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Factor pair latin squares

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Abstract

Sudoku has risen in popularity over the past few years. The rules are simple, yet the solutions are often less than trivial. Mathematically, these puzzles are interesting in their own right. This paper will generalize the idea of a sudoku puzzle to define a new kind of $n \times n$ array. We define a latin square of order n as an $n \times n$ array where every row and every column contain every symbol $1, 2, \ldots, n$ exactly once. We say (a, b) is an ordered factor pair of the integer n if $n = a \times b$. An (a, b)-sudoku latin square is a latin square where in addition to each row and column containing every symbol exactly once, each $a \times b$ rectangle also contains every symbol exactly once when the $n \times n$ array is tiled with $a \times b$ rectangles in the natural way. A factor pair latin square of order n (denoted FPLS(n)) is an (a, b)-sudoku latin square for every ordered factor pair (a, b) of n. This paper will mainly be concerned with the existence of such designs.

1 Introduction

1.1 History

In recent years, sudoku puzzles have become extremely popular. The modern day sudoku puzzle first appeared in 1979 as a puzzle called "Number Place" in Dell Magazine [6]. They were designed by Howard Garns [2], a freelance puzzle constructor

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and retired architect. A *sudoku puzzle* is a 9×9 square grid in which every cell contains exactly one *symbol* (typically denoted with integers 1 through 9) in such a way that each 1×9 row contains each symbol exactly once, each 9×1 column contains each symbol exactly once, and each 3×3 sub-square tiling the grid starting at the top left (often called *blocks* or *regions*) contains each symbol exactly once. For ease of speech, we say a subset S of n cells of an $n \times n$ grid is *latin* if S contains each symbol exactly once. The property that every row and every column is latin is an important property. Arrays of n symbols in which every row and every column is latin are called *latin squares*.

Generalizations of sudoku puzzles have been popular over the years. The most closely related to this topic are Retransmission Permutation Arrays. The existence of such designs have been studied by people such as Wanless and Zhang in [14] as well as Dinitz, Paterson, Stinson, and Wei in [5]. Retransmission Permutation Arrays are $n \times n$ grids used to resolve problems in overlapping channel transmissions.

There has also been a great deal of literature written on orthogonal arrays based on sudoku puzzles such as in [8], which deals with sudoku-like arrays, codes, and orthogonality. Other sudoku arrays have been studied by Lorch in [10]. Samurai sudoku-based space filling curves have also been a variation of sudoku-arrays that has been used to pool data from multiple sources in [15].

Other generalizations of sudoku puzzles such as Magic sudoku variants have been studied in [1]. Some variations include sudoku using partially ordered sets in [4], modular magic squares in [11], and strongly symmetric self-orthogonal diagonal sudoku squares in [12].

We propose yet another way to extend the idea of the traditional sudoku puzzle. A positive integer n is said to have *ordered factor pairs* (a, b) if $n = a \times b$. Given an ordered factor pair (a, b) of a positive integer n, the $n \times n$ grid can then be partitioned into $a \times b$ regions in a very natural way, namely starting in the top left corner. An (a, b)*-sudoku latin square* of order n is a latin square on the symbol set $\{1, 2, \ldots n\}$ where each $a \times b$ region in the natural tiling contains all of the symbols exactly once [7].

There exists an (a, b)-sudoku latin square for each ordered factor pair (a, b) of a positive integer n, as will be shown in Section 2 by giving an explicit construction. Taking this definition a step further, we say a *factor pair latin square* of order n, denoted FPLS(n), is a square that is an (a, b)-sudoku latin square of order n for every ordered factor pair (a, b) of n.

Certainly as the number of ordered factor pairs increase, the problem gets more complex. In order to solidify this idea, perhaps an example is in order. First, observe that the number six has ordered factor pairs $1 \times 6, 6 \times 1, 2 \times 3$, and 3×2. Therefore, a FPLS(6) would have the following regions being Latin: $1 \times 6, 6 \times 1, 3 \times 2$, and 2×3. An example of a FPLS(6) is presented in Figure 1.

Of course, the first question that arises with such a definition is whether these designs exist. Originally, we believed that there exists a factor pair latin square for every order n; however, as we will see in Section 3, this is not true since there exist

1	2	3	4	5	6
4	5	6	1	2	3
3	6	2	5	1	4
5	1	4	6	3	2
2	4	1	3	6	5
6	3	5	2	4	1

Figure 1: FPLS(6)

several infinite families of positive integers n such that there does not exist a factor pair latin square of order n. Therefore, finding necessary and sufficient conditions for the existence of such designs is non-trivial.

There are other natural questions to ask as we are striving to expand the definition of a sudoku puzzle. One such way is to change the shape of the regions. These designs are called *Gerechte designs* [3]. Gerechte designs have been useful in designing agricultural experiments. Moreover, an $n \times n$ square that satisfies multiple Gerechte designs at the same time is called a *multiple Gerechte design* [2]. This paper will focus on finding some necessary (and sometimes sufficient) conditions for the existence of factor pair latin squares, which are special types of multiple Gerechte designs.

