AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **64(3)** (2016), Pages 437–443

Additive structure of difference sets and a theorem of Følner

Norbert Hegyvári

ELTE TTK, Eötvös University, Institute of Mathematics
H-1117 Pázmány st. 1/c, Budapest
Hungary
hegyvari@elte.hu

Imre Z. Ruzsa

Alfréd Rényi Institute
Hungarian Academy of Sciences, Pf.127, H-1365
Hungary
ruzsa@renyi.hu

Abstract

A theorem of Følner asserts that for any set $A \subset \mathbb{Z}$ of positive upper density there is a Bohr neigbourhood B of 0 such that $B \setminus (A - A)$ has zero density. We use this result to deduce some consequences about the structure of difference sets of sets of integers having a positive upper density.

1 Introduction

This paper is about the structure of the difference set $D(A) := A - A$ of sets of integers having positive density. By density we mean the upper asymptotic density defined by

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap [-n, n]|}{2n+1} > 0.$$

For sets $X, Y \subseteq \mathbb{Z}$ we mean

$$X + Y = \{x + y : x \in X; y \in Y\}$$

and

$$X \cdot Y = \{x \cdot y : x \in X; y \in Y\}.$$

When $X = \{x\}$ we write $x \cdot Y$.

We define a *Bohr set* as a set of the form

$$B(S, \varepsilon) = \{m \in \mathbb{Z} : \max_{s \in S} \|sm\| < \varepsilon\}, \tag{1.1}$$

where S is a finite set of real numbers. Here $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$, the absolute fractional part.

Recall that every Bohr set has positive density, and for every pair of sets S, S' and for every k, $0 < k \cdot \varepsilon' \leq \varepsilon$, we have

$$k \cdot B(S, \varepsilon') \subseteq B(S, \varepsilon), \tag{1.2}$$

and

$$B(S \cup S', \varepsilon) = B(S, \varepsilon) \cap B(S', \varepsilon) \tag{1.3}$$

(see e.g. [9] p. 165).

These sets are just the basic neighbourhoods of 0 in the Bohr topology. We say that $B(S, \varepsilon)$ is a k, ε-neighbourhood if $|S| = k$ (or a k-neighbourhood if ε is unimportant).

Bogolyubov [4] proved in the case of integers, and Følner [5], [6] generalized for general commutative groups, that the second difference set $D(D(A)) = A - A + A - A$ of a set having positive upper Banach density is always a Bohr neighborhood of 0.

In Bogolyubov's theorem four copies of A are used. Three suffice with a small change. If r, s, t are nonzero integers satisfying $r + s + t = 0$ and A is a set of integers having positive Banach density, then $S = rA + sA + tA$ is a Bohr neighbourhood of 0, see [3]. Here $rA = \{ra : a \in A\}$. The condition $r + s + t = 0$ is necessary to exclude trivial counterexamples; so there is no really "symmetric" result here. (A further comment on this is given in Section 3). The case $r = s = 1$, $t = -2$ immediately generalizes Bogolyubov's theorem.

On the other hand, a theorem of Kříž ([8]) implies that there is a set A with positive upper density whose difference set contains no Bohr set.

In the positive direction in [4] Følner proved that there is a Bohr set which is almost a subset of $A - A$; the exceptional set has zero density.

In this paper we give some applications of Følner's theorem. We investigate $A + A + A$ and $A + A - A$ and Bohr sets. In [1] Bergelson investigated the additive structure of $D(A)$. He also proved that for every k there exists an infinite set B of integers for which $A - A \supseteq B + B + \cdots + B$, k times, provided A has positive upper density. His proof of this theorem is based on an ergodic theorem, namely the Fürstenberg correspondence theorem In [7] the first author gave a purely combinatorial proof of this result. Here we give a third proof of it using Følner's theorem. See also Theorem 2.5 in [2].

