that none of its neighbours, nor itself, is in S. That is, $N[v] \cap S = \emptyset$, and in such a case, we say $\overline{S} = V(G) - S$ *encompasses* v or v is encompassed by $\overline{S}$. The next elementary lemma will help us identify which subsets are not dominating.

Lemma 2.2 *For a graph G and $S \subseteq V(G)$, S is not dominating if and only if there exists a vertex $v \in \overline{S}$ which is encompassed by $\overline{S}$.* ∎

We now determine the number of dominating sets of a certain size in graphs with mild restrictions (no components of cardinality at most 2) by counting the number of subsets of vertices with a given cardinality which contain the closed neighbourhood of at least one of its vertices.

Lemma 2.3 *For a graph G of order n with no K_2 components let W be the set of all stems of G. Then for $k \in \mathbb{N}$, where $0 \leq k \leq n - \gamma(G)$,*

$$d(G, n-k) = \sum_{\substack{S \subseteq V-W \\ |N[S]| \leq k}} (-1)^{|S|} \binom{n - |N[S]|}{k - |N[S]|}.$$

Proof Fix $k \in \{0, 1, \ldots, n - \gamma(G)\}$. For each $v \in V$, let A_v be the collection of k-subsets which encompass v. By Lemma 2.2, $d(G, n-k) = \binom{n}{k} - |\cup_{v \in V} A_v|$. If a k-subset S encompasses any stem s_i then it also encompasses any leaf $l \in N[s_i]$. As G has no K_2 components, all leaves are in $V - W$. Hence $A_{s_i} \subseteq A_l$ and $\displaystyle\bigcup_{v \in V} A_v = \bigcup_{v \in V-W} A_v$. Furthermore for any $S \subset V - W$, $\cap_{v \in S} A_v$ is the collection of k-subsets which encompass all of S. Therefore $|\displaystyle\bigcap_{v \in S} A_v| = \binom{n - |N[S]|}{k - |N[S]|}$, and by inclusion-exclusion,

$$d(G, n-k) \;=\; \binom{n}{k} - \left| \bigcup_{v \in V-W} A_v \right|$$

$$= \binom{n}{k} - \sum_{\substack{S\subseteq V-W\\ S\neq\emptyset}} (-1)^{|S|-1}\left|\bigcap_{v\in S} A_v\right|$$
$$= \sum_{\substack{S\subseteq V-W\\ |N[S]|\leq k}} (-1)^{|S|}\binom{n-|N[S]|}{k-|N[S]|}.$$

■

In Lemma 2.3 we omit graphs with K_2 components because each vertex in a K_2 is both a stem and a leaf. We will further this restriction and omit graphs with isolated vertices as these graphs will arise in the next section when considering graphs that are domination equivalent to paths.

In the next lemma we will use Lemma 2.3 to determine a formula for $d(G, n-3)$ for a graph G of order n with no isolated vertices and no K_2 components. Before we begin, we define some graph parameters and subsets. An *r-loop* is an induced r-cycle in G such that all but one vertex has degree two in G. Examples of r-loops can be found in Figure 1; the vertices s_3, v_{11}, and v_{12} form a 3-loop, and the vertices $s_3, v_5, v_6, \ldots, v_{10}$ form a 7-loop. Further, we use the following notation, all with respect to a graph G:

- T_r: the set of vertices of degree r in G *which are not stems.*
- ω: the number of stems in G.
- $W = \{s_1, s_2, \ldots, s_\omega\}$: the set of all stems in G.
- S_i: the set of leaves attached to stem s_i.
- $\mathcal{L}_r$: the set of r-loops.
- $\mathcal{L}_r^i$: the set of r-loop subgraphs which contain stem s_i.
- $\mathcal{C}_r$: the set of components which are cycles of order r.

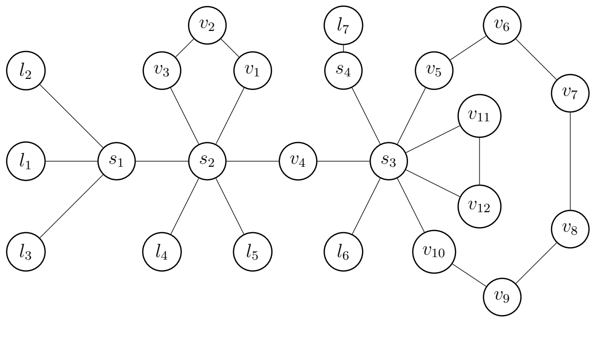


Figure 1: An example of a graph

As an example, consider the graph in Figure 1. The set of stems W is $\{s_1, s_2, s_3, s_4\}$ and $\omega = 4$. There are no degree zero vertices so $T_0 = \emptyset$. There are seven degree one vertices (leaves), none of which are stems, so $T_1 = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7\}$. There are 13 degree two vertices, one of which (s_4) is a stem, so $T_2 = \{v_i | 1 \leq i \leq 12\}$. The sets of leaves are $S_1 = \{l_1, l_2, l_3\}$, $S_2 = \{l_4, l_5\}$, $S_3 = \{l_6\}$, and $S_4 = \{l_7\}$.

Theorem 2.4 *For a graph G of order n where G has no isolated vertices and no K_2 components,*

$$d(G, n-3) = \binom{n}{3} - \left(|T_1| \cdot (n-2) + |T_2| - \sum_{i=1}^{\omega} \binom{|S_i|}{2} - |\mathcal{L}_3| - 2|\mathcal{C}_3|\right). \qquad (1)$$

Proof We will determine and sum $(-1)^{|S|}\binom{n-|N[S]|}{k-|N[S]|}$ for each $S \subseteq V - W$ such that $|N[S]| \leq 3$ in order to use Lemma 2.3. If $|N[S]| \leq 3$ then $|S| \leq 3$, thus we will consider the cases $|S| = 0, 1, 2$ and 3. Note the case $|S| = 0$ is trivial and yields the summand $\binom{n}{3}$.

If $|S| = 1$, then $S = \{v\}$ where v non-stem vertex and $\deg(v) \leq 2$. Hence v is either in T_0, T_1 or T_2. As G has no isolated vertices, the case $|S| = 1$ yields the summand $-(|T_1| \cdot (n-2) + |T_2|)$.

If $|S| = 2$, let $S = \{u, v\}$ for non-stem vertices u and v. Note $N[S] = 2$ or 3. However, if $N[S] = 2$ then u and v are either isolated or form a K_2, which would contradict our assumptions. Hence $N[S] = 3$ and $N[S] = \{u, v, x\}$. There must be at least one edge from either u or v to x. Without loss of generality, let $x \in N[u]$. Furthermore neither u nor v can be isolated. Hence $N[S]$ induces one of the subgraphs shown in Figure 2.

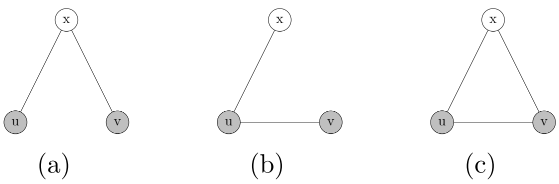


Figure 2: Every subgraph induced by $N[S]$ when $|S| = 2$

Note that the degree of u and v in G is the same as their degree in the subgraph induced by their closed neighbourhood. In Figure 2 (a), u and v are leaves in G on the same stem. We can enumerate all such subgraphs in G by $\sum_{i=1}^{\omega}\binom{|S_i|}{2}$. In Figure 2 (b), v is a leaf with stem u, which contradicts u and v being non-stem vertices. In Figure 2 (c), u and v are in an induced 3-cycle; as we do not know the degree of x in G, this subgraph is either a 3-loop or 3-cycle component. We can enumerate all such subgraphs in G by $|\mathcal{L}_3| + 3|\mathcal{C}_3|$. The summand for the case $|S| = 2$ is $\sum_{i=1}^{\omega}\binom{|S_i|}{2} + |\mathcal{L}_3| + 3|\mathcal{C}_3|$.