2 Constructions

Provided $n = ab$, and n, a, and b are positive integers, an *(a, b)-sudoku latin square* of order n is a latin square on the symbol set $\{1, 2, \ldots n\}$ where each $a \times b$ region in the natural tiling contains all of the symbols exactly once [7]. A *factor pair latin square* of order n is a square that is an (a, b)-sudoku latin square of order n for every ordered factor pair (a, b) of n. Since we are using (a, b)-sudoku latin squares to define a factor pair latin square, it is good to start out with the existence of (a, b)-sudoku latin squares.

Theorem 2.1. *There exists an (a, b)-sudoku latin square for each ordered factor pair (a, b) of order n.*

Proof. This proof will be constructive. That is to say that it will give an algorithm for creating an (a, b)-sudoku latin square followed by a proof that the algorithm works. We will use a cyclical row/column shifting method similar to Keedwell in [9].
Algorithm. Construct an $ab \times ab$ array as in Figure 2 as follows:

1. Fill in the top left $a \times b$ so that each symbol appears exactly once. Call this $a \times b$ matrix A.

2. Let the $a \times a$ matrix
$$P = \left[\begin{array}{c|c} 0 & 1 \\ \hline I_{a-1,a-1} & 0 \end{array}\right]$$
define the permutation matrix of order a that cyclically shifts the rows by one when applied on the left of a matrix and the $b \times b$ matrix
$$Q = \left[\begin{array}{c|c} 0 & I_{b-1,b-1} \\ \hline 1 & 0 \end{array}\right]$$
denote the permutation matrix of order b that cyclically shifts the columns by one when applied on the right of a matrix.

3. For the $1 \leq i \leq b$ and $1 \leq j \leq a$, to compute the rectangle in the i^{th} row (with respect to the rectangles) and j^{th} column (with respect to the rectangles), multiply $P^{j-1}AQ^{i-1}$.

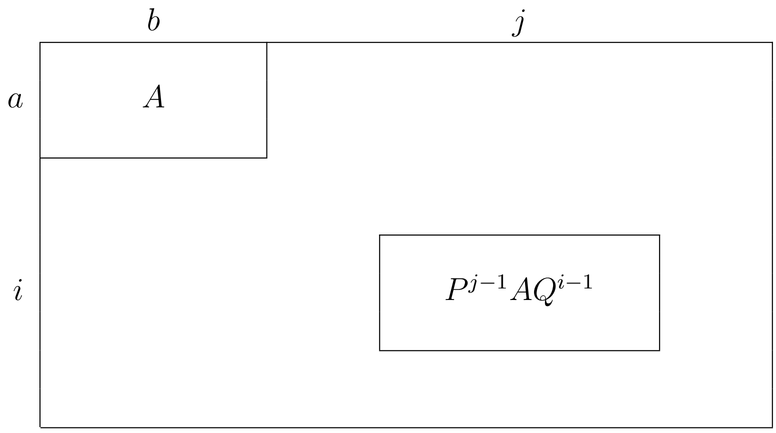

Figure 2: (a, b)-sudoku Latin Square Construction

Justification. Since every symbol occurs exactly once in the top left $a \times b$ rectangle, the top left rectangle is clearly latin. Since P^{j-1} is a permutation matrix of order a which rotates the rows cyclically down by j, each of the first a rows in the $n \times n$ array are latin. Also, since the original top left $a \times b$ rectangle was latin to begin with, rectangles in the first row of $a \times b$ rectangles are also latin. In a similar fashion, since Q^{i-1} is a permutation matrix of order b which rotates the columns cyclically right by i, each of the columns in the $n \times n$ array is latin. Similarly, the remaining rows of the $n \times n$ array are latin since the first a rows were latin. $\square$

However, there is not a similar theorem for factor pair latin squares. As we will see in Section 3, there does not exist a factor pair latin square for every order n. This section will be mostly be concerned with constructing infinite families of factor pair latin squares.

2.1 Power of Primes

In this section, we will show that there exists a factor pair latin square of every prime order. Given a prime number p, the only two ordered factor pairs that need to be latin are the $1 \times p$ rows and the $p \times 1$ columns. By the definition of a latin square, every latin square of order p is a factor pair latin square of order p.

On a less trivial note, it is natural to look at the prime factorization of a number n, since factor pair latin squares are designs that are concerned with all of the ordered factors of the number n. The *Cayley table of a quasigroup* is a latin square with a headline, a sideline, and a binary operation defined. The following theorem ensures that a factor pair latin square can be constructed if n is a power of a prime number.

Theorem 2.2. *Let p be a prime number and let α be a positive integer. Then there exists a factor pair latin square of order p^α.*

Proof. This proof will be constructive. That is to say that it will give an algorithm for creating a factor pair latin square of order $n = p^\alpha$, followed by a proof that the algorithm does what is intended.