2 Structure of sum-differences

We have already mentioned in the introduction that $D(D(A))$ always contains a Bohr set, while the set $D(A)$ does not necessary contain a Bohr set. Now we investigate the three-fold sum-differences of A. This generalizes Bogolyubov's theorem in a different direction.

Theorem 2.1 *There is a symmetric set A of integers such that $0 \in A$, $\overline{d}(A) > 0$ and the set $A + A + A$ does not contain a Bohr set.*

On the other hand we prove that $A + A - A$ is always a Bohr neighborhood of many $a \in A$.

Theorem 2.2 *Assume that $\overline{d}(A) > 0$. There exists a subset A' of A, such that $d(A \setminus A') = 0$ and for every $a' \in A'$, the set $A + A - A - a'$ is a Bohr neighbourhood of 0.*

We remark that the arguments used in our proof (see Section 3) actually yield the following property. For every A and X, with $\overline{d}(A) > 0, \overline{d}(X) > 0$, there exists a subset X' of X such that $d(X \setminus X') = 0$, and for every $x' \in X'$, the set $X + A - A - x'$ is a Bohr neighbourhood of 0. We leave the details of this to the interested reader.

Let $f : \mathbb{N}_+ \to \mathbb{N}_+$ be any function and $C \subseteq \mathbb{N}$; $C \neq \emptyset$. We will use the following notation:

$$FS_f(C) := \Big\{ \sum_{c_i \in X} w_i c_i : X \subseteq C,\ |X| < \infty;\ w_i \in [1, f(i)] \cap \mathbb{N} \Big\}.$$

Let the sum be zero when X is the empty set.

Furthermore write

$$FP(C) := \Big\{ \prod_{c_i \in X} c_i : X \subseteq C;\ X \neq \emptyset,\ |X| < \infty \Big\}.$$

Clearly we have

$$FS_f(\{c_1, c_2, \ldots c_n\}) = FS_f(\{c_1, c_2, \ldots c_{n-1}\}) + \{0, c_n, \ldots, f(n)c_n\}, \tag{2.1}$$

and

$$FP(\{c_1, c_2, \ldots c_n\}) = FP(\{c_1, c_2, \ldots c_{n-1}\}) \cdot \{1, c_n\}, \tag{2.2}$$

for every $\{c_1, c_2, \ldots c_n\} \subseteq \mathbb{N}$; $n \geq 2$; or equivalently,

$$FP(\{c_1, c_2, \ldots c_n\}) = FP(\{c_1, c_2, \ldots c_{n-1}\}) \cup c_n \cdot FP(\{c_1, c_2, \ldots c_{n-1}\}).$$

Theorem 2.3 *Let A be a set of integers $\overline{d}(A) > 0$. Let $f : \mathbb{N}_+ \to \mathbb{N}_+$ be any function. There exists an infinite set C of integers, such that*

$$A - A \ \supseteq \ FS_f(C) \cup FP(C).$$

This will give a third proof of Bergelson's theorem (see [1]). In fact we can conclude that $A - A$ contains both an additive and a multiplicative structure.

3 Proofs

Proof of Theorem 2.1.

By the theorem of Kříž [8] we know the existence of a set X of positive integers for which $\overline{d}(X) > 0$, and the set $X - X$ does not contain a Bohr set. Let

$$Y = \{4x + 1 : x \in X\},$$

and

$$A = Y \cup -Y \cup \{0\}.$$

Since $\overline{d}(Y) = \frac{1}{4}\overline{d}(X) > 0$, we have $\overline{d}(A) > 0$ and the set A is symmetric and contains 0.

Now we prove that $A+A+A$ does not contain a Bohr set. Assume to the contrary that there is a $B(S, \varrho) \subseteq A + A + A$. Then by (1.2), $4 \cdot B(S, \varrho/4) \subseteq A + A + A$. Now notice that $4k \in A + A + A$ if and only if $4k \in Y - Y = 4(X - X)$. So we conclude that $B(S, \varrho/4) \subseteq X - X$ which contradicts the fact that $X - X$ does not contain a Bohr set. □

Proof of Theorem 2.2.

