AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **68(2)** (2017), Pages 285–303

Towards a characterization of graphs with distinct betweenness centralities

Ruth Lopez

Mathematics and Statistics
California State University, Long Beach
Long Beach, CA, U.S.A.

Jacob Worrell

Psychological and Brain Sciences
Indiana University – Bloomington, IN
Bloomington, IN, U.S.A.

Henry Wickus

Mathematics and Computer Science
DeSales University
Center Valley, PA, U.S.A.

Rigoberto Flórez

Mathematics and Computer Science
The Citadel
Charleston, SC, U.S.A.

Darren A. Narayan

School of Mathematical Sciences
Rochester Institute of Technology
Rochester, NY, U.S.A.

Abstract

In a social network individuals have prominent centrality if they are intermediaries between the communication of others. The *betweenness centrality* of a vertex measures the number of intersecting geodesics between two other vertices. Formally, the *betweenness centrality* of a vertex v is the ratio of the number of shortest paths between two other vertices u and w which contain v to the total number of shortest paths between u and w. In this paper, we consider the problem of characterizing all graphs with distinct betweenness centralities. This results in a specialized class of graphs with unusual symmetries including a trivial automorphism group.

We begin by solving the problem for all graphs with less than or equal to seven vertices. Then we consider the general problem by investigating the density and minimality of graphs with distinct betweeness centralities Finally, we investigate the problem of creating infinite families of graphs with this property.

1 Introduction

In a social network individuals have varying levels of centrality according to the flow of information. Individuals placed at the intersection of shortest lines of communication between others have a high degree of "betweenness" centrality. The study of centrality in networks in general has received much attention in recent years with the rising presence of social media sites such as Facebook, Twitter, Instagram, and LinkedIn. The concept of centrality based on intersecting geodesics was introduced independently by Anthonisse [1] and Freeman [7]. Numerous means for quantifying centrality have emerged including degree centrality, closeness centrality, eigenvector centrality, leverage centrality, betweenness centrality, and others [2, 3, 7, 8, 9, 15, 18, 21]. In this paper we focus on (vertex) betweenness centrality.

In social networks, certain people have central roles in communication which give an elevated level of authority. Suppose the flow of information in an organization follows shortest paths of communication and that some action (i.e. mediation or approval) is required by each person on these paths. The number of actions a person must perform is linked to both the topology of the network as well as their location within it. The number of actions a person has to perform can be described by a property well known in social network literature called betweenness centrality. We give a formal definition of betweenness centrality in the next subsection.

1.1 Betweenness centrality

The *betweenness centrality* of a vertex v is the ratio of the number of shortest paths between two other vertices u and w which contain v to the total number of shortest paths between u and w. This idea was introduced by Anthonisse [1] and Freeman [7] in the context of social networks. This concept has since appeared frequently in both social network and neuroscience literature [3, 4, 8, 9, 14, 19, 21, 23]. The betweenness centrality of graphs has been computed for various families of graphs including complete bipartite graphs, wheel graphs, cocktail party graphs, ladder graphs, and cycles [16].

We first give some background with some elementary results.

Definition 1.1 *The betweenness centrality of a vertex v in a graph G denoted $bc_G(v)$, measures the frequency at which v appears on a shortest path between two other distinct vertices x and y. Let σ_{xy} be the number of shortest paths between distinct* *vertices x and y, and let $\sigma_{xy}(v)$ be the number of shortest paths between x and y that contain v. Then $bc_G(v) = \sum_{x,y} \frac{\sigma_{xy}(v)}{\sigma_{xy}}$ (for all distinct vertices v, x, and y).*

In our first lemma, we restate an elementary result on the lower and upper bounds of the betweenness centrality of a vertex. This was found by Gago et al. and Grassi et al. [11] and [13].

Lemma 1.1 *For a given graph G with n vertices, $0 \leq bc(v) \leq (n-1)(n-2)$ for all vertices v in G. Furthermore, these bounds are tight.*

It is clear that if a vertex has a betweenness centrality of zero, it means that the vertex is likely to be less vital to the network than a vertex with a higher betweenness centrality. Gago et al. and Grassi et al. [11] and [13] provided a classification for vertices to have a betweenness centrality of zero. We restate this as our next lemma. We recall that the *closed neighborhood of a vertex* is the subgraph induced by a vertex and its neighbors.

Lemma 1.2 *Given a vertex v, $bc(v) = 0$ if and only if the closed neighborhood of v forms a complete subgraph.*

Gago, Hurajová, and Madaras [10] investigated graphs where the betweenness centrality of all of the vertices were the same. In this paper we investigate the other extreme.

We consider the problem of constructing graphs where the betweenness centralities of all vertices are distinct. The motivation for this problem is to consider an organizational network where each person has a distinct level of authority and hence the levels of authority form a total order. In addition, the resulting graphs form an interesting and unusual family - one with no symmetries (in terms of either placement or intersecting geodesics). This results in a specialized class of graphs with unusual symmetries including a trivial automorphism group. Determining a complete characterization of this class of graphs turns out to be a difficult problem.

In Section 2, we consider the problem of characterizing graphs with distinct betweenness centralities including general properties such as extrema, density, and minimality. In Section 3, we investigate extensions of graphs with distinct betweenness centralities to infinite families of graphs of the same type. Finally in Section 4, we state a series of related open problems.

2 Distinct betweenness centralities

We consider the problem of characterizing graphs with distinct betweenness centralities. We introduce a family of graphs that will appear throughout the paper. The pendant ladder graph PL_n is the Cartesian Product $P_2 \times P_n$ with a pendant edge attached to a corner vertex (PL_n has $2n+1$ vertices). The graph PL_3 is shown in Figure 1. This graph has distinct betweenness centralities.

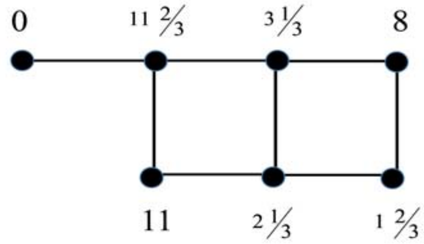

Figure 1. The graph PL_3 distinct betweenness centralities

A graph G with distinct betweenness centralities must have the property that G has a trivial automorphism group. If G has a non-trivial automorphism where two vertices are switched. Then these two vertices will have the same betweenness centrality.

However, it is possible for a graph to have a trivial automorphism group and not have distinct betweenness centralities. An example is shown in Figure 2.

Figure 2. A graph with six vertices with a trivial automorphism group

This graph has three vertices whose closed neighborhood is a complete subgraph, which by Lemma 1.2 will have a betweenness centrality of zero.

Proposition 2.1 *Let G be a graph with distinct betweenness centralities. Then the following properties hold:*

(i) G has a trivial automorphism group.