Algorithm. Let $\mathbb{Z}_p^\alpha$ denote the words of length α from the alphabet $\mathbb{Z}_p$. Consider the following quasigroup with entries from $\mathbb{Z}_p^\alpha$. Label the headline with $v_1, v_2, \ldots, v_n$ such that $\bigcup_{i=1}^n v_i = \mathbb{Z}_p^\alpha$ and the v_i's are ordered lexicographically. Label the sideline with $u_1, u_2, \ldots, u_n$ where u_i is obtained from v_i by writing the corresponding v_i backwards. Let the entry of cell (a, b) be $a + b \pmod{p}$.

Justification. Define a block to be the projection of some rectangle that we are interested in onto the headline or sideline. Let $\beta \in \{0, 1, \ldots, \alpha\}$ and note that each block in the headline consists of a word w of length β concatenated with every word of length $\alpha - \beta$. The sideline has every word of length β concatenated with some fixed word v of length $\alpha - \beta$. So, within each block, v and w are set words that run through every combination of words of size $|v|$ and $|w|$ respectively. That is to say that entries in the same row of a given $p^{\alpha-\beta} \times p^\beta$ block must agree in the first $\alpha - \beta$ components and differ in the last β components, while the opposite is true for entries in the same column. This forces all of the entries in each of the blocks to be distinct. This is exemplified in Figure 3. □

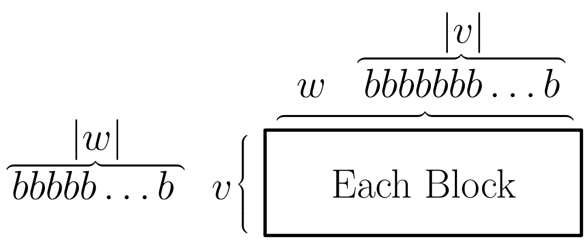

Figure 3: Power of Primes

2.2 Twice a Prime

One of the goals is to get constructions for as many factor pair latin squares as we can. To that avail, if n can be decomposed into twice a prime number, then a factor pair latin square of order n can be constructed. That is there exists a factor pair latin square of order $n = 2p$, where p is a prime number.

Theorem 2.3. *Let $p > 2$ be a prime number. Then there exists a factor pair latin square of order $2p$.*

Proof. This proof will be constructive. That is to say that when p is a prime number an algorithm is presented for creating a factor pair latin square of order $2p$. This will be followed by a proof that the algorithm does what is intended.

Algorithm. Since $n = 2p$, we will be looking at elements from $\mathbb{Z}_2 \times \mathbb{Z}_p$. That is to say that every element in the $2p \times 2p$ array will be an ordered pair (a, b) where a is an element of $\mathbb{Z}_2$ and b is an element of $\mathbb{Z}_p$. For ease of construction, we will construct two different $2p \times 2p$ arrays. The first array, $\mathscr{Z}_2$, will contain symbols from $\mathbb{Z}_2$, while the second array, $\mathscr{Z}_p$, will contain symbols from $\mathbb{Z}_p$. Let x_{ij} denote the symbol in cell (i, j) of $\mathscr{Z}_2$ and y_{ij} denote the symbol in cell (i, j) of $\mathscr{Z}_p$. Cell (i, j) of the constructed factor pair latin square of order $2p$ will be filled with the ordered pair (x_{ij}, y_{ij}). Construct $\mathscr{Z}_2$ and $\mathscr{Z}_p$ as follows:
To construct $\mathscr{Z}_2$, fill in cell (i, j) with the symbol $(i + j) \pmod 2$. To construct $\mathscr{Z}_p$:

- Let $1 \le i \le p - 1$ and $1 \le j \le n$. For rows 1 through $p - 1$, fill cell (i, j) with the symbol $2\left\lfloor \frac{i-1}{2} \right\rfloor + (j - 1) \pmod p$.
- Let $p + 2 \le i \le 2p - 2$ and $1 \le j \le n$. For rows $p + 2$ through $2p - 2$, fill cell (i, j) with the symbol $2\left\lfloor \frac{i-(p+2)}{2} \right\rfloor + j \pmod p$.
- Let $1 \le j \le 2p$. For rows $i = p$ and $i = 2p$, fill in cell (i, j) with the symbol $(p - 1) + 2\left\lfloor \frac{j-1}{2} \right\rfloor \pmod p$.
- Let $1 \le j \le 2p$. For rows $i = p + 1$ and $i = 2p - 1$, fill in cell (i, j) with the symbol $(p - 2) + 2\left\lfloor \frac{j}{2} \right\rfloor \pmod p$.