If $|S| = 3$, then $|N[S]| = 3$ and $N[S] = S$. No vertex in S is isolated. Now suppose $v \in S$ was a leaf. Then as $N(v) \subseteq N[S] = S$, then the corresponding stem of v is also in S which contradicts $S \subseteq V - W$. Therefore $N[S]$ is a 3-cycle component and the summand for the case $|S| = 3$ is $-|\mathcal{C}_3|$.

Taking the sum of each of the cases gives use the right hand side of equation (1). ■

We now introduce a new collection of graphs that play a pivotal role in the next section.

Definition 2.5 *Let $\mathcal{G}_k$ denote the set of all graph G with the property that every vertex is either a stem or has degree at most k.*

Our focus will be when $k = 2$. Two familiar families of graphs in $\mathcal{G}_2$ are paths and cycles. Another example was shown in Figure 1. For a graph G of order n without isolated vertices, clearly $G \in \mathcal{G}_2$ if and only if $\omega + |T_1| + |T_2| = n$. Note if $G \in \mathcal{G}_2$ and G has an r-loop then the one vertex of the r-loop which is not degree two must be a stem.

In the next lemma we will extend our work of Theorem 2.4 and determine $d(G, n-4)$ for a graph $G \in \mathcal{G}_2$ of order n with no isolated vertices and no K_2 components (we will make essential use of this in the next section). The proof is similar to that of Theorem 2.4, but more involved (details can be found in [8]). Before we begin, we will partition T_2 into subsets based on the number of neighbouring stems.

- V_0: The subset of T_2 with no adjacent stems.
- V_1^i: The subset of T_2 adjacent to exactly one stem, stem i.
- V_2^{ij}: The subset of T_2 adjacent to exactly two stems, stems i and j (this set is denoted V_2 when G only has two stems).

Theorem 2.6 *Let $G \in \mathcal{G}_2$ be a graph of order n with no isolated vertices and no K_2 components. Then*

$$d(G, n-4) = \binom{n}{4} - |T_1|\binom{n-2}{2} - |T_2|(n-3) + \alpha_1 + \alpha_2 + \alpha_3,$$

where

$$\begin{aligned}
\alpha_1 &= \sum_{i=1}^{\omega} \tbinom{|S_i|}{2}(n-3) + |\mathcal{L}_3|(n-4) + |\mathcal{C}_3|(2n-9), \\
\alpha_2 &= \sum_{i=1}^{\omega} \tfrac{|S_i|}{2}(|T_1| - |S_i|) + \sum_{i \neq j} |V_2^{ij}|(|S_i| + |S_j|) + \sum_{i=1}^{\omega} |V_1^i||S_i|, \text{ and} \\
\alpha_3 &= |V_0| + \sum_{i=1}^{\omega} \tfrac{|V_1^i|}{2} + \sum_{i \neq j} \tbinom{|V_2^{ij}|}{2} - \sum_{i=1}^{\omega} \tbinom{|S_i|}{3} - \sum_{i=1}^{\omega} |\mathcal{L}_3^i||S_i| - |\mathcal{C}_4|.
\end{aligned}$$

■

We now turn specifically to the coefficients of the domination polynomials of paths, with interest in the top four.

Theorem 2.7 [4]

(i) For every $n \geq 2$, $d(P_n, n-1) = n$.

(ii) For every $n \geq 3$, $d(P_n, n-2) = \binom{n}{2} - 2$.

(iii) For every $n \geq 4$, $d(P_n, n-3) = \binom{n}{3} - (3n-8)$.

(iv) For every $n \geq 5$, $d(P_n, n-4) = \binom{n}{4} - (2n^2 - 13n + 20)$.

We will also need to know when -2 is root of a domination polynomial, as this plays a role in our characterization of graphs that are domination equivalent to paths. The domination polynomial is multiplicative across components (that is, $D(G_1 \cup \cdots G_m, x) = D(G_1, x)D(G_2, x) \cdots D(G_m, x))$ and $D(K_2, -2) = 0$. Therefore, $D(G, -2) \neq 0$ implies G has no K_2 components. This observation is vital to determining the domination equivalence classes of paths.

It is well known (and easy to see [6]) that P_n satisfies the recurrence

$$D(G_n, x) = x(D(G_{n-1}, x) + D(G_{n-2}, x) + D(G_{n-3}, x))$$

for $n \geq 3$ (other families, such as the cycles C_n, do as well). We show that for $n \geq 9$, -2 is *never* a root of $D(P_m, x)$, by showing that, given a sequence of graphs satisfying such a recurrence if the $D(G_i, -2)$ is non-zero, increasing in absolute value and of alternating sign for the four consecutive indices $i = N, N+1, N+2, N+3$, then $D(G_m, -2) \neq 0$ for $m \geq N$. This allows us to show that any $G \sim_{\mathcal{D}} G_m$ does not have any K_2 components, since $D(K_2, -2) = 0$.

Lemma 2.8 *Fix $k \geq 1$. Suppose we have a sequence of graphs $(G_n)_{n \geq 1}$ that satisfies the recurrence*

$$D(G_n, x) = x(D(G_{n-1}, x) + D(G_{n-2}, x) + D(G_{n-3}, x))$$

for $n \geq 3$. If for some $N \in \mathbb{N}$

$$0 < |D(G_N, -2)| < |D(G_{N+1}, -2)| < |D(G_{N+2}, -2)| < |D(G_{N+3}, -2)|$$

and $D(G_N, -2)$, $D(G_{N+1}, -2)$, $D(G_{N+2}, -2)$, $D(G_{N+3}, -2)$ have alternating sign, then $D(G_m, -2) \neq 0$ for $m \geq N$.

Proof Substituting $x = -2$ into the recurrence, we find that

$$D(G_n, -2) = -2(D(G_{n-1}, -2) + D(G_{n-2}, -2) + D(G_{n-3}, -2)).$$

By induction we will show that $D(G_n, -2)$ is increasing in absolute value and alternating in sign for all $n \geq N+3$. As $|D(G_N, -2)| > 0$ then this will imply $D(G_n, -2) \neq 0$ for $m \geq N$. By the hypotheses, the result is true for $n = N$.

Suppose for some $k \geq N$, $D(G_{N+3}, -2), \ldots, D(G_k, -2)$ alternate in signs and increase in absolute value. Then we will first show $D(G_{k+1}, -2)$ has opposite sign to $D(G_k, -2)$. First assume $D(G_k, -2) > 0$ (a similar argument holds when $D(G_k, -2) < 0$). Then $D(G_{k-1}, -2) < 0$ and $D(G_{k-2}, -2) > 0$. By our induction assumption, the absolute value of $D(G_m, -2)$ is strictly increasing for $N + 3 \leq m \leq k$. Therefore

$$D(G_k, -2) + D(G_{k-1}, -2) + D(G_{k-2}, -2) > 0.$$

When we multiply the left side of the above inequality by -2, from the recurrence relation for $D(G_k, -2)$ we will obtain $D(G_{k+1}, -2)$. The signs continue to alternate.