(ii) There is at most one vertex whose closed neighborhood is a complete subgraph.

However the combination of these two necessary conditions is not sufficient. We give an example in Figure 3, where vertices v_3 and v_4 have a betweenness centrality of $\frac{5}{3}$.

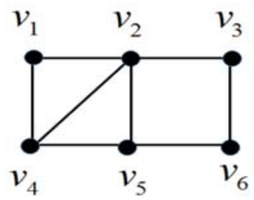

Figure 3. A graph that meets conditions (i) and (ii) but does not have distinct betweenness centralities

2.1 Extremal properties

In the next two theorems we show that a smallest connected graph with distinct betweenness centralities has seven vertices.

Theorem 2.1 *Let n be the number of vertices in a graph. If $2 \leq n \leq 6$, then there are no connected graphs with distinct betweenness centralities.*

Proof. We consider different cases.

- $n = 2$: The only connected graph with two vertices is K_2 which has two vertices with degree one.
- $n = 3$: There are two non-isomorphic connected graphs with exactly 3 vertices: P_3 and K_3. The graph P_3 has two vertices of degree one and in K_3 will have three vertices with betweenness centrality zero, by Lemma 1.2.
- $n = 4$: There are six non-isomorphic connected graphs with four vertices: K_4; $K_4 - e$ (e is an arbitrary edge); $K_4 - P_3$; $K_{1,3}$; C_4; and P_4. All of these graphs have rotational or reflectional symmetry and therefore have a non-trivial automorphism group.
- $n = 5$: All of the 21 non-isomorphic connected graphs with five vertices have a non-trivial automorphism group. A list can be found at [24].
- $n = 6$: We apply Proposition 2.1 to the 112 non-isomorphic connected graphs with six vertices which were given by Cvetković and Petrić [5]. Proposition 2.1 (i) rules out all of the graphs except for graphs 12, 19, 24, 25, 33, 46, 59, 60, 67, 77, 85, 87, 95, and 98 since all of the other graphs clearly have non-trivial automorphism groups. Proposition 2.1 (ii) rules out 11 graphs: 12, 19, 24, 33, 59, 60, 77, 85, 87, 95, and 98. This leaves three graphs: 25, 46, and 67. Graph 67 was shown in Figure 3 and it was noted that there are two vertices with the same betweenness centrality. The final two cases are discussed below.

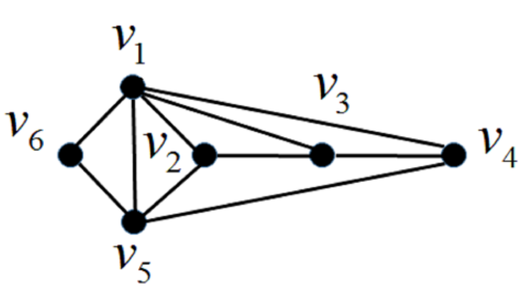

Figure 4. Graph 25

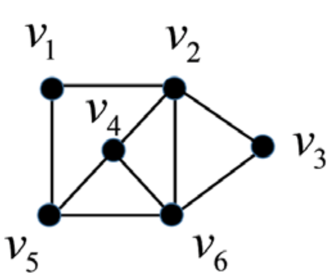

Figure 5. Graph 46

In graph 25 (see Fig. 4) vertices v_2, v_3, and v_4 have a betweenness centrality of $\frac{2}{3}$.

In graph 46 (see Fig. 5) vertices v_1 and v_4 have a betweenness centrality of $\frac{2}{3}$.

Hence there are no graphs with six vertices or less that have distinct betweenness centralities.

This completes the proof. ■

We obtained all 853 non-isomorphic connected graphs with seven vertices from McKay [17]. We note that we need only consider connected graphs since all graphs on five vertices or less have at least two vertices with the same betweenness centrality and the two graphs on six vertices with distinct betweenness centralities have a vertex with a betweenness centrality of zero. Each of the 853 non-isomorphic connected graphs with seven vertices were checked for distinct betweenness centrality using the package MatlabBGL by Gleich [12]. The result was that there are exactly 21 graph with distinct betweenness centralities. An independent verification using Mathematica gave an identical result.

Theorem 2.2 *There are exactly* 21 *graphs on seven vertices with distinct betweenness centralities. These are as follows:*

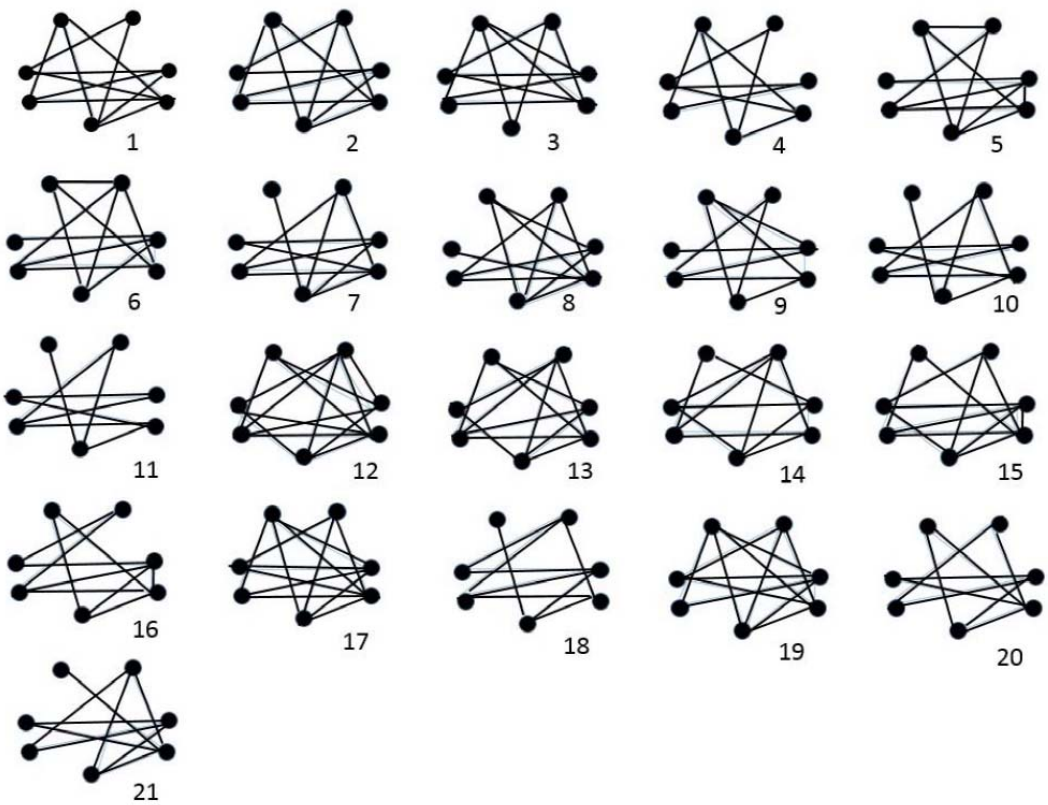

Figure 6. The 21 graphs of order seven with distinct betweenness centralities

2.2 Density

Since every graph with distinct betweenness centralities is an asymmetric graph we can use known results about the density of asymmetric graphs to give properties regarding the density of graphs with distinct betweenness centralities. We recall a theorem of Quintas [20].