Justification. Notice that in $\mathscr{Z}_p$, each $2 \times p$ block and each $p \times 2$ block has symbols 1 through $2p$ exactly twice. This fact is obvious for the first two rows, and is true for every subsequent pair of two rows from the third row through the $(p - 1)^{\text{st}}$ row as well as the $(p + 2)^{\text{nd}}$ row through the $(2p - 2)^{\text{nd}}$ row since we are simply repeating the symbols (that is to say the third row is the same as the fourth row and so on). Similarly, the p^{th} row and the $(p+1)^{\text{st}}$ row as well as the $(2p)^{\text{th}}$ row and the $(2p-1)^{\text{st}}$ row have each symbol exactly twice since for every symbol, it's partner is next to it except for the symbol $2p - 2$, which is the first symbol of the next row. Moreover, each repeated symbol is next to it's partner (it's partner is either to the left, right, above or below it) except for the $(p + 1)^{\text{st}}$ row and the $(2p - 1)^{\text{st}}$ row, which have exactly one symbol that wraps around the grid. In any case, since the $(i, j)^{\text{th}}$ entry of

$\mathcal{Z}_2$ is $(i + j) \pmod 2$, every ordered pair occur exactly once within each row $(1 \times 2p)$, column $(2p \times 1)$, and block ($2 \times p$ and $p \times 2$) of the grid. □

3 Negative Results

3.1 Generalizing Order Twelve

At the onset of this problem, we believed that a factor pair latin square existed for every order. With the proper time and patience, one can show that there does not exist a factor pair latin square of order twelve, since the ordered factor pairs of twelve are $\{1 \times 12\ ,\ 12 \times 1,\ 2 \times 6,\ 6 \times 2,\ 3 \times 4,\ 4 \times 3\}$.

The idea of this proof can be generalized to higher orders; however, for a concrete example, there does not exist a factor pair latin square of order 12 by the following theorem if we let $a = 2, b = 6, c = 3, d = 4, f = 4$, and $g = 3$. In general, we have the following theorem that describes in terms of ordered factor pairs what orders are inadmissible by this technique.

Theorem 3.1. *There does not exist a factor pair latin square if:*

- $n = a \cdot b = c \cdot d = f \cdot g$,
- $a < c < f, g < d < b$,
- $g \left\lfloor \frac{d}{g} \right\rfloor < d$,
- $g \left\lceil \frac{d}{g} \right\rceil \geq b$.

Proof. Fill in the first $a \times b$ rectangle with the n symbols. Since $g \left\lfloor \frac{d}{g} \right\rfloor < d$, the shaded region in Figure 4 is nonempty. Moreover, the restriction that $g \left\rceil \frac{d}{g} \right\rceil \geq b$ dictates that the cells in the first $a \times b$ region but not the first $c \times d$ region must all be contained within the last $g \times f$ region to intersect the first $a \times b$ region. Suppose symbol α appears in the shaded region in Figure 4. It cannot appear again in the top $c \times d$ region, so it can't appear in the portion of the $a \times b$ rectangle that lies within this $c \times d$ region. Therefore, α must appear in the far right portion of the $a \times b$ region, outside of the $c \times d$ region. But this location of α and the location of α in the shaded region are distinct and both lie in the right most $f \times g$ region in Figure 4, giving a contradiction. □

This all started with the observation that there does not exist a factor pair latin square of order twelve. Moreover, the following corollary generalizes this to say that there cannot be a factor pair latin square of any order that is divisible by twelve.

Corollary 3.2. *If m satisfies the conditions of Theorem 3.1 and $n \equiv 0 \pmod m$ then there does not exist a factor pair latin square of order n.*

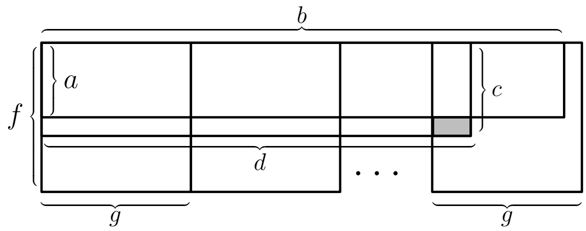

Figure 4: Three Conditions

Proof. Let m satisfy the conditions of Theorem 3.1. That is to say that

- $m = a \cdot b = c \cdot d = f \cdot g$,
- $a < c < f, g < d < b$,
- $g \left\lfloor \frac{d}{g} \right\rfloor < d$,
- $g \left\lceil \frac{d}{g} \right\rceil \geq b$.

Moreover, since $n \equiv 0 \pmod{m}$, $n = km$ for some positive integer k. That is to say that there exist three ordered factor pairs such that

- $n = a \cdot bk = c \cdot dk = f \cdot gk$,
- $a < c < f, gk < dk < bk$,
- $gk \left\lfloor \frac{dk}{gk} \right\rfloor = gk \left\lfloor \frac{d}{g} \right\rfloor < dk$,
- $gk \left\lceil \frac{dk}{gk} \right\rceil = gk \left\lceil \frac{d}{g} \right\rceil \geq bk$.

Since each inequality is simply multiplied by k, the above inequalities also satisfy the conditions for Theorem 3.1. Hence, there does not exist a factor pair latin square of order n. □

3.2 Generalizing Order Twenty

Factor pair latin squares can be constructed using methods from Section 2 for orders thirteen, fourteen, sixteen, seventeen, and nineteen. Moreover, there exists a factor pair latin square of order fifteen as shown in Figure 5 as well as a factor pair latin square of order eighteen as shown in Figure 6. One might notice, however, that order twenty is missing from this list. The following theorem gives a reason as to why that is the case. We can see that there does not exist a factor pair latin square of order 20 by letting $a = 4, b = 5, c = 5, d = 4, f = 10, g = 2, h = 2$, and $j = 10$.