We now show that $|D(G_{k+1}, -2)| > |D(G_k, -2)|$. We consider the two cases: $D(G_k, -2) > 0$ and $D(G_k, -2) < 0$. If $D(G_k, -2) > 0$ then $D(G_{k-1}, -2) < 0$, $D(G_{k-2}, -2) > 0$, and $D(G_{k-3}, -2) < 0$. By our induction assumption, the absolute value of $D(G_m, -2)$ is strictly increasing. Therefore

$$D(G_{k-1}, -2) + D(G_{k-2}, -2) + D(G_{k-3}, -2) < D(G_{k-1}, -2) + D(G_{k-2}, -2) < 0.$$

By the recurrence relation for $D(G_k, -2)$ we deduce

$$\begin{aligned} D(G_k, -2) &= -2(D(G_{k-1}, -2) + D(G_{k-2}, -2) + D(G_{k-3}, -2)) \\ &> -2(D(G_{k-1}, -2) + D(G_{k-2}, -2)). \end{aligned}$$

Since $D(G_{k+1}, -2) < 0$, we have

$$\begin{aligned} |D(G_{k+1}, -2)| &= -D(G_{k+1}, -2) \\ &= -(-2(D(G_k, -2) + D(G_{k-1}, -2) + D(G_{k-2}, -2))) \\ &= 2D(G_k, -2) + 2D(G_{k-1}, -2) + 2D(G_{k-2}, -2) \\ &> D(G_k, -2) - 2(D(G_{k-1}, -2) + D(G_{k-2}, -2)) \\ &\quad +2D(G_{k-1}, -2) + 2D(G_{k-2}, -2) \\ &= D(G_k, -2) \\ &= |D(G_k, -2)|. \end{aligned}$$

Therefore $|D(G_{k+1}, -2)| > |D(G_k, -2)|$ and our claim is true. A similar argument holds when $D(G_k, -2) < 0$. ∎

Using the base cases $D(P_1, x) = x$, $D(P_2, x) = x^2+2x$ and $D(P_3, x) = x^3+3x^2+x$, calculations will show that $D(P_i, -2) \neq 0$ for $9 \leq i \leq 12$, $D(P_{13}, -2) = -32$, $D(P_{14}, -2) = 64$ and $D(P_{15}, -2) = -96$. From this and the previous Lemma, we conclude:

Corollary 2.9 *If $n \geq 9$, -2 is not a zero of $D(P_n, x)$.* ∎

3 Equivalence Classes of Paths

We have done the necessary background work to proceed onto our characterization of those graphs that are domination equivalent to path P_n.

We first observe that any graph $G \sim_{\mathcal{D}} P_n$ does not have any 4-cycle components. This follows from the multiplicativity of the domination polynomial over components and the following two lemmas.

Lemma 3.1 [2] *If n is a positive integer, then*

$$D(C_n, -1) = \begin{cases} 3 & n \equiv 0 \mod 4 \\ -1 & \text{otherwise} \end{cases}.$$

Lemma 3.2 [3] *Let F be a forest. Then $D(F, -1) \in \{1, -1\}$ and therefore $D(P_n, -1) \in \{1, -1\}$.*

Corollary 3.3 *If a graph G is $\mathcal{D}$-equivalent to P_n with a component H, then $|D(H, -1)| = 1$, and so G does not have any 4-cycle components.*

In the next lemma we use the results from Theorem 2.4, Theorem 2.6 and Corollary 2.9 to show, for large enough n, that any graph $G \sim_{\mathcal{D}} P_n$ must be the disjoint union of one path and some number of cycles.

Lemma 3.4 *For $n \geq 9$, if $G \sim_{\mathcal{D}} P_n$ then $G = H \cup C$ where $H \in \{P_k, \widetilde{P_k}\}$ and C is a disjoint union of cycles.*

Proof Let G be a graph with $D(G, x) = D(P_n, x)$ where $n \geq 9$. Then $d(G, i) = d(P_n, i)$ for all i. Furthermore, by Theorem 2.7 we have:

(i) $d(G, n-1) = n$.

(ii) $d(G, n-2) = \binom{n}{2} - 2$.

(iii) $d(G, n-3) = \binom{n}{3} - (3n - 8)$.

(iv) $d(G, n-4) = \binom{n}{4} - (2n^2 - 13n + 20)$.

By Theorem 2.1 the number of isolated vertices in G is $n - d(G, n-1) = 0$. By Corollary 2.9, $D(G, -2) \neq 0$, and again by Theorem 2.1 the number of leaves is $|T_1| = \binom{n}{2} - d(G, n-2) = 2$. By Theorem 2.4, since G has no K_2 components and no isolated vertices,

$$d(G, n-3) = \binom{n}{3} - \left(|T_1| \cdot (n-2) + |T_2| - \sum_{i=1}^{\omega} \binom{|S_i|}{2} - |\mathcal{L}_3| - 2|\mathcal{C}_3| \right).$$

Furthermore, from $|T_1| = 2$ and (iii) we know

$$n - 4 = |T_2| - \sum_{i=1}^{\omega} \binom{|S_i|}{2} - |\mathcal{L}_3| - 2|\mathcal{C}_3|.$$

By rearranging for $|T_2|$ we get

$$|T_2| = n - 4 + \sum_{i=1}^{\omega} \binom{|S_i|}{2} + |\mathcal{L}_3| + 2|\mathcal{C}_3|.$$

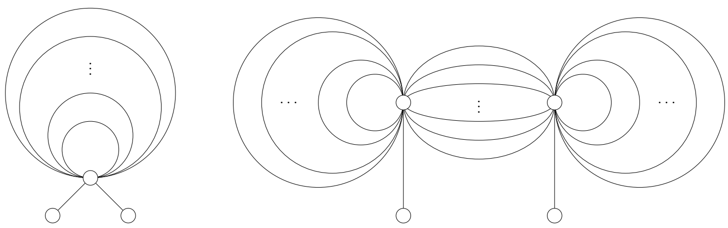
We claim that for G, $|\mathcal{L}_3| = 0$, $|\mathcal{C}_3| = 0$ and $G \in \mathcal{G}_2$ (recall that $\mathcal{G}_2$ is the set of all graphs with maximum non-stem degree two, and that ω is the number of stems in G). We will show our claim is true using the fact that $n = \omega + \sum_{i \in \mathbb{N}} |T_i|$ so $n \geq \omega + |T_1| + |T_2|$. Since $|T_1| = 2$, it follows that $T_2 \leq n - (2 + \omega)$. Also, if $n = \omega + |T_1| + |T_2|$ then $G \in \mathcal{G}_2$. As G has two leaves, it either has one or two stems. We prove the claim for the case where G has two stems (the case where G has one stem is simpler and is omitted).

Suppose G has two stems. Then $\omega = 2$, $|S_1| = 1$, $|S_2| = 1$, and $|T_2| \leq n - 4$. Thus

$$|T_2| = n - 4 + 0 + |\mathcal{L}_3| + 2|\mathcal{C}_3|.$$

Since $|\mathcal{L}_3| + 2|\mathcal{C}_3| \geq 0$, it follows that $|T_2| \geq n - 4$, and therefore $|T_2| = n - 4$. Furthermore, $|\mathcal{L}_3| + 2|\mathcal{C}_3| = 0$ so $|\mathcal{L}_3| = 0$ and $|\mathcal{C}_3| = 0$. As $\omega + |T_1| + |T_2| = n$, we have $G \in \mathcal{G}_2$.