Theorem 2.3 *If K is an asymmetric graph having p vertices and q edges, then*
(i) $p = 1$ or $p \geq 6$ and
(ii) $m_p \leq q \leq M_p$, where

$$m_p = \begin{cases} 0 & \text{if} \quad p = 1 \\ 6 & \text{if} \quad p = 6, 7 \\ p - \sum_{n=1}^{N} a_n - w & \text{if} \quad p \geq 8 \end{cases}$$

and

$$M_p = \begin{cases} 0 & \text{if} \quad p = 1 \\ 9 & \text{if} \quad p = 6 \\ 15 & \text{if} \quad p = 7 \\ \frac{p(p-3)}{2} + \sum_{n=1}^{n} a_n + w & \text{if} \quad p \geq 8 \end{cases}$$

where a_n equals the number of asymmetric trees having less than or equal to n vertices and w is the number of asymmetric trees having $n+1$ vertices.

We can refine the above theorem to give properties about the density of graphs with distinct betweenness centralities.

We first note that although the complement of an asymmetric graph is asymmetric, the complement of a graph with distinct betweenness centralities may not have distinct betweenness centralities. We consider the graph shown in Figure 7 and its complement.

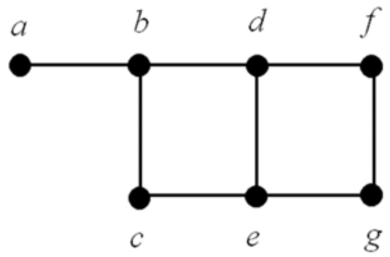

Figure 7. A graph with distinct betweenness centralities whose complement does not have distinct betweenness centralities

Let G be the graph shown in Figure 7. The betweenness centralities for G and $\overline{G}$ are shown in Table 1 below.

v	$bc_G(v)$	$bc_{\overline{G}}(v)$
a	0	5.6667
b	11.6667	1.6667
c	3.3333	1.6667
d	11.0000	0
e	8.0000	0.6667
f	2.3333	2.6667
g	1.6667	3.6667

Table 1.

Theorem 2.4 *Let G be a graph with n vertices and distinct betweenness centralities. Then $n = 1$ or $n \geq 7$ and*

(i) When $n = 1$ we have $|E(G)| = 0$.

(ii) When $n = 7$ we have $8 \leq |E(G)| \leq 14$.

(iii) When $n = 8$ we have $7 \leq |E(G)| \leq 21$.

(iv) When $n = 9$ we have $8 \leq |E(G)| \leq 29$.

(v) When $n \geq 10$ we have $n - 1 \leq |E(G)| \leq \frac{n(n-1)}{2} - \left(n - 2 - \left(\left\lfloor \frac{n-1}{6} \right\rfloor - 1\right)\right)$.

Proof. Case (i) is trivial and Case (ii) follows by inspection of the 21 graphs with seven vertices that have distinct betweenness centralities in Theorem 2.2. For cases (ii)-(v) we note that the lower bound follows from the fact that there can be at most one isolated vertex and no component of G can be a tree with more than one vertex. As a result $|E(G)| \geq n - 1$.

(iii) $n = 8$: Assume $|E(G)| \geq 22$. Since G has distinct betweenness centralities, G is asymmetric. This implies that $\overline{G}$ is asymmetric and $|E(G)| \leq 6$ and since $n = 8$ this is impossible by [20]. Calculation of the betweenness centralities for the graph shown in Figure 8 shows this upper bound is tight.

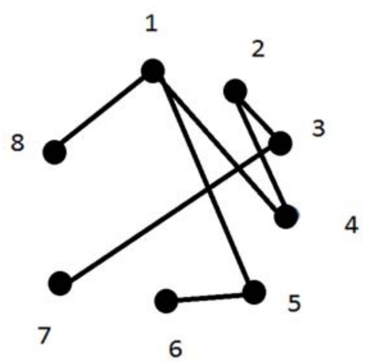

Figure 8. A graph whose complement has distinct betweenness centralities

(iv) $n = 9$: Assume $|E(G)| \geq 30$. Since G has distinct betweenness centralities, G is asymmetric. This implies that $\overline{G}$ is asymmetric and $|E(G)| \leq 6$ and since $n = 9$ this is impossible by [20]. As noted by a referee, the calculation of betweenness centralities for the complement of the graph shown in Figure 9, shows this upper bound is tight.

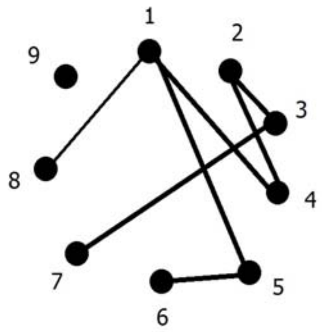

Figure 9. A graph whose complement has distinct betweenness centralities

(iv) $n \geq 10$: The proof will be to show that if G has a large number of edges then the complement will contain a component of size between 2 and 6. Then the complement will not be asymmetric [20], and therefore the original graph will not be asymmetric. Consider the complement. It can have at most one isolated vertex. We could form a bound on the number of edges by having the remaining $n-1$ vertices be part of a tree, which would result in $n-2$ edges. However depending on the tree it may be possible to remove some edges so that the tree breaks up so that each component has order at least 6 (the minimum order for a graph to be asymmetric) [20]. However if $\overline{G}$ has less than $n-2-\left(\left\lfloor \frac{n-1}{6} \right\rfloor - 1\right)$ edges it must have a component of size between 2 and 6 which would make the graph asymmetric. Hence $\left|E(\overline{G})\right| \geq n-2-\left(\left\lfloor \frac{n-1}{6} \right\rfloor - 1\right) \Rightarrow |E(G)| \leq \frac{n(n-1)}{2} - \left(n-2-\left(\left\lfloor \frac{n-1}{6} \right\rfloor - 1\right)\right)$. ■

2.3 Minimality

An undirected graph G with at least two vertices is a *minimal asymmetric* graph if G is asymmetric and no proper induced subgraph of G on at least two vertices is asymmetric. A recent paper by Schweitzer and Schweitzer [22] identifies all 18 minimal asymmetric graphs. It turns out that only two of these graphs have distinct betweenness centralities. The first is the pendant ladder graph shown in Figure 2 (which is also graph number 11 in Theorem 2.2). The second graph is shown in Figure 10.