Let $B = B(S, \varepsilon)$ be a Bohr set for which

$$d(B(S, \varepsilon) \setminus (A - A)) = 0,$$

the existence of which is given by Følner's theorem. Since $\{B(S, \varepsilon)+x : x \in \mathbb{Z}\}$ is an open covering of $\mathbb{Z}$ in the (compact) Bohr topology, there is a finite set T for which

$$B(S, \varepsilon) + T = \mathbb{Z}.$$

For $t \in T$ write $A_t = A \cap (B + t)$. Some of these sets have positive upper density; let A' be the union of such sets A_t. Clearly $A \setminus A'$ is contained in the union of finitely many A_t of density 0, so it has density 0 itself.

Put $B' = B(S, \varepsilon/3)$. We now show $A + A - A \supset A' + B'$. This is equivalent to $A + A - A \supset A_t + B'$ whenever $\overline{d}(A_t) > 0$.

Take arbitrary $a \in A_t$, $b \in B'$. Consider the set $a + b - A_t$. This has positive upper density and

$$a + b - A_t \subset A_t - A_t + B' \subset (B' + t) - (B' + t) + B' = B' + B' - B' \subset B.$$

Hence $a + b - A_t$ is contained, up to a subset of density 0, in $A - A$, so we can find $a' \in A_t$ such that $a+b-a' \in A-A$, and consequently $a+b \in a'+A-A \subset A+A-A$ as wanted. □

Proof of Theorem 2.3.

We start our proof by quoting Følner's theorem again. We have a Bohr set for which the exceptional set has density zero, i.e. for some $B = B(S, \varepsilon)$, $E := B(S, \varepsilon) \setminus (A - A)$, $d(E) = 0$.

We will prove the existence of the infinite set C inductively.

Let $K_1 := f(1)$. Since any Bohr set has positive density and the exceptional set has zero density, and also using (1.2), it follows that one can find an element c_0 from $B(S, \varepsilon/K_1)$ such that $ic_1 \notin E$, for $i = 1, 2, \dots K_1$. So we have

$$FS_f(\{c_1\}) \cup FP(\{c_1\}) = \{0, c_1, \dots, K_1c_1\} \subseteq B \setminus E \subseteq A - A.$$

Assume now that the elements $c_1 < c_2 < \cdots < c_n$ have been defined with the property

$$\mathcal{F}_n := FS_f(\{c_1, c_2, \dots, c_n\}) \cup FP(\{c_1, c_2, \dots, c_n\}) \subseteq B \setminus E \subseteq A - A.$$

Write $FP(\{c_1, c_2, \dots, c_n\}) = \{p_1 < p_2 < \cdots < p_m\}$, and let $K := \max\{f(n+1), p_m\}$. Define

$$\varepsilon_1 = \frac{1}{K} \min\{\varepsilon - \|xs\| : x \in FS_f(\{c_1, c_2, \dots, c_n\}); s \in S\}, \tag{3.1}$$

and let $B_1 := B(S, \varepsilon_1)$. Note that $B(S, \varepsilon_1) \subseteq B = B(S, \varepsilon)$.

By (3.1) we have that for every non-negative integer $i \le K$, for every $u \in FS_f(\{c_1, c_2, \dots, c_n\})$, for every $c \in B_1$ and $s \in S$,

$$\|s(u + ic)\| < \varepsilon$$

holds; hence

$$FS_f(\{c_1, c_2, \dots c_n\}) + \{0, c, 2c, \dots K \cdot c\} \subseteq B.$$

Now we claim that there exists an element $c \in B_1$, with $c > c_1$, for which

$$FS_f(\{c_1, c_2, \dots c_n\}) + \{0, c, 2c, \dots K \cdot c\} \subseteq B \setminus E \subseteq A - A$$

also holds.