For a graph in $\mathcal{G}_2$, a T_2 vertex can only be adjacent to stems or other T_2 vertices. Therefore the T_2 vertices form paths between stems, r-loops, and disjoint cycles in $\mathcal{G}_2$ graphs. As $G \in \mathcal{G}_2$, G will be the disjoint union of some number of cycles and a subgraph H which has one of the two forms shown in Figure 3. (These two forms were noted for graphs domination equivalent to paths in [1], but we shall need more than was used there about the types of subgraphs present to limit the possibilities).



(a) One stem (b) Two stems

Figure 3: The two possible structures of H

Recall from Section 2, we partitioned T_2 into subsets based on the number of neighbouring stems.

- V_0: The subset of T_2 with no adjacent stems.
- V_1^i: The subset of T_2 adjacent to exactly one stem, stem i.
- V_2^{ij}: The subset of T_2 adjacent to exactly two stems, stems i and j (this set is denoted V_2 when G only has two stems).

We wish to show that the subgraph H of G is either a path or a path with an edge between its stems. This is equivalent of showing H has two stems with either one path between them or two, with one being an edge, and no r-loops. If G has exactly two stems, and no r-loops, then the number of paths between the stems is exactly $\frac{1}{2}(|V_1^1|+|V_1^2|)+|V_2|$. Furthermore, if $|V_1^1| \leq 1$ and $|V_1^2| \leq 1$ then H has no r-loops. Therefore it is sufficient to show H has two stems and either $|V_1^1| = |V_1^2| = 0$ and $|V_2| = 1$, or $|V_1^1| = |V_1^2| = 1$ and $|V_2| = 0$. We will show this by examining $d(G, n-4)$.

By Theorem 2.6, as G has no K_2 components, no isolated vertices, and $G \in \mathcal{G}_2$, we have that

$$d(G, n-4) = \binom{n}{4} - |T_1|\binom{n-2}{2} - |T_2|(n-3) + \alpha_1 + \alpha_2 + \alpha_3,$$

where

$$\begin{aligned}
\alpha_1 &= \sum_{i=1}^{\omega} \tbinom{|S_i|}{2}(n-3) + |\mathcal{L}_3|(n-4) + |\mathcal{C}_3|(2n-9), \\
\alpha_2 &= \sum_{i=1}^{\omega} \tfrac{|S_i|}{2}(|T_1| - |S_i|) + \sum_{i \neq j} |V_2^{ij}|(|S_i| + |S_j|) + \sum_{i=1}^{\omega} |V_1^i||S_i|, \text{ and} \\
\alpha_3 &= |V_0| + \sum_{i=1}^{\omega} \tfrac{|V_1^i|}{2} + \sum_{i \neq j} \tbinom{|V_2^{ij}|}{2} - \sum_{i=1}^{\omega} \tbinom{|S_i|}{3} - \sum_{i=1}^{\omega} |\mathcal{L}_3^i||S_i| - |\mathcal{C}_4|.
\end{aligned}$$

Since $|\mathcal{L}_3| = 0$, $|\mathcal{L}_3^i| = 0$ for every i. Furthermore, $|\mathcal{C}_3| = 0$, and by Corollary 3.3 $|\mathcal{C}_4| = 0$. We again consider the two cases, where G has one stem and G has two stems. Note that $|T_2| = |V_0| + \sum_{i=1}^{\omega} |V_1^i| + \sum_{i \neq j} |V_2^{ij}|$.

Again there are two cases to consider, depending on whether G has one or two stems. We focus on the case of two stems, leaving the case of one stem to the reader.

Suppose that G has two stems. We find that $\omega = 2$,$|S_1| = 1$, $|S_2| = 1$, and $|T_2| = n-4$. As there are only two stems, let the set of T_2 vertices which are adjacent to both be denoted V_2. Furthermore $|V_0| + |V_1^1| + |V_1^2| + |V_2| = |T_2| = n-4$. Using this we can simplify α_1, α_2, and α_3 to be

$$\alpha_1 = 0, \quad \alpha_2 = 1 + 2|V_2| + \sum_{i=1}^{2} |V_1^i|, \quad \text{and} \quad \alpha_3 = |V_0| + \sum_{i=1}^{2} \tfrac{|V_1^i|}{2} + \tbinom{|V_2|}{2}.$$

Since $|V_0| + |V_1^1| + |V_1^2| + |V_2| = n-4$, it follows that $\alpha_1 + \alpha_2 + \alpha_3 = n - 3 + \sum_{i=1}^{2} \frac{|V_1^i|}{2} + |V_2| + \binom{|V_2|}{2}$ and

$$d(G, n-4) = \binom{n}{4} - \left(2n^2 - 13n + 21 - \sum_{i=1}^{2} \frac{|V_1^i|}{2} - |V_2| - \binom{|V_2|}{2}\right).$$

However by item (iv), $d(G, n-4) = \binom{n}{4} - (2n^2 - 13n + 20)$ and therefore

$$\sum_{i=1}^{2} \frac{|V_1^i|}{2} + |V_2| + \binom{|V_2|}{2} = 1.$$

As each summand is non-negative and $|V_2| + \binom{|V_2|}{2}$ is a non-negative integer, the only solutions to this are $\sum_{i=1}^{2} \frac{|V_1^i|}{2} = 1, |V_2| = 0$ or $\sum_{i=1}^{2} \frac{|V_1^i|}{2} = 0, |V_2| = 1$.

In the case $\sum_{i=1}^{2} \frac{|V_1^i|}{2} = 1, |V_2| = 0$, then because G has no K_2 components and there are no vertices adjacent to both stems ($|V_2| = 0$) it follows that $|V_1^i| \geq 1$ for each i. Furthermore, because $\sum_{i=1}^{2} \frac{|V_1^i|}{2} = 1$, we have $|V_1^1| = |V_1^2| = 1$, and $|V_2| = 0$. In the case $\sum_{i=1}^{2} \frac{|V_1^i|}{2} = 0, |V_2| = 1$ since $|V_1^i| \geq 0$ for each i, then we have $|V_1^1| = |V_1^2| = 0$, and $|V_2| = 1$. Both cases result in H having one path between its two stems and no r-loops. As we do not specify the degree of the stems, this allows for the possibility of an edge to be between them, and proves our result. ■

Let $n \in \mathbb{Z}$, $n \neq 0$ and p be a prime factor of n. Then there is a nonnegative integer a such that $p^a \mid n$ but $p^{a+1} \nmid n$; we set $\mathrm{ord}_p(n) = a$. In other words, a is the exponent of p in the prime decomposition of n. Furthermore, define $\mathrm{ord}_p(0) = 0$. In a similar method used by Akbari and Oboudi [2] we will determine $\mathrm{ord}_3(D(P_n, -3))$ in order to show that if a graph G is $\mathcal{D}$-equivalent to a path, then G is the disjoint union of a path and at most two cycles.

Lemma 3.5 *[2] For $n \in \mathbb{N}$,*

$$\mathrm{ord}_3(D(C_n, -3)) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & n \equiv 0 \mod 3 \\ \lceil \frac{n}{3} \rceil \text{ or } \lceil \frac{n}{3} \rceil + 1 & n \equiv 1 \mod 3 \\ \lceil \frac{n}{3} \rceil & n \equiv 2 \mod 3 \end{cases}.$$

Using a similar approach, we can prove the following statement.