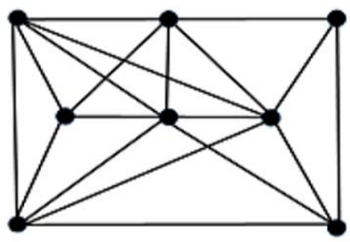

Figure 10. A graph with distinct betweenness centralities

In Theorem 2.2 we presented a complete list of the 21 graphs with seven vertices that have distinct betweenness centralities. It turns out that graph 11 is a subgraph in 19 of the 21 graphs. The other two graphs that appear as subgraphs of others are graphs 4 and 20. We summarize the relationships in the partially ordered set shown in Figure 11, where the relation is by subgraph containment.

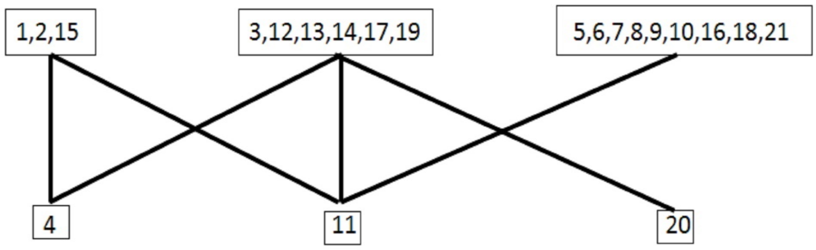

Figure 11. A partially ordered set showing structure of the 21 graphs

3 Infinite families of graphs

We consider the problem of creating an infinite family of graphs with distinct betweenness centralities. One approach would be to start with a graph that has distinct betweenness centralities and show that this graph implies the existence of another graph with distinct betweenness centralities. Another would be to fix each graph in an infinite family and obtain a closed formula of the betweenness centrality for each vertex, and show they are all distinct. In this section we present infinite families created using both of these methods.

We begin with the first approach which involves an extension from a graph with distinct betweenness centralities to other graphs with the same property. If we start with a graph G_0 with a vertex v of degree 1, and we successively append vertices $v_1, v_2, ..., v_k$ of a path to v, the betweenness centrality of the vertices in G_0 changes by a constant amount with each new vertex added.

Lemma 3.1 (Constant) *Let G_0 be a graph with a vertex v of degree 1 where all of the betweenness centralities are distinct. Let G_k be the graph where a path P_k with vertices $v_1, v_2, ..., v_k$ is appended to the vertex v. Then for every vertex $u \in G_k - v$, $bc_{G_{k+1}}(u) = bc_{G_k}(u) + \sum\limits_{a \neq v,u} \frac{\sigma_{va}(u)}{\sigma_{va}}$ for every $k \geq 1$.*

Proof. Consider any $u \in G_k - v$. Since the location of v_1 in G_{k+1} is the same as the location of v in G_k, we have $bc_{G_{k+2}}(u) - bc_{G_{k+1}}(u) = bc_{G_{k+1}}(u) - bc_{G_k}(u)$ and hence $bc_{G_{k+1}}(u) = bc_{G_k}(u) + \sum\limits_{a \neq v,u} \frac{\sigma_{va}(u)}{\sigma_{va}}$. ■

Our next lemma involves an elementary property from geometry which will be useful later in Theorem 3.1.

Lemma 3.2 *Let $L_1, L_2, ..., L_n$ be a finite set of lines in the x, y plane. Then there exists an x value after which no two lines intersect.*

Proof. The lines $L_1, L_2, ..., L_n$ can only intersect at a finite number of points. Let (x, y) be the intersection point with the maximum x value. ■

We show in our next example how a graph with distinct betweenness centralities can be extended to an infinite family of graphs with the same property.

Example 3.1 *Let $G_0 = PL_4$. The vertices in G have distinct betweenness centralities, and eight of the nine are non-integers: $bc_{G_0}(v_0) = 0$, $bc_{G_0}(v_1) = 16\frac{1}{6}$, $bc_{G_0}(v_2) = 19\frac{2}{3}$, $bc_{G_0}(v_3) = 15\frac{1}{2}$, $bc_{G_0}(v_4) = 2\frac{2}{3}$, $bc_{G_0}(v_5) = 4\frac{1}{3}$, $bc_{G_0}(v_6) = 14\frac{1}{6}$, $bc_{G_0}(v_7) = 13\frac{1}{3}$, and $bc_{G_0}(v_8) = 2\frac{1}{6}$. By Lemma 3.1 the change in betweenness centralities of these eight vertices stays constant as vertices are added. Let G_t be the graph G_0 with the path P_t appended to the vertex with degree 1 in G_0. For this graph, $bc_{G_t}(v_1) = 16\frac{1}{6} + t(14)$, $bc_{G_t}(v_2) = 19\frac{2}{3} + 6t(\frac{47}{6}) = 47t + \frac{59}{3}$, $bc_{G_t}(v_3) = 15\frac{1}{2} + 6t(\frac{11}{3}) = 22t + \frac{31}{2}$, $bc_{G_t}(v_4) = 2\frac{2}{3} + 6t(\frac{1}{2}) = 3t + \frac{8}{3}$, $bc_{G_t}(v_5) = 4\frac{1}{3} + 6t(\frac{13}{6}) = 13t + \frac{13}{3}$, $bc_{G_t}(v_6) = 14\frac{1}{6} + 6t(\frac{7}{3}) = 14t + \frac{85}{6}$, $bc_{G_t}(v_7) = 13\frac{1}{3} + 6t(\frac{3}{2}) = 9t + \frac{40}{3}$, $2\frac{1}{6} + 6t(0) = \frac{13}{6}$, and $bc_{Gt}(v_8) = 2\frac{1}{6}$. Since $\operatorname{lcm}(6, 3, 2) = 6$ the total changes over six iterations will be integers. The betweenness centralities for all other vertices will be integers, none of which are the same, so all of the vertices in the graph will have distinct betweenness centralities. Letting t vary over the positive multiples of 6 creates an infinite class of graphs $bc_{G_{6t}}$ which have distinct betweenness centralities.*

We next present general methods for extending a graph with distinct betweenness centralities to an infinite family of graphs with the same property.