Theorem 3.3. *There does not exist a factor pair latin square if:*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
6	7	8	9	10	11	12	13	14	15	1	2	3	4	5
11	12	13	14	15	1	2	3	4	5	6	7	8	9	10
4	5	9	2	3	7	1	10	15	8	13	14	6	11	12
10	14	15	8	12	13	5	6	11	3	4	9	1	2	7
7	1	6	11	13	2	4	9	12	14	3	5	10	15	8
2	3	4	1	6	5	8	7	10	9	12	15	11	13	14
5	8	10	7	9	12	14	15	13	11	2	1	4	3	6
12	13	11	15	14	3	6	1	2	4	8	10	5	7	9
9	15	14	10	4	8	3	11	5	6	7	13	2	12	1
3	6	1	5	2	4	9	12	7	13	14	8	15	10	11
8	11	12	13	7	10	15	14	1	2	5	3	9	6	4
13	4	2	12	1	9	10	5	6	7	15	11	14	8	3
14	9	5	3	11	15	13	4	8	12	10	6	7	1	2
15	10	7	6	8	14	11	2	3	1	9	4	12	5	13

Figure 5: FPLS(15)

- $n = a \times b = c \times d = f \times g = h \times j$,
- $h < a < c < f, g < d < b < j$,
- $b \mid j, g \mid d, h \mid a, c \mid f, g \mid 2b$
- $h > c - a, g > b - d$,
- $(a - h)\left(g\left\lceil \frac{b}{g} \right\rceil - b\right) + \left(h\left\lceil \frac{c}{h} \right\rceil - c\right)(2d - b) > (f - 2a)\, g.$

Proof. First and foremost, it should be noted that if the above conditions are satisfied, then there exists a configuration of rectangles as seen in Figure 7 in the top left corner of the proposed factor pair latin square. This proof will show that if the above conditions are satisfied, then there are not enough symbols to fill up Region 10 as depicted in Figure 7. First, fill in the $h \times j$ rectangle in Figure 7. Since the top left $a \times d$ rectangle is in common with the other rectangles in the top left, Region 1 and Region 11 must have the same symbol set. That is to say that Region 10 and Region 1 cannot share any of the same symbols since Region 10 and Region 11 are in the same $f \times g$ rectangle together. Moreover, Regions 2 and 6 are in the same $f \times g$ rectangle as Region 10; so, they also cannot share any symbols. Similarly, Regions $3, 4, 7, 8$, and 9 cannot share any symbols, nor can they share any symbols with region 10, since they occur in the same $a \times b$ rectangle. By this fact and by the intersection of the top right $a \times b$ rectangle along with the top right $c \times d$ rectangle, the union of Regions $1, 2, 3$, and 4 must have the same symbol set as Region 12. That is to say that the symbols from Regions 5 and 6 have to be divided among Regions 13 and 14. Also, the symbols in Regions 7 and 8 must all occur in Region 14 as well, since the symbols in Regions $5, 6, 7$, and 8 must all occur in Regions 13 and 14 and Regions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
12	13	11	10	17	14	18	16	15	2	4	5	7	1	8	6	3	9
8	15	7	16	18	9	14	3	6	1	13	17	5	4	10	12	2	11
4	10	5	2	12	11	1	17	13	14	16	18	3	6	9	7	8	15
14	16	9	1	8	7	4	2	11	3	15	6	12	17	18	13	10	5
6	17	18	15	13	3	10	12	5	7	8	9	11	16	2	4	1	14
3	7	17	14	1	4	9	15	16	12	10	2	8	18	11	5	13	6
5	11	8	12	2	10	6	13	18	4	1	14	9	15	3	17	7	16
9	18	6	13	16	15	5	11	8	17	3	7	10	2	1	14	12	4
10	1	2	3	7	17	12	4	14	16	9	15	6	13	5	11	18	8
15	12	16	5	11	8	17	10	3	18	6	13	4	7	14	1	9	2
13	14	4	9	6	18	2	1	7	8	5	11	16	12	17	3	15	10
11	9	15	6	10	2	13	14	1	5	7	3	17	8	4	18	16	12
7	5	12	17	3	16	8	18	4	15	2	10	1	11	6	9	14	13
18	4	13	8	14	1	11	9	12	6	17	16	15	10	7	2	5	3
2	3	10	7	15	5	16	6	17	13	14	4	18	9	12	8	11	1
17	6	1	18	9	13	15	5	2	11	12	8	14	3	16	10	4	7
16	8	14	11	4	12	3	7	10	9	18	1	2	5	13	15	6	17

Figure 6: FPLS(18)