Lemma 3.6 *For $n \in \mathbb{N}$*

$$\mathrm{ord}_3(D(P_n, -3)) = \begin{cases} \lceil \frac{n}{3} \rceil & n \equiv 0 \mod 3 \\ \lceil \frac{n}{3} \rceil & n \equiv 1 \mod 3 \\ \lceil \frac{n}{3} \rceil \text{ or } \lceil \frac{n}{3} \rceil + 1 & n \equiv 2 \mod 3 \end{cases}.$$

■

The next straightforward lemma gives the domination numbers of paths and cycles, and will help us to restrict the number of disjoint cycles in G if $G \sim_{\mathcal{D}} P_n$.

Lemma 3.7 *For every $n \geq 1$, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$, and for all $n \geq 3$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$.* ■

From Lemma 3.4 we know that if G is $\mathcal{D}$-equivalent to P_n then G is the disjoint union of H and some number of cycles where $H \in \{P_k, \widetilde{P_k}\}$ and $k \leq n$. In the next lemma we will show the number of cycles is at most two.

Lemma 3.8 *For $n \in \mathbb{N}$ For $n \geq 9$, if $G \sim_{\mathcal{D}} P_n$ then $G = H \cup C$ where $H \in \{P_k, \widetilde{P_k}\}$, $k \leq n$, and C is a disjoint union of at most two cycles.*

Proof Let $n = 3m + r$ and G be a graph with $D(G, x) = D(P_{3m+r}, x)$ where $r \in \{0, 1, 2\}$. By Lemma 3.4,

$$G = P_{3m_1+r_1} \cup C_{3m_2+r_2} \cup \ldots \cup C_{3m_k+r_k},$$

where $3m + r = \sum_{i=1}^{k}(3m_i + r_i)$ and for each i, $r_i \in \{0, 1, 2\}$. In this proof we will begin by restricting the number of non-zero r_i, and then restrict the number of r_i which are zero. By Lemma 3.7 we know

$$\gamma(G) = \sum_{i=1}^{k} \left\lceil \frac{3m_i + r_i}{3} \right\rceil = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} \left\lceil \frac{r_i}{3} \right\rceil.$$

Since $3m + r = \sum\limits_{i=1}^{k}(3m_i + r_i)$, it follows that $\sum\limits_{i=1}^{k} m_i = m + \frac{r}{3} - \sum\limits_{i=1}^{k} \frac{r_i}{3}$ and

$$\gamma(G) = m + \frac{r}{3} + \sum_{i=1}^{k} \left(\left\lceil \frac{r_i}{3} \right\rceil - \frac{r_i}{3} \right).$$

Since $\gamma(G) = \gamma(P_{3m+r})$ and $\gamma(P_{3m+r}) = \lceil \frac{3m+r}{3} \rceil = m + \lceil \frac{r}{3} \rceil$, then we have

$$\sum_{i=1}^{k} \left(\left\lceil \frac{r_i}{3} \right\rceil - \frac{r_i}{3} \right) = \left\lceil \frac{r}{3} \right\rceil - \frac{r}{3}.$$

Let $f(r_i) = \lceil \frac{r_i}{3} \rceil - \frac{r_i}{3}$. Because $r_i \in \{0, 1, 2\}$, it follows that $f(0) = 0, f(1) = \frac{2}{3}$, and $f(2) = \frac{1}{3}$. Now consider the number of $r_i \neq 0$ for the cases $r = 0, 1,$ and 2:

- If $r = 0$ then $\sum f(r_i) = 0$ and no $r_i \neq 0$.
- If $r = 1$ then $\sum f(r_i) = \frac{2}{3}$ and at most two $r_i \neq 0$.
- If $r = 2$ then $\sum f(r_i) = \frac{1}{3}$ and at most one $r_i \neq 0$.

We now count those r_i with $r_i = 0$. For a graph H, let $g(H) = \mathrm{ord}_3(D(H, -3)) - \gamma(H)$. Using Lemma 3.5, Lemma 3.6 and the fact that $\gamma(C_{3m+r}) = \gamma(P_{3m+r}) = \lceil \frac{3m+r}{3} \rceil$ we can obtain $g(P_{3m+r})$ and $g(C_{3m+r})$:

$$g(P_{3m+r}) = \begin{cases} 0 & r = 0 \\ 0 & r = 1 \\ 0 \text{ or } 1 & r = 2 \end{cases}, \quad g(C_{3m+r}) = \begin{cases} 1 & r = 0 \\ 0 \text{ or } 1 & r = 1 \\ 0 & r = 2 \end{cases}.$$

For simplicity we will denote $g(P_{3m+r})$ and $g(C_{3m+r})$ with $g_P(r)$ and $g_C(r)$. Because G is the disjoint union of a path and cycles, $\gamma(G)$ is just the sum of domination numbers of each of the paths and cycles. Similarly $\mathrm{ord}_3(D(G, -3))$ is just the sum of the orders of each of its components. From this we get the following equality:

$$g_P(r) = g_P(r_1) + \sum_{i=2}^{k} g_C(r_i).$$

Now consider the number of $r_i = 0$ for the cases $r = 0, 1,$ and 2.

- If $r = 0$ then $g_P(r_1) + \sum_{i=2}^{k} g_C(r_i) = 0$ and no $r_i = 0$ for $i \geq 2$.
- If $r = 1$ then $g_P(r_1) + \sum_{i=2}^{k} g_C(r_i) = 0$ and no $r_i = 0$ for $i \geq 2$.
- If $r = 2$ then $g_P(r_1) + \sum_{i=2}^{k} g_C(r_i) = 0$ or 1 and at most one $r_i = 0$ for $i \geq 2$.

Together with the three cases counting the number of nonzero r_i, we can easily see there are at most two r_i for $i \geq 2$. Therefore there are at most two cycle components. ■

We have narrowed the number of cycle components to two in graphs which are $\mathcal{D}$-equivalent to paths. From Lemma 3.1 we know $D(C_n, -1)$. We will now evaluate $D'(C_n, -1)$, $D''(C_n, -1)$, and $D'''(C_n, -1)$ as well as $D(P_n, -1)$, $D'(P_n, -1)$, $D''(P_n, -1)$, and $D'''(P_n, -1)$.

Lemma 3.9 *[2] For $n \in \mathbb{N}$*

$$D'(C_n, -1) = \begin{cases} -n, & n \equiv 0 \mod 4 \\ n, & n \equiv 1 \mod 4 \\ 0, & n \equiv 2 \mod 4 \\ 0, & n \equiv 3 \mod 4 \end{cases}.$$

Lemma 3.10 *[2] For $n \in \mathbb{N}$*

$$D''(C_n, -1) = \begin{cases} \frac{1}{4} n\,(n-4)\,, & n \equiv 0 \mod 4 \\ -\frac{1}{2} n\,(n-1)\,, & n \equiv 1 \mod 4 \\ \frac{1}{4} n\,(n+2)\,, & n \equiv 2 \mod 4 \\ 0, & n \equiv 3 \mod 4 \end{cases}.$$

The proofs of Lemmas 3.11 to 3.15 are similar and are left to the reader.