Lemma 3.3 (Extension) *Let G_0 be a graph with a vertex v of degree 1 where all of the betweenness centralities are distinct. Let G_k be the graph where a path P_k with vertices v_1,v_2, ..., v_k is appended to the vertex v. Then there exists some sufficiently large k beyond which the betweenness centralities of the vertices in G_0 are all distinct in G_k.*

Proof. Consider any vertex $u \in G_k - v$. By Lemma 3.1 we have that the betweenness centrality values for each u change by a constant amount at each iteration. For each vertex u we have that $bc_{G_{k+1}}(u) = bc_{G_k}(u) + \sum\limits_{a \neq v,u} \frac{\sigma_{va}(u)}{\sigma_{va}}$. Let y_u be a line with slope $\sum\limits_{a \neq v,u} \frac{\sigma_{va}(u)}{\sigma_{va}}$ and y-intercept $bc_{G_0}(u)$. By Lemma 3.2, there exists some value k after which no two lines intersect. Hence there exists some sufficiently large k beyond which the betweenness centralities of the vertices in G_0 are all distinct in G_k. ■

Theorem 3.1 *Let G be a graph with n vertices with a vertex v of degree 1 and $n - 1$ vertices with distinct betweenness centrality values that are not integers. Let G_n be the graph with $V(G_n) = V(G) \cup \{v_i|\ 1 \leq i \leq n\}$ and $E(G_n) = E(G) \cup \{(v, v_1), (v_1, v_2), ..., (v_{n-1}, v_n)\}$. There will be an infinite number of graphs of the form G_n (for sufficiently large n) with distinct betweenness centralities.*

Proof. By Lemma 3.3, each of the $n-1$ vertices with distinct betweenness centrality values will increase by a fixed amount. Let these amounts be $\frac{a_1}{b_1}, \frac{a_2}{b_2}, ..., \frac{a_{n-1}}{b_{n-1}}$. Let $N = \operatorname{lcm}(b_1, b_2, .., b_{n-1})$. After every N iterations the cumulative additions to the original betweenness centrality values will be integers. Hence the betweenness centrality values will all be non-integers. We note that $bc_{G_n}(v_i) = 2(i - 1)(n - i)$ which are all positive integers and all different. Since the appended vertices all have betweenness centrality values that are integers and the vertices in G all have betweenness centrality values that are non-integers, all of the vertices in G_n must have distinct betweenness centralities. ■

3.1 Prayer Flag Graphs

We next use the second approach for obtaining an infinite family of graphs with distinct betweenness centralities. We give a precise calculation of the betweenness centrality of each vertex in each graph of an infinite family. Recall that a ladder graph is the graph $P_2 \times P_l$ with vertices $v_1, v_3, ..., v_{2l-1}$ on the "top" row and vertices $v_2, v_4, ..., v_{2l}$ on the "bottom" row. Then a pendant ladder graph PL_n is obtained by adding a single vertex v_0 and an edge from v_0 to v_1. A *prayer flag graph* PF_n is PL_n with the following set of edges removed: $\{v_{4k}, v_{4k+2}\}$ for all $1 \leq k \leq \frac{l-3}{2}$. An example is shown in Figure 12. In our next proof we will refer to the six vertices $v_{2l-5}, ..., v_{2l}$ as 'the block'.

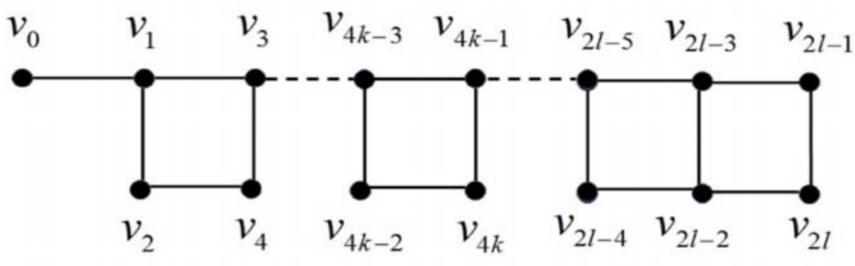

Figure 12. A prayer flag graph

Theorem 3.2 *Let $G = PF_l$. Then:*

$bc(v_0) = 0$

$bc(v_{2l-1}) = 1 + \frac{2}{3}(2l-4) = \frac{4}{3}l - \frac{5}{3} = \frac{1}{3}(4l-5)$

$bc(v_{2l-3}) = 2\left(\frac{2}{3} + 2l - 4 + \frac{1}{2}\right) + \frac{4}{3}(2l-4) + (2l-4) = \frac{26}{3}l - 15$

$bc(v_{2l-5}) = 20l - 50 + \frac{5}{3} = 20l - \frac{145}{3}$

$bc(v_{4k-1}) = 2(4k)(2l-4k) + 4k - 2$ when $4 \leq 4k \leq 2l-6$

$bc(v_{4k-3}) = 2\left((4k-3)(2l-(4k-2)+1) + \frac{1}{2}(2l-(4k-1)+1)\right)$ when $1 \leq 4k - 3 \leq 2l-9$.

$bc(v_{4k}) = 2l - (4k-1) = 2l - 4k + 1$ when $4 \leq 4k \leq 2l-6$

$bc(v_{4k-2}) = 4k - 2$ when $2 \leq 4k \leq 2l-6$

$bc(v_{2l-4}) = \frac{5}{6}(2l-4)$

$bc(v_{2l-2}) = \frac{8l}{3}$

$bc(v_{2l}) = \frac{5}{3}$

Proof.

- $bc(v_0) = 0$

- $bc(v_{2l-1}) = 1 + \frac{2}{3}(2l-4) = \frac{4}{3}l - \frac{5}{3} = \frac{1}{3}(4l-5)$

  v_{2l-1} falls on $\frac{1}{3}$ shortest paths between v_i where $0 \leq i \leq 2l-5$ and v_{2l}. It falls on $\frac{1}{2}$ of the shortest paths between v_{2l-3} and v_{2l}.

  Then we double to account for both directions.

- $bc(v_{2l-3}) = 2\left(\frac{2}{3} + 2l - 4 + \frac{1}{2}\right) + \frac{4}{3}(2l-4) + (2l-4) = \frac{26}{3}l - 15$

  v_{2l-3} falls on $\frac{2}{3}$ shortest paths between v_i where $0 \leq i \leq 2l-5$ and v_{2l}.

  v_{2l-3} falls on $\frac{1}{2}$ shortest paths between v_i where $0 \leq i \leq 2l-5$ and v_{2l-2}.

  v_{2l-3} falls on all of the shortest paths between v_i where $0 \leq i \leq 2l-5$ and v_{2l-1}.

  v_{2l-3} falls on $\frac{2}{3}$ of the shortest paths between v_{2l-4} and v_{2l-1}.

  v_{2l-3} falls on $\frac{1}{2}$ of the shortest paths between v_{2l-2} and v_{2l-1}.

  Then doubling gives the desired result.

- $bc(v_{2l-5}) = 20l - 50 + \frac{5}{3} = 20l - \frac{145}{3}$

  v_{2l-5} falls on all of the shortest paths between v_i where $0 \leq i \leq 2l-6$ and v_j where $2l-4 \leq j \leq 2l$.

  v_{2l-5} falls on $\frac{1}{3}$ of the shortest paths between v_{2l-4} and v_{2l-1}.

  v_{2l-5} falls on $\frac{1}{2}$ of the shortest paths between v_{2l-4} and v_{2l-3}.