7 and 8 share an $a \times b$ rectangle with Region 13. The symbols in region 10 must all occur in Region 5, since Regions $1, 2, 3, 4, 6, 7, 8$, and 9 all share some rectangle with Region 10. In addition, the symbols in Region 10 must also occur in Regions 13 and 14, since symbols in Region 5 must occur in Regions 13 and 14. Moreover, the symbols in Region 10 must all occur in Region 14, since Regions 10 and 13 share a rectangle. So, the symbols in Regions $7, 8$, and 10 are all distinct and must all occur in region 14. Since $(a-h)\left(g\left\lceil\frac{b}{g}\right\rceil - b\right) + \left(h\left\lceil\frac{c}{h}\right\rceil - c\right)(2d-b) > (f-2a)\,g$, the number of cells in Region 14 is smaller than the number of unique symbols in Region $7, 8$, and 10; so, there are not enough symbols to properly fill this design. $\square$

As before, Theorem 3.3 can be extended to say that if order m satisfies the conditions of Theorem 3.3 and $n \equiv 0 \pmod{m}$, then there does not exist a factor pair latin square of order n.

Corollary 3.4. *If m satisfies the conditions of Theorem 3.3 and $n \equiv 0 \pmod{m}$, then there does not exist a factor pair latin square of order n.*

Proof. Say m satisfies the conditions of Theorem 3.3. That is, there exist a set of four ordered factor pairs such that

- $n = a \times b = c \times d = f \times g = h \times j$,
- $h < a < c < f, g < d < b < j$,
- $b \mid j, g \mid d, h \mid a, c \mid f, g \mid 2b$

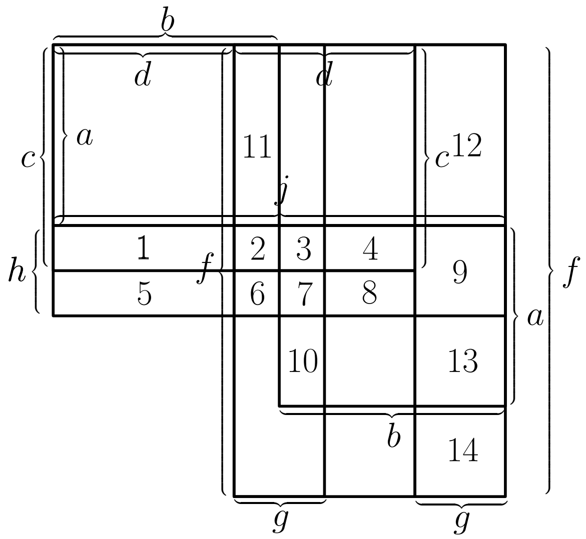

Figure 7: Generalization of Order Twenty

- $h > c - a, g > b - d$,
- $(a-h)\left(g\left\lceil\frac{b}{g}\right\rceil - b\right) + \left(h\left\lceil\frac{c}{h}\right\rceil - c\right)(2d-b) > (f-2a)\,g$.

Let $n \equiv 0 \pmod{m}$. That is to say that there exists a positive integer k such that $n = km$. Moreover, there exist a set of four ordered factor pairs such that

- $n = a \times bk = c \times dk = f \times gk = h \times jk$,
- $h < a < c < f, gk < dk < bk < jk$,
- $bk \mid jk, gk \mid dk, h \mid a, c \mid f, gk \mid 2bk$
- $h > c - a, gk > bk - dk$,
- $(a-h)\left(gk\left\lceil\frac{bk}{gk}\right\rceil - bk\right) + \left(h\left\lceil\frac{c}{h}\right\rceil - c\right)(2dk-bk) > (f-2a)\,gk$.

Since all of the inequalities are either the same as before or multiples of k, n also satisfies the conditions of Theorem 3.3. Hence, there does not exist a factor pair latin square of order n. $\square$

3.3 General Order Twenty-Eight

A factor pair latin square of order twenty-one has been constructed in Figure 8. One might also notice that Section 2 gives us constructions for factor pair latin squares for orders twenty-two, twenty-three, and twenty-five through twenty-seven. Also, there cannot exist a factor pair latin square of order twenty-four since twenty-four is divisible by twelve. Twenty eight; however, is one that is not covered by these

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
15	16	17	18	19	20	21	1	2	3	4	5	6	7	8	9	10	11	12	13	14
8	9	10	11	12	13	14	15	16	17	18	19	20	21	1	2	3	4	5	6	7
19	20	21	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
12	13	14	15	16	17	18	19	20	21	1	2	3	4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	1	2	3	4
4	11	18	7	14	21	3	10	17	6	13	20	2	9	16	5	12	19	1	8	15
16	5	12	19	1	8	15	4	11	18	7	14	21	3	10	17	6	13	20	2	9
10	17	6	13	20	2	9	16	5	12	19	1	8	15	4	11	18	7	14	21	3
2	1	4	3	6	5	8	7	10	9	16	11	12	13	14	15	20	21	17	18	19
7	14	9	10	11	12	13	2	18	15	17	21	19	20	6	1	4	3	8	5	16
18	15	19	17	21	16	20	3	14	4	6	8	1	5	2	10	9	12	7	11	13
3	8	11	9	4	15	1	21	19	2	20	13	7	17	18	14	5	16	6	12	10
13	21	20	14	7	18	17	6	12	5	3	10	9	16	11	8	19	2	4	15	1
6	12	2	5	10	19	16	11	4	1	14	15	18	8	3	7	13	17	21	9	20
9	3	1	2	8	4	5	13	7	11	10	6	14	12	21	18	16	20	15	19	17
11	10	13	6	15	7	12	17	1	20	21	18	16	19	9	4	2	5	3	14	8
20	18	16	21	17	14	19	9	3	8	5	4	15	2	7	12	11	10	13	1	6
14	4	5	16	3	1	2	20	8	19	9	17	11	10	13	21	15	6	18	7	12
17	7	15	12	18	11	6	14	21	13	2	3	4	1	20	19	8	9	10	16	5
21	19	8	20	13	9	10	18	15	16	12	7	5	6	17	3	1	14	11	4	2