Assume to the contrary that for every $c \in B_1$ with $c > c_1$ there is at least one element $x \in FS_f(\{c_1, c_2, \dots c_n\})$ and one integer $j \in [1, \dots, K]$ for which $x + jc \in E$. Since $d(B_1 \setminus [1, c_n]) > 0$, by the pigeonhole principle there is then an $x_0 \in FS_f(\{c_1, c_2, \dots c_n\})$, $j_0 \in [1, \dots, K]$ and a $B_1' \subseteq B_1$, such that $\underline{d}(B_1) > 0$ and $x_0 + j_0B_1' \subseteq E$, contradicting the fact that $d(E) = 0$ and $\underline{d}(x_0 + j_0B_1') > 0$.

Let c_{n+1} be any such c. Since $K \ge p_m$ and $0 \in FS_f(\{c_1, c_2, \dots, c_n\})$ we have

$$c_{n+1} \cdot FP(\{c_1, c_2, \dots, c_n\}) \subseteq \{0, c_{n+1}, 2c_{n+1}, \dots, K \cdot c_{n+1}\} \subseteq B \setminus E.$$

Then by $(2, 2)$ and by the inductive hypothesis, $FP(\{c_1, c_2, \dots, c_n, c_{n+1}\}) \subseteq B \setminus E$. Moreover $K > f(n+1)$,

$$\begin{aligned} FS_f(\{c_1, c_2, \dots c_n, c_{n+1}\}) &\subseteq FS_f(\{c_1, c_2, \dots c_n\}) \\ &\qquad + \{0, c_{n+1}, 2c_{n+1}, \dots, K \cdot c_{n+1}\} \\ &\subseteq B \setminus E. \end{aligned}$$

Thus we have that

$$\mathcal{F}_{n+1} \subseteq B \setminus E \subseteq A - A,$$

as we wanted.

So our desired set is

$$C := \{c_1 < c_2 < \dots c_n < c_{n+1} < \dots\}.$$

□

4 Further problems and results

We mention some open problems and announce some new results without proofs.

Bogolyubov's proof is effective: given the density of A one can specify k, η so that $A + A - A - A$ contains a Bohr k, η-set. Følner's proof is not effective, and the reason is that an effective version does not hold:

For every $\alpha < 1/2$, $k \in \mathbb{N}$ and $\eta > 0$ there is an $A \subset \mathbb{Z}$, $\overline{d}(A) > \alpha$ such that $\overline{d}(V \setminus (A - A)) > 0$ for every Bohr k, η-set V.

Our proof of Theorem 2.2 about $A + A - A$ used Følner's theorem, and so it is not effective. We cannot decide whether an effective version holds. However, we can solve positively a related finite question. The result is as follows:

Let $\alpha > \varepsilon > 0$ be given. There exist k, η depending on α and ε with the following property. For every $A \subset \mathbb{Z}_m$, $|A| \geq \alpha m$, the set $S = A + A - A - a$ contains a Bohr k, η-set for all but εm elements $a \in A$.

Here $\mathbb{Z}_m$ is the group of residues modulo m and Bohr sets are defined as in (1.1) with the modification that only rational numbers for $s \in S$ of the form k/m can be used.

Assume $\overline{d}(A) > 0$. Is $A - A$ a Bohr neighbourhood of *something*? We know it may not be a neighbourhood of 0, and 0 is the most natural difference. For the analogous finite question we can give a negative answer, which is as follows:

Let $\alpha < 1/2$, k, η be given. For all large m there is an $A \subset \mathbb{Z}_m$, $|A| \geq \alpha m$, such that $A - A - x$ does not contain a Bohr k, η-set for any $x \in \mathbb{Z}_m$.

We close by posing the following open question.

Is $A - A$ a Bohr neighbourhood of 0 under the stronger assumption that A has positive lower Banach density? (In this case A is syndetic, that is, has bounded gaps).

Here we cannot solve the related finite problem either, and do not have any heuristic reasoning in any direction.

Acknowledgements

We thank the reviewers for their careful reading and for much advice. This note is supported by OTKA grants K 81658, K 100291.

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(Received 5 June 2015; revised 24 Jan 2016)