  Then doubling gives the desired result.

- $bc(v_{4k-1}) = 2(4k)(2l-4k) + 4k - 2$ when $4k \leq 2l-6$

  v_{4k-1} falls on all of the shortest paths between v_i where $0 \leq i \leq 4k$, $i \neq 4k-1$ and v_i where $4k+1 \leq i \leq 2l$.

  v_{4k-1} falls on $\frac{1}{2}$ of the shortest paths between v_i where $0 \leq i \leq 4k-3$ and v_{4k}.

  Then doubling gives the desired result.

- $bc(v_{4k-3}) = 2\left((4k-3)(2l-(4k-2)+1) + \frac{1}{2}(2l-(4k-1)+1)\right) = 44k - 10l + 16kl - 32k^2 - 16$ when $1 \leq 4k-3 \leq 2l-9$.

  v_{4k-3} falls on all of the shortest paths between v_i where $0 \leq i \leq 4k-4$, and v_i where $4k-2 \leq i \leq 2l$.

  v_{4k-3} falls on $\frac{1}{2}$ of the shortest paths between v_{4k-2} and v_i where $4k-1 \leq i \leq 2l$ and v_{4k+2}.

  Then doubling gives the desired result.

- $bc(v_{4k}) = 2l - (4k-1) = 2l - 4k + 1$ when $4k \leq 2l-6$

  v_{4k} falls on $\frac{1}{2}$ of the shortest paths between v_i where $4k-1 \leq i \leq 2l$, $i \neq 4k$ and v_{4k+2}.

  Then doubling gives the desired result.

- $bc(v_{4k-2}) = 4k - 2$ when $4k \leq 2l - 6$

  v_{4k-2} falls on $\frac{1}{2}$ of the shortest paths between v_{4k} and v_i where $0 \leq i \leq 4k - 3$. Then doubling gives the desired result.

- $bc(v_{2l-4}) = \frac{5}{6}(2l - 4)$

  v_{2l-4} falls on $\frac{1}{2}$ of the shortest paths between v_{2l-2} and v_i where $0 \leq i \leq 2l-5$.
  v_{2l-4} falls on $\frac{1}{3}$ of the shortest paths between v_{2l} and v_i where $0 \leq i \leq 2l - 5$.
  Then doubling gives the desired result.

- $bc(v_{2l-2}) = \frac{8l}{3}$

  v_{2l-2} falls on $\frac{2}{3}$ of the shortest paths between v_i where $0 \leq i \leq 2l - 5$ and v_{2l}.
  v_{2l-2} falls on the shortest path between v_{2l-4} and v_{2l}.
  v_{2l-2} falls on $\frac{1}{2}$ of the shortest paths between v_{2l-3} and v_{2l}.
  v_{2l-2} falls on $\frac{1}{2}$ of the shortest paths between v_{2l-4} and v_{2l-3}.
  v_{2l-2} falls on $\frac{2}{3}$ of the shortest paths between v_{2l-4} and v_{2l-1}.
  Summing these gives $\frac{2}{3}(2l - 4) + 1 + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} = \frac{4}{3}l$.
  Then doubling gives the desired result.

- $bc(v_{2l}) = \frac{5}{3}$
  v_{2l} falls on $\frac{1}{2}$ of the shortest paths between v_{2l-2} and v_{2l-1}.
  v_{2l} falls on $\frac{1}{3}$ of the shortest paths between v_{2l-4} and v_{2l-1}.
  Then doubling gives the desired result.

This completes the proof. ■

Lemma 3.4 *Let $t \in \mathbb{Z}^+$. When $l = (6t + 1)$, block vertices have betweenness centrality values that are non-integers.*

Proof. Let $l = (6t + 1)$. Then
$bc(v_{2l-1}) = \frac{1}{3}(4l - 5) = \frac{1}{3}(4(6t + 1) - 5) = 8t - \frac{1}{3}$
$bc(v_{2l-3}) = \frac{26}{3}(6t + 1) - 15 = 52t - \frac{19}{3}$
$bc(v_{2l-5}) = 10l - \frac{145}{6} = 10(6t + 1) - \frac{145}{6} = 60t - \frac{85}{6}$
$bc(v_{2l-4}) = \frac{5}{6}(2(6t + 1) - 4) = 10t - \frac{5}{3}$
$bc(v_{2l-2}) = \frac{8l}{3} = \frac{8(6t+1)}{3} = 16t + \frac{8}{3}$
$bc(v_{2l}) = \frac{5}{3}$ ■

Lemma 3.5 *Let $t \in \mathbb{Z}^+$. When $l = (6t + 1)$, non-block vertices have betweenness centrality values that are integers.*

Proof. Let $l = (6t + 1)$.
$bc(v_{4k-2}) = 4k - 2$
$bc(v_{4k-1}) = (4k)(2l - 4k) = -4k\,(4k - 2l)$ when $4k - 1 \leq 2l - 7$
$bc(v_{4k}) = 2l - (4k - 1) = 2l - 4k + 1$
$bc(v_{4k+1}) = 2(4k + 1)(2l + 1 - (4k + 2)) + (2l - (4k + 3))$
$= 6l - 20k + 16kl - 32k^2 - 5$ when $4k - 1 \leq 2l - 9$ ■

Lemma 3.6 *The block vertices all have distinct betweenness centralities.*

Proof. When $t \in \mathbb{Z}^+$., We have
$bc(v_{2l}) = \frac{5}{3} < bc(v_{2l-1}) = 8t - \frac{1}{3} < bc(v_{2l-4}) = 10t - \frac{5}{3} < bc(v_{2l-2}) = 16t + \frac{8}{3} < bc(v_{2l-3}) = 52t - \frac{19}{3} < bc(v_{2l-5}) = 120t - \frac{85}{3}$. ■

Lemma 3.7 $\{bc(v_{4k-2}), bc(v_{4k-1})\} \cap \{bc(v_{4k}), bc(v_{4k+1})\} = \emptyset$.

Proof. $\{bc(v_{4k-2}), bc(v_{4k-1})\}$ are all even since $bc(v_{4k-2}) = 4k - 2$ and $bc(v_{4k-1}) = (4k)(2l - 4k) = -4k\,(4k - 2l)$ when $4k - 1 \leq 2l - 7$.