Figure 8: FPLS(21)

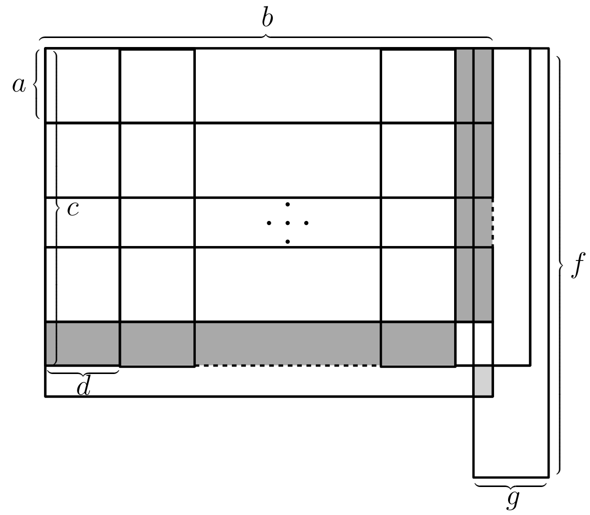

Figure 9: Generalization of Order Twenty-Eight

constructions. The following section will give reasoning for why there does not exist a factor pair latin square of order twenty-eight.

We turn our attention to showing that there does not exist a factor pair latin square of order twenty-eight. In fact, we will display another infinite family of forbidden orders of which twenty eight is the first member ($a = 4, b = 7, c = 7, d = 4, f = 14$, and $g = 2$).

Theorem 3.5. *There does not exist a factor pair latin square if*

- $n = a \times b = c \times d = f \times g$,
- $a < c < f, g < d < b$,
- $d \nmid b$,
- $d \left\lfloor \frac{b}{d} \right\rfloor < g \left\lfloor \frac{b}{g} \right\rfloor < d \left\lceil \frac{b}{d} \right\rceil \leq g \left\lceil \frac{b}{g} \right\rceil$.

Proof. If the above conditions hold, the top left corner of the $n \times n$ grid looks like Figure 9. Let $s = \left\lfloor \frac{c}{a} \right\rfloor = \left\lfloor \frac{b}{d} \right\rfloor$. That is to say that each symbol occurs s times in the first s $a \times b$ blocks aligned along the vertical axis. Then the r symbols in the rightmost dark gray area in Figure 9 occur $s-1$ times in the intersection. Also, as each symbol occurs s times in the first s $c \times d$ blocks aligned along the horizontal axis, each of these symbols must occur in the lowermost dark gray area. Similarly, any symbol not occurring in the rightmost dark gray area occurs s times in the intersection, so cannot occur in the lowermost dark gray area. So, the dark gray regions in Figure 9 must have the same symbol set. Fill in the last $c \times d$ region that intersects the top left $a \times b$ with symbols. Suppose some symbol α occurred in the light gray area in Figure 9. The symbol α cannot occur in the rightmost dark gray region, as each of these symbols occur in the lowermost dark gray region, which share an $a \times b$ region with the light gray region. That is, the symbol α must also occur in the $f \times g$ region that intersects the last $c \times d$ region that intersects the top left $a \times b$ region, or it must be in the repeated symbol set within the same $a \times b$ rectangle as the light gray region, which is a contradiction. □

Theorem 3.5 can be rephrased in terms of the factor pairs. We will establish this in the following corollary.

Corollary 3.6. *Let $k > 1$. There does not exist a FPLS$(4\,(4k+3))$.*

Proof. Let $n = 4\,(4k+3)$ for some integer $k > 1$. Then $a = 4$, $b = 4k+3$, $c = 4k+3$, $d = 4$, $f = 2(4k+3)$, and $g = 2$ satisfy the conditions of Theorem 3.5; therefore, there does not exist a factor pair latin square of order $4\,(4k+3)$ □

As before, we can extend this to orders which are multiples of a number that is deemed inadmissible by Theorem 3.5.