Lemma 3.11 *For $n \in \mathbb{N}$*

$$D'''(C_n, -1) = \begin{cases} -\frac{1}{16}n^3 + \frac{3}{4}n^2 - 2n, & n \equiv 0 \mod 4 \\ \frac{3}{16}n^3 - \frac{9}{8}n^2 + \frac{15}{16}n, & n \equiv 1 \mod 4 \\ -\frac{3}{16}n^3 + \frac{3}{4}n, & n \equiv 2 \mod 4 \\ \frac{1}{16}n^3 + \frac{3}{8}n^2 + \frac{5}{16}n, & n \equiv 3 \mod 4 \end{cases}.$$

■

Lemma 3.12 *For $n \in \mathbb{N}$*

$$D(P_n, -1) = \begin{cases} 1, & n \equiv 0 \mod 4 \\ -1, & n \equiv 1 \mod 4 \\ -1, & n \equiv 2 \mod 4 \\ 1, & n \equiv 3 \mod 4 \end{cases}.$$

■

Lemma 3.13 *For $n \in \mathbb{N}$*

$$D'(P_n, -1) = \begin{cases} 0, & n \equiv 0 \mod 4 \\ \frac{n+1}{2}, & n \equiv 1 \mod 4 \\ 0, & n \equiv 2 \mod 4 \\ -\frac{n+1}{2}, & n \equiv 3 \mod 4 \end{cases}.$$

■

Lemma 3.14 *For $n \in \mathbb{N}$*

$$D''(P_n, -1) = \begin{cases} -\frac{1}{8}n(n+4), & n \equiv 0 \mod 4 \\ -\frac{1}{8}(n-1)^2, & n \equiv 1 \mod 4 \\ \frac{1}{8}(n+2)^2, & n \equiv 2 \mod 4 \\ \frac{1}{8}(n-3)(n+1), & n \equiv 3 \mod 4 \end{cases}.$$

■

Lemma 3.15 *For $n \in \mathbb{N}$*

$$D'''(P_n, -1) = \begin{cases} \frac{1}{16}n^3 - n, & n \equiv 0 \mod 4 \\ -\frac{9}{16}n^2 + \frac{3}{8}n + \frac{3}{16}, & n \equiv 1 \mod 4 \\ -\frac{1}{16}n^3 + \frac{1}{4}n, & n \equiv 2 \mod 4 \\ \frac{9}{16}n^2 + \frac{3}{8}n - \frac{3}{16}, & n \equiv 3 \mod 4 \end{cases}.$$

■

We now present our main result, the equivalence class of paths. The next theorem will show $[P_n] = \{P_n, \widetilde{P_n}\}$ for $n \geq 9$. However, first we will discuss the $[P_n]$ for $n \leq 8$ as shown in Table 1. For $n \neq 4, 7, 8$, $[P_n] = \{P_n, \widetilde{P_n}\}$ (P_n and $\widetilde{P_n}$ are isomorphic when $n \leq 3$). Note that $D(P_n, -2) = 0$ when $n = 4, 7$ and 8, and in fact P_4, P_7, and P_8 are each $\mathcal{D}$-equivalent to graphs with K_2 components. Note that $[P_7]$ and $[P_8]$ each have four graphs. Hhowever, two of the graphs arise from the other two by adding in an irrelevant edge (an edge between two stems).

Theorem 3.16 *Let F_i ($i \geq 3$) denote the graph that consists of a cycle C_i with a pendant edge (that is, one of the vertices v_i of the cycle is attached to a new vertex of degree 1), and let H_i denote the graph formed from F_i and K_2 by adding in an edge between the stem in F_i and a vertex of K_2. Then*

- $[P_n] = \{P_n\}$ *if* $n \leq 3$,
- $[P_4] = \{P_4, 2P_2\}$,
- $[P_n] = \{P_n, \widetilde{P_n}, F_{n-3} \cup K_2, H_{n-3}\}$ *for* $n = 7,\ 8$, *and*
- $[P_n] = \{P_n, \widetilde{P_n}\}$ *otherwise.*

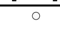
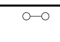
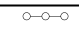
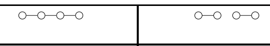
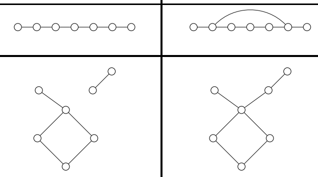
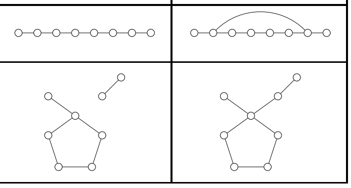
n	$D(P_n, x)$	$[P_n]$
1	x	
2	$x^2 + 2x$	
3	$x^3 + 3x^2 + x$	
4	$x^4 + 4x^3 + 4x^2$	
5	$x^5 + 5x^4 + 8x^3 + 3x^2$	
6	$x^6 + 6x^5 + 13x^4 + 10x^3 + x^2$	
7	$x^7 + 7x^6 + 19x^5 + 22x^4 + 8x^3$	
8	$x^8 + 8x^7 + 26x^6 + 40x^5 + 26x^4 + 4x^3$	

Table 1: The domination equivalence classes for paths up to length eight

Proof From previous remarks, it suffices to only consider $n \geq 9$. Let G be a graph which is $\mathcal{D}$-equivalent to P_n. By Lemma 3.8, $G = H \cup C$ where $H \in \{P_{n_1}, \widetilde{P_{n_1}}\}$ with $n_1 \leq n$ and C is the disjoint union of at most two cycles. Therefore either $G = H$, $G = H \cup C_{n_2}$ or $G = H \cup C_{n_2} \cup C_{n_3}$. It is sufficient to show the latter two cases always yield a contradiction. We will do so by evaluating $D(P_n, -1), \ldots, D'''(P_n, -1)$ and $D(G, -1), \ldots, D'''(G, -1)$ for all cases $n_1, n_2, n_3 \equiv 0, 1, 2, 3 \pmod 4$ and showing each case contradicts $D(G, x) = D(P_n, x)$. By Lemma 3.1 there can be no cycles with order congruent to 0 (mod 4). Tthere are 12 cases to consider for one cycle (without loss of generality $n_1 \equiv 0, 1, 2, 3 \pmod 4$ and $n_2 \equiv 0, 1, 2, 3 \pmod 4$)) and similarly 24 cases for two cycles. In each case, we can derive a contradiction.

We begin with the situation of only one cycle, so that, without loss of generality, $n_3 = 0$ and $G = P_{n_1} \cup C_{n_2}$. As the domination polynomial is multiplicative across components, taking the first three derivatives of $D(G, x)$ we obtain the following system of equations:

$$D(Gn, -1) = D(P_{n_1}, -1) D(C_{n_2}, -1) \qquad (PC0)$$

$$D'(G, -1) = D'(P_{n_1}, -1) D(C_{n_2}, -1) + D(P_{n_1}, -1) D'(C_{n_2}, -1) \qquad (PC1)$$

$$\begin{aligned} D''(G, -1) = {} & D''(P_{n_1}, -1) D(C_{n_2}, -1) + 2D'(P_{n_1}, -1) D'(C_{n_2}, -1) \\ & + D(P_{n_1}, -1) D''(C_{n_2}, -1) \end{aligned} \qquad (PC2)$$

$$\begin{aligned} D'''(G, -1) = {} & D'''(P_{n_1}, -1) D(C_{n_2}, -1) + 3D''(P_{n_1}, -1) D'(C_{n_2}, -1) \\ & + 3D'(P_{n_1}, -1) D''(C_{n_2}, -1) + D(P_{n_1}, -1) D'''(C_{n_2}, -1) \end{aligned} \qquad (PC3)$$

Of the 12 cases to consider, we will only illustrate the proofs for two of them.