$\{bc(v_{4k}), bc(v_{4k+1})\}$ are all odd since $bc(v_{4k}) = 2l - (4k - 1) = 2l - 4k + 1$ and $bc(v_{4k+1}) = 6l - 20k + 16kl - 32k^2 - 5$ when $4k - 1 \leq 2l - 9$. ■

Lemma 3.8 *All of the vertices of the form v_{4k-2} that are not in the block have distinct betweenness centralities.*

Proof. We know that $bc(v_{4k-2}) = 4k - 2$, which is monotonically increasing, so all betweenness centralities must be distinct. ■

Lemma 3.9 *All of the vertices of the form v_{4k} that are not in the block have distinct betweenness centralities.*

Proof. We know that $bc(v_{4k}) = 2l - 4k + 1$, which is monotonically decreasing, so all betweenness centralities must be distinct. ■

Lemma 3.10 *All of the vertices of the form v_{4k-1} that are not in the block have distinct betweenness centralities.*

Proof. We know that $bc(v_{4k-1}) = -4k\,(4k - 2l)$ when $4k - 1 \leq 2l - 7$, which is monotonically decreasing, so all betweenness centralities must be distinct. ■

Lemma 3.11 *All vertices of the form v_{4k-3} that are not in the block have distinct betweenness centralities.*

Proof. We have established that $bc(v_{4k-3}) = 44k - 10l + 16kl - 32k^2 - 16$. Assume that there exist integers k and h such that $bc(v_{4k-3}) = bc(v_{4h-3})$. Then $44k - 10l + 16kl - 32k^2 - 16 = 44h - 10l + 16hl - 32h^2 - 16$. Then $44k - 10l + 16kl - 32k^2 - 16 - (44h - 10l + 16hl - 32h^2 - 16) = 0$. By simplifying, $44k - 10l + 16kl - 32k^2 - 16 - (44h - 10l + 16hl - 32h^2 - 16) = 4\,(h - k)\,(8h + 8k - 4l - 11)$. Then we look for solutions to the equation $4\,(h - k)\,(8h + 8k - 4l - 11) = 0$. This occurs when $h - k =$

0 or $8h + 8k - 4l - 11 = 0$. We note that if $k = h$ then $v_k = v_h$. Next we consider $8h + 8k - 4l - 11 = 0$. This would imply that $l = 2h + 2k - \frac{11}{4}$, which contradicts the fact that l is an integer. Hence the only time we can have $bc(v_{4k-3}) = bc(v_{4h-3})$ is when $k = h$, or equivalently if $k \neq h$ then $bc(v_{4k-3}) \neq bc(v_{4h-3})$ which is what we wanted to prove. ■

Lemma 3.12 *All of the vertices in the block have distinct non-integer betweenness centralities when* $l \equiv 1 \bmod 6$.

Proof. Recall the betweenness centralities for vertices in the block are: $bc(v_{2l-1}) = \frac{1}{3}(4l - 5)$; $bc(v_{2l-3}) = \frac{1}{3}(19l - 31)$; $bc(v_{2l-5}) = 10l - \frac{145}{6}$; $bc(v_{2l-4}) = \frac{5}{6}(2l - 4)$; $bc(v_{2l-2}) = \frac{8l}{3}$; and $bc(v_{2l}) = \frac{5}{3}$. If $l = (6k + 1)$ we have $bc(v_{2l-1}) = 8k - \frac{1}{3}$; $bc(v_{2l-3}) = 52k - \frac{19}{3}$; $bc(v_{2l-5}) = 60k - \frac{85}{6}$; $bc(v_{2l-4}) = 10k - \frac{5}{3}$; $bc(v_{2l-2}) = 16k + \frac{8}{3}$; and $bc(v_{2l}) = \frac{5}{3}$. The result then follows. ■

We show in our next three lemmas that no vertex of the form v_{4k-2} has the same betweenness centrality as a vertex of the form v_{4k-1}.

We first show that the betweenness centrality values of the bottom left corner vertices are bounded from above by $2l - 8$.

Lemma 3.13 $bc(v_{4k-2}) \leq 2l - 8$.

Proof. Note that since v_{4k-2} is not in the block we must have $k \leq \frac{l-3}{2}$. Then $bc(v_{4k-2}) = 4k - 2 \leq 4\left(\frac{l-3}{2}\right) - 2 = 2l - 8$. ■

In the next lemma we show that the betweenness centrality values of the upper right corner vertices are bounded from below by the value at $k = 1$.

Lemma 3.14 *We have that* $(2)\,(4k)(2l - 4k) + 4k - 2$ *is minimized when* $k = 1$.

Proof. Case 1: $k = 1$

$(2)\,(4k)(2l - 4k) + 4k - 2 = 16l - 30$

Case 2. $k > 1$

We will show that $(2)\,(4k)(2l-4k)+4k-2-(16l-30) = -4\,(k-1)\,(8k - 4l + 7) \geq 0$.

Since $k > 1$, it suffices to show that $8k - 4l + 7 \leq 0$.

Since $l \geq 2k + 3$, $4l \geq 4(2k + 3) = 8k + 12 \geq 8k + 7$. ■

Lemma 3.15 $(2)\,(4k)(2l - 4k) + 4k - 2 > 2l - 8$ *when* $k = 1$.

Proof. $(2)\,(4(1))(2l - 4(1)) + 4(1) - 2 > 2l - 8$

$\Rightarrow (2)\,(4(1))(2l - 4(1)) + 4(1) - 2 = 16l - 30$

Since $16l - 30 > 2l - 8$ the smallest of the betweenness centralities of vertices of the form v_{4k-1} is strictly greater than the largest of the of the betweenness centralities of vertices of the form v_{4k-2}. ■

We show in our next series of lemmas that no vertex of the form v_{4k} has the same betweenness centrality as a vertex of the form v_{4k-3}.

Lemma 3.16 *$bc(v_{4k}) = 2l - (4k-1) = 2l - 4k + 1$ is largest when $k = 1$.*

Proof. This follows since the betweenness centralities of vertices of the form v_{4k} are monotonically decreasing as k increases. ■

Lemma 3.17 *$bc(v_{4k-3}) = 44k - 10l + 16kl - 32k^2 - 16$. is minimized when $k = 1$.*

Proof. Let $k = 1$. Then $bc(v_4) = 44 - 10l + 16l - 32 - 16 = 6l - 4$. We claim that $6l - 4$ is the smallest betweenness centrality of any of the vertices of the form v_{4k-3}. We need to show that $44k - 10l + 16kl - 32k^2 - 16 \geq 6l - 4$ for all $k \geq 1$. $44k - 10l + 16kl - 32k^2 - 16 - (6l - 4) = 4(k-1)(4l - 8k + 3) \geq 0$. We consider two cases.

Case 1: $k = 1$. Then $6l - 4 \geq 6l - 4$.

Case 2. $k \neq 0$. This implies that $(4l - 8k + 3) \geq 0 \Rightarrow 8k - 4l - 3 \leq 0$

We note that $k \leq \frac{l-3}{2}$. If $k = \frac{l-3}{2}$ is the case, we have $8\left(\frac{l-3}{2}\right) - 4l - 3 = -15$. If k is any smaller $8k - 4l - 3 < 0$.