Corollary 3.7. *If m satisfies the conditions of Theorem 3.5 and $n \equiv 0 \pmod{m}$ then there does not exist a factor pair latin square of order n.*

Proof. If m satisfies Theorem 3.5, then there exist three ordered factor pairs such that

- $m = a \times b = c \times d = f \times g$,
- $a < c < f, g < d < b$,
- $d \nmid b$,
- $d \left\lfloor \frac{b}{d} \right\rfloor < g \left\lfloor \frac{b}{g} \right\rfloor < d \left\lceil \frac{b}{d} \right\rceil \leq g \left\lceil \frac{b}{g} \right\rceil$.

Moreover, if n is a multiple of m, then $n = mk$ for some positive integer k. Furthermore, there exist three ordered factor pairs of n such that

- $n = a \times bk = c \times dk = f \times gk$,
- $a < c < f, gk < dk < bk$,
- $dk \nmid bk$,
- $dk \left\lfloor \frac{bk}{dk} \right\rfloor < gk \left\lfloor \frac{bk}{gk} \right\rfloor < dk \left\lceil \frac{bk}{dk} \right\rceil \leq gk \left\lceil \frac{bk}{gk} \right\rceil$.

Since every inequality is either the same as m or it is a multiple of k. In either case, the above set of inequalities satisfy the conditions of Theorem 3.5. Henceforth, if n is a multiple of a number that satisfies Theorem 3.5, then there does not exist a factor pair latin square of order n. □

The theorems discussed in this section give us an infinite family of inadmissible orders for factor pair latin squares.

4 Multiple Gerechte Designs

A *gerechte design* is an $n \times n$ grid partitioned into n regions (possibly of different shapes and possibly disconnected) with n cells in each region such that each row, column, and region is latin. A *multiple gerechte design* is a latin square for which multiple gerechte designs are satisfied [2]. Factor pair latin squares are particular kinds of multiple gerechte designs.

A *gerechte skeleton* of order n is an $n \times n$ array whose n^2 cells are partitioned into n regions containing n cells each. E. R. Vaughan has shown in [13] that deciding whether a gerechte skeleton has a completion is NP-complete; however, if the gerechte skeleton is restricted to contiguous regions, the answer is unknown. Similarly, if the regions are required to be rectangles, the solution is unknown. The problem of finding a completion to a factor pair latin square is even more specific, since we are requiring multiple particular gerechte skeletons.

Perhaps more importantly to the design of experiments, a further question is whether or not a design has an orthogonal mate. Two latin squares of size n, $L = a_{i,j}$ on symbol set S and $L' = b_{i,j}$ on symbol set S', are said to be *orthogonal* if every element in $S \times S'$ occurs exactly once among the n^2 pairs $(a_{i,j}, b_{i,j})\,, 1 \leq i, j \leq n$. A set of latin squares are *mutually orthogonal* if every pair of latin squares in the set are mutually orthogonal [7].

A natural question is how many mutually orthogonal factor pair latin squares can be found of a given order. The following theorem gives a maximum number of mutually orthogonal factor pair latin squares.

Theorem 4.1 (Modified from [2]). *Let d denote the maximum size of the intersection between any two different rectangles in a factor pair latin square of order n. There exists at most $n - d$ mutually orthogonal factor pair latin squares of order n.*

Proof. Say that rectangles A_1 and A_2 have the biggest intersection. Moreover, let $d = |A_1 \cap A_2|$. Let c be a cell in $A_1 \setminus A_2$. By renaming the cells in each of the mutually orthogonal factor pair latin squares, we can say that cell c of each of the mutually orthogonal factor pair latin squares contains symbol 1. Moreover, symbol 1 must occur exactly once in A_2 and not in A_1 in each of the factor pair latin squares; however, each subsequent factor pair latin square must have symbol 1 in a different cell within A_2, since symbol 1 has already occurred in cell c. Hence, there can be at most $|A_2 \setminus A_1|$ mutually orthogonal factor pair latin squares. □

5 Open Problems

This paper has discussed some aspects of factor pair latin squares. Namely, it has addressed some existential problems; however, necessary and sufficient conditions have not been shown. Progress has been made towards that goal; however, it remains an open problem.

Open Problem 1. *What are the necessary and sufficient conditions for a factor pair latin square of order n to exist?*

Simpler problems can be tackled first, however. Namely, are there other constructions for when a factor pair latin square of order n exists? Powers of primes and twice a prime number have been dealt with in Section 2, but what about a product of two distinct primes?

Open Problem 2. *Does there exist a factor pair latin square of order $3p$ where p is a prime numbers?*

Open Problem 3. *Does there exist a factor pair latin square of order pq where p and q are both prime numbers?*

Asymptotically, it seems that perhaps the only admissible orders for factor pair latin squares are primes, powers of primes, and numbers that have prime factorization pq where p and q are both prime numbers.

Open Problem 4. *Asymptotically, does there exist any factor pair latin squares other than those that have prime factorization p^α or pq where p and q are prime numbers and α is a positive integer?*

We have also done some work in mapping the problem of finding factor pair latin squares for a given order n; however, the complexity has not been studied.

Open Problem 5. *What is the complexity of completing a partially filled factor pair latin square of order n?*

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