Case 2: $n_1 \equiv 0,\ n_2 \equiv 2 \pmod 4$
As $n \equiv n_1 + n_2 \pmod 4$, $n \equiv 2 \pmod 4$, and so equation $(PC2)$ reduces to

$$\frac{(n+2)^2}{8} = \frac{n_1(n_1+4)}{8} + \frac{n_2(n_2+2)}{4},$$

and equation $(PC3)$ reduces to

$$-\frac{n^3}{16} + \frac{n}{4} = -\frac{n_1^3}{16} + n_1 - \frac{3n_2^3}{16} + \frac{3n_2}{4}.$$

We will now substitute $n = n_1 + n_2$ into the reduced equation $(PC2)$:

$$\begin{array}{rcl} 0 &=& \frac{1}{8}(n_1+n_2+2)^2 - (\frac{1}{8}n_1(n_1+4) + \frac{1}{4}n_2(n_2+2)) \\ 0 &=& (n_1+n_2+2)^2 - n_1(n_1+4) - 2n_2(n_2+2) \\ 0 &=& n_1^2 + n_2^2 + 4 + 2n_1n_2 + 4n_1 + 4n_2 - n_1^2 - 4n_1 - 2n_2^2 - 4n_2 \\ 0 &=& -n_2^2 + 2n_1n_2 + 4. \end{array}$$

Therefore $n_1 = (n_2^2 - 4)/2n_2$. By substituting this and $n = n_1 + n_2$ into the reduced equation $(PC3)$ and multiplying by n_2 we obtain

$$0 = -\frac{1}{4}n_2^4 - 2n_2^2 + 12.$$

We obtain the solutions $n_2 = -2, 2, -2\sqrt{3}i$ or $2\sqrt{3}i$. Because n_2 is the border of the cycle, we have $n_2 \geq 3$, a contradiction for all four solutions.

Case 7: $n_1 \equiv 2,\ n_2 \equiv 1 \pmod 4$
Since $n \equiv n_1 + n_2 \pmod 4$, we have $n \equiv 3 \pmod 4$, and so equation $(PC1)$ reduces to

$$-\frac{n+1}{2} = -n_2$$

and equation $(PC2)$ reduces to

$$\frac{1}{8}(n-3)(n+1) = -\frac{1}{8}(n_1+2)^2 + \frac{1}{2}n_2(n_2-1).$$

Therefore $n = 2n_2 - 1$. As $n = n_1 + n_2$, it follows that $n_1 = n_2 - 1$. By substituting this into the reduced equation $(PC2)$ and multiplying both sides by 8 we obtain

$$(2n_2 - 4)(2n_2) = -(n_2+1)^2 + 4n_2(n_2-1).$$

Bringing everything to one side and simplifying we are left with

$$(n_2 - 1)^2 = 0.$$

Therefore $n_2 = 1$. However, because $n_2 \geq 3$, this is a contradiction.

Since each of the 12 cases results in a contradiction, G is not a disjoint union of H and one cycle, where $H \in \{P_{n_1}, \widetilde{P_{n_1}}\}$. We will now consider whether G can be a disjoint union of H and two cycles; this yields (without loss of generality) 24 cases, a number of which can be handled quickly, although some are more involved than the others. We only present two of the cases, leaving the rest to the reader (again, details can be found in [8]). Here we have $G = P_{n_1} \cup C_{n_2} \cup C_{n_3}$. In a similar manner to the case with one cycle we obtain a system of equations by taking the first three derivatives of $D(G, x)$.

$$D(G,-1) = D(P_{n_1},-1)D(C_{n_2},-1)D(C_{n_3},-1) \qquad (PCC0)$$

$$\begin{aligned} D'(G,-1) = {} & D'(P_{n_1},-1)D(C_{n_2},-1)D(C_{n_3},-1)+ \\ & D(P_{n_1},-1)D'(C_{n_2},-1)D(C_{n_3},-1) + \\ & D(P_{n_1},-1)D(C_{n_2},-1)D'(C_{n_3},-1) \end{aligned} \qquad (PCC1)$$

$$\begin{aligned} D''(G,-1) = {} & D''(P_{n_1},-1)D(C_{n_2},-1)D(C_{n_3},-1)+ \\ & D(P_{n_1},-1)D''(C_{n_2},-1)D(C_{n_3},-1)+ \\ & D(P_{n_1},-1)D(C_{n_2},-1)D''(C_{n_3},-1)+ \\ & 2D'(P_{n_1},-1)D'(C_{n_2},-1)D(C_{n_3},-1)+ \\ & 2D'(P_{n_1},-1)D(C_{n_2},-1)D'(C_{n_3},-1)+ \\ & 2D(P_{n_1},-1)D'(C_{n_2},-1)D'(C_{n_3},-1) \end{aligned} \qquad (PCC2)$$

$$\begin{aligned} D'''(G,-1) = {} & D'''(P_{n_1},-1)D(C_{n_2},-1)D(C_{n_3},-1)+ \\ & D(P_{n_1},-1)D'''(C_{n_2},-1)D(C_{n_3},-1)+ \\ & D(P_{n_1},-1)D(C_{n_2},-1)D'''(C_{n_3},-1)+ \\ & 3D''(P_{n_1},-1)D'(C_{n_2},-1)D(C_{n_3},-1)+ \\ & 3D'(P_{n_1},-1)D''(C_{n_2},-1)D(C_{n_3},-1)+ \\ & 3D''(P_{n_1},-1)D(C_{n_2},-1)D'(C_{n_3},-1)+ \\ & 3D'(P_{n_1},-1)D(C_{n_2},-1)D''(C_{n_3},-1)+ \\ & 3D(P_{n_1},-1)D''(C_{n_2},-1)D'(C_{n_3},-1)+ \\ & 3D(P_{n_1},-1)D'(C_{n_2},-1)D''(C_{n_3},-1)+ \\ & 6D'(P_{n_1},-1)D'(C_{n_2},-1)D'(C_{n_3},-1) \end{aligned} \qquad (PCC3)$$

Case 2: $n_1 \equiv 0,\ n_2 \equiv 1,\ n_3 \equiv 2 \pmod 4$

As $n \equiv n_1 + n_2 + n_3 \pmod 4$, we have $n \equiv 3 \pmod 4$, and equation $(PCC1)$ reduces to

$$-\frac{n+1}{2} = -n_2$$

and equation $(PCC2)$ reduces to

$$\frac{1}{8}(n-3)(n+1) = -\frac{1}{8}n_1(n_1+4) - \frac{1}{4}n_3(n_3+2) + \frac{1}{2}n_2(n_2-1).$$

Therefore $n = 2n_2 - 1$. Since $n = n_1 + n_2 + n_3$, we have $n_2 = n_1 + n_3 + 1$ and $n = 2n_1 + 2n_3 + 1$. By substituting this into the reduced equation $(PCC2)$ and multiplying both sides by 8 we obtain

$$(2n_1+2n_3-2)(2n_1+2n_3+2) = -n_1(n_1+4)-2n_3(n_3+2)+4(n_1+n_3+1)(n_1+n_3),$$

which simplifies to

$$\begin{aligned} 4(n_1+n_3)^2-4 &= -n_1(n_1+4)-2n_3(n_3+2)+4(n_1+n_3)^2+4(n_1+n_3) \\ -4 &= -n_1^2-4n_1-2n_3^2-4n_3+4n_1+4n_3 \\ 0 &= -n_1^2-2n_3^2+4. \end{aligned}$$

Since $n_3 \geq 3$, there are no solutions, which is a contradiction.