Hence $bc(v_{4k-3}) \geq 6l - 4$. ■

Lemma 3.18 *The largest of the betweenness centralities of vertices of the form v_{4k} is strictly less than the smallest of the betweenness centralities of vertices of the form v_{4k-3}.*

Proof. The largest of the betweenness centralities of vertices of the form v_{4k} is $2l-3$. The smallest of the betweenness centralities of vertices of the form v_{4k-3} is $6l - 4$. Clearly $6l - 4 > 2l - 3$ when $l \geq 1$. The result then follows. ■

4 Conclusion

Our paper shows the existence of infinite families of graphs with distinct betweenness centralities and investigates the problem of characterizing this class of graphs. In Proposition 2.1 we presented two necessary conditions for a graph to have distinct betweenness centralities. We pose the problem of finding conditions that are necessary and sufficient, which would give a complete characterization of this family.

Problem 4.1 *Determine necessary and sufficient conditions for a graph to have distinct betweenness centralities.*

We also pose some ideas involving special cases which are certainly more tractable. In Figure 1 we gave an example of a pendant ladder graph with 7 vertices that has distinct betweenness centralities, and we showed later that many graphs that have distinct betweenness centralities contain this graph. It was conjectured that the family of pendant ladders has distinct betweenness centralities [6]. While this holds for PL_n for values $3 \leq n \leq 9$ it oddly fails to hold for PL_{10} where two vertices have betweenness centralities of 90.2357. While we showed that this family can be tweaked to create prayer flag graphs that can have distinct betweenness centralities, it is plausible that some infinite subsequence of pendant ladder graphs have distinct betweenness centralities. We were able to show that an infinite subsequence of prayer flag graphs has distinct betweenness centralities. This property may hold for the entire sequence. It would also be intriguing to investigate other families of graphs that are similar in structure to pendant ladder graphs or prayer flag graphs.

Using a computer search and MatlabBGL [12] we found that 890 of 11,117 non-isomorphic graphs on eight vertices have distinct betweenness centralities. It would be an interesting problem to determine a set of minimal graphs analogous to the ones shown in Figure 11. This could then be generalized to graphs with more than eight vertices.

Finally, in Theorem 3.1 we showed a method for extending certain graphs with distinct betweenness centralities to families of graphs with the same property. It would be an interesting problem to determine if this method can be generalized.

Acknowledgements

The authors are grateful to the efforts of two anonymous referees whose comments greatly improved this paper. In particular we would like to thank one of these referees whose example in Figure 10 helped us state Theorem 2.4 (iv) with a tight upper bound. This research was supported by a National Science Foundation Research Experiences for Undergraduates Site Award Grant (#1358583). Darren Narayan was also supported by NSF Award #1019532.

References

[1] J. M. Anthonisse, *The rush in a directed graph*, Technical report, Amsterdam: University of Amsterdam Mathematical Centre, 1971.

[2] S. P. Borgatti, M. G. Everett and J. C. Johnson, *Analyzing social networks*, SAGE Publications Limited, 2013.

[3] U. Brandes, S. Borgatti and L. Freeman, Maintaining the Duality of Closeness and Betweenness Centrality, *Social Networks* 44 (2016), 153–159.

[4] E. Bullmore and O. Sporns, Complex brain networks: graph theoretical analysis of structural and functional systems, *Nature* **10** (2009), 186–196.

[5] D. Cvetković and M. Petrić, A table of connected graphs on six vertices, *Discrete Math.* 50 (1984), 37–49.

[6] B. Ek, *Efficiency and betweenness centrality of graphs and some applications*, Masters Thesis (RIT), 2014.

[7] L. Freeman, A set of measures of centrality based upon betweenness, *Sociometry* **40** (1977), 35–41.

[8] L. Freeman, Centrality in social networks conceptual clarification, *Social Networks* 1 (1978/79), 215–239.

[9] L. C. Freeman, S. P. Borgatti and D. R. White, Centrality in valued graphs: A measure of betweenness based on network flow, *Social Networks* 13(2) (1991), 141–154.

[10] S. Gago, J. Coroničová Hurajová and T. Madaras, On Betweenness-Uniform Graphs, *Czechoslovak Math. J.* 63 (138) (2013), 629–642.

[11] S. Gago, J. Hurajov and T. Madaras, Notes on the betweenness centrality of a graph, (English summary), i *Mathematica Slovaca* **62** (2012), 1–12.

[12] D. Gleich, `https://www.mathworks.com/matlabcentral/fileexchange/10922-matlabbgl`.

[13] R. Grassi, R. Scapellato, S. Stefani and A. Torriero, *Networks, Topology and Dynamics*, Springer (Berlin), (2009), 161–175.

[14] M. Guye, G. Bettus, F. Bartolomei and P. J. Cozzone, Graph theoretical analysis of structural and functional connectivity of MRI in normal pathological brain networks, *Magn. Reson. Mater. Phy.* **23** (2010), 409–421.

[15] K. E. Joyce, P. J. Laurienti, J. H. Burdette and S. Hayasaka, A new measure of centrality for brain networks, (2010), *PLoS ONE* 5(8): e12200. doi:10.1371/journal.pone.0012200.

[16] S. Kumar, K. Balakrishnan and M. Jathavedan, Betweenness centrality in some classes of graphs, *Int. J. Combin.* (Hindawi), (2014), Art. 241723, `http://dx.doi.org/10.1155/2014/241723`.

[17] B. D. McKay, `cs.anu.edu.au/people/bdm/data/graphs.html`.

[18] J. Nieminen, On centrality in a graph, *Scandinavian J. Psychology* 15 (1974), 322–336.

[19] A. S. Pandit, P. Expert, R. Lambiotte, V. Bonnelle, R. Leech, F. Turkheimer and D. J. Sharp. Traumatic brain injury impairs small-world topology, *Neurology* **80** (2013), 1826–1833.

[20] L. Quintas, Extrema concerning asymmetric graphs, *J. Combin. Theory* **3** (1967), 57–82.

[21] D. Schoch and U. Brandes, Re-conceptualizing centrality in social networks, *Europ. J. Appl. Math.* 27 (2016), 971–985.

[22] Pascal Schweitzer and Patrick Schweitzer, Minimal asymmetric graphs, arXiv:1605.01320

[23] D. R. White and S. P. Borgatti, Betweenness centrality measures for directed graphs, *Social Networks* 16(4) (1994), 335–346.

[24] `http://wiki.smp.uq.edu.au/G-designs/index.php/Graphs_with_five_vertices`

(Received 25 Aug 2016; revised 17 Jan 2017, 12 Mar 2017)