Case 3: $n_1 \equiv 0,\ n_2 \equiv 2,\ n_3 \equiv 2 \pmod 4$

As $n \equiv n_1+n_2+n_3 \pmod 4$, we have $n \equiv 0 \pmod 4$, and so equation $(PCC2)$ reduces to

$$-\frac{1}{8}n(n+4) = -\frac{1}{8}n_1(n_1+4)-\frac{1}{4}n_3(n_3+2)-\frac{1}{4}n_2(n_2+2),$$

and equation $(PCC3)$ reduces to

$$\frac{1}{16}n^3-n = \frac{1}{16}n_1^3-n_1+\frac{3}{16}n_3^3-\frac{3}{4}n_3+\frac{3}{16}n_2^3-\frac{3}{4}n_2.$$

We will now substitute $n = n_1+n_2+n_3$ into the the reduced equation $(PCC2)$:

$$n_2^2+n_3^2-2n_1n_2-2n_1n_3-2n_2n_3 = 0.$$

Therefore if we isolate for n_1 we find

$$n_1 = \frac{(n_2-n_3)^2}{2(n_2+n_3)}.$$

By substituting this and $n = n_1+n_2+n_3$ into the the reduced equation $(PCC3)$, multiplying by $64n_2+64n_3$, and simplifying we obtain

$$n_2^4-8n_2^3n_3+30n_2^2n_3^2-8n_2n_3^3+n_3^4-16n_2^2-32n_2n_3-16n_3^2 = 0.9 \tag{9}$$

We have plotted the non-negative solutions to equation (9) along with the line $n_3 = 8-n_2$ in Figure 4. We will show that any line $n_3 = k-n_2$ which intersects the set of non-negative solutions to equation (9) must have $k \leq 8$. Therefore we will be able to bound all solutions to equation (9) with the bounds $n_3 \leq 8-n_2$and $n_3, n_2 \geq 3$.

We will show the line $n_3 = k-n_2$ only intersects the set of non-negative solutions to equation (9) if $k \leq 8$. First substitute $n_3 = k-n_2$ into equation (9) to obtain

$$48n_2^4-96kn_2^3+60k^2n_2^2-12k^3n_2+k^4-12k^2 = 0.$$

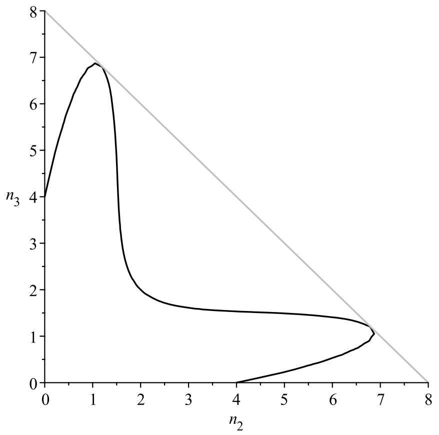


Figure 4: Solutions to $n_2^4 - 8n_2^3n_3 + 30n_2^2n_3^2 - 8n_2n_3^3 + n_3^4 - 12n_2^2 - 32n_2n_3 - 12n_3^2 = 0$

The solutions are

$$n_2 = \frac{1}{2}k \pm \frac{1}{12}\sqrt{18k^2 \pm 6k\sqrt{-3k^2 + 192}}.$$

Therefore n_2 is real only if $-3k^2 + 192 \geq 0$ and hence $k \leq 8$. Therefore the only remaining viable solutions are the 6 integer pairs bounded by $n_2, n_3 \geq 3$ and $n_3 \leq 8 - n_2$. As none are solutions, this is a contradiction.

Since each case results in a contradiction, G is not a disjoint union of H and one cycle nor two cycles, where $H \in \{P_{n_1}, \widetilde{P_{n_1}}\}$. We conclude that G has no cycle components and $G \in \{P_n, \widetilde{P_n}\}$. ■

4 Concluding Remarks

We have determined the domination equivalence classes for paths. Along the way of our proof, we have also determined the number of dominating sets of large cardinality in families of graphs, as a function of various graph parameters. We expect that these formulae (and the techniques used to prove them) will be of use in other problems related to domination-equivalence. One extension of our results is to consider trees in general (we remark that the only domination unique trees are stars, as they are unique because they are the only trees with all but one vertex being a leaf, and any other tree has at least two distinct stems, which can be joined to form an irrelevant edge).

Open problem 1. For a tree T, characterize the domination-equivalence class of T.

Results in [13] show that we can build dominating equivalent graphs by adding an (irrelevant) edge between stems. We have shown large enough paths are domination-unique up to this operation.

Open problem 2. Which other graphs in $\mathcal{G}_2$ are domination-unique up to the addition and removal of irrelevant edges?

Finally, it would be useful to know from the domination polynomial whether a graph is connected. However the addition of irrelevant edges make it possible for a connected and disconnected graph to be domination-equivalent. This leads us to our last open problem.

Open problem 3. Is there a disconnected graph which is domination-equivalent to a connected graph but which cannot be made connected through the addition of irrelevant edges?

Acknowledgements

The authors would like to thank the referees for their comments. The second author would like to acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (grant number RGPIN 170450–2013).

References

[1] S. Akbari, S. Alikhani and Y. H. Peng, Characterization of graphs using domination polynomials, *European J. Combin.* **31** (2010), 1714–1724.

[2] S. Akbari and M. R. Oboudi, Cycles are determined by their domination polynomials, *Ars Combin.* **116** (2014), 353–358.

[3] S. Alikhani, The Domination Polynomial of a Graph at -1, *Graphs Combin.* **29** (2013), 1175–1181.

[4] S. Alikhani and Y. H. Peng, Dominating sets and domination polynomials of paths, *Int. J. Math. Math. Sci.* **10** (2009), 1–10.

[5] S. Alikhani and Y. H. Peng, Domination polynomials of cubic graphs of order 10, *Turkish J. Math.* 35 (2011), 355–366.

[6] S. Alikhani and Y. H. Peng, Introduction to domination polynomial of a graph, *Ars Combin.* **114** (2014), 257–266.

[7] B. M. Anthony and M. E. Picollelli, Complete r-partite Graphs Determined by their Domination Polynomial, *Graphs Combin.* **31** (2015), 1993–2002.

[8] I. Beaton and J. Brown, The Domination Equivalence Classes of Paths, (preprint); arXiv:1710.03871.

[9] F. Dong, K. Koh and K. Teo, *Chromatic Polynomials and Chromaticity of Graphs*, World Scientific, London, 2005.

[10] T. Haynes, S. Hedetniemi and P. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 2005.

[11] J. Hooker, R. Garfinkel and C. Chen, Finite dominating sets for network location problems, *Oper. Res.* **4** (1991), 100–118.

[12] S. Jahari and S. Alikhani, On D-equivalence classes of some graphs, *Bull. Georg. Natl. Acad. Sci.* **10** (2015), 12–19.

[13] T. Kotek, J. Preen, F. Simon, P. Tittmann and M. Trinks, Recurrence relations and splitting formulas for the domination polynomial, *Electron. J. Combin.* **19** (3) (2012), #P47, 27pp.

(Received 19 Oct 2017; revised 10 Apr 2018, 26 Nov 2018)