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Domination number of the directed cylinder

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Abstract

Let $\vec{P_m}\Box\vec{C_n}$ be the Cartesian product of the directed path $\vec{P_m}$ and the directed cycle $\vec{C_n}$. In this paper, we give the exact value of the domination number and the signed 2-independence number of $\vec{P_m}\Box\vec{C_n}$ for any integers m and n.

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1 Introduction

All digraphs considered in this paper are finite, without loops and multiple arcs. For notation and terminology not defined here, we generally follow [2]. Specifically, let G be a digraph with vertex set $V(G)$ and arc set $A(G)$. We say that u is an *in-neighbor* of v and v is an *out-neighbor* of u if uv is an arc of G. For a vertex $v \in V(G)$, the sets of in-neighbors and out-neighbors of v are called the *open in-neighborhood* $N_G^-(v)$ and *open out-neighborhood* $N_G^+(v)$ of v, respectively. The *closed in-neighborhood* of v is $N_G^-[v] = N_G^-(v) \cup \{v\}$. The numbers $d_G^-(v) = |N_G^-(v)|$ and $d_G^+(v) = |N_G^+(v)|$ are the *indegree* and *outdegree* of v, respectively. We omit the subscript G whenever no ambiguity on G is possible. For $S \subseteq V(G)$, $G[S]$ denotes the subdigraph induced by S.

Given two digraphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$, the Cartesian product $G_1 \Box G_2$ is the digraph with vertex set $V_1 \times V_2$ and $(x_1, x_2)(y_1, y_2) \in A(G_1 \Box G_2)$ if and only if $x_1 = y_1$ and $x_2y_2 \in A_2$ or $x_2 = y_2$ and $x_1y_1 \in A_1$, where $x_i, y_i \in V_i$ for $i = 1, 2$. Throughout this paper, we denote the sets of vertices of the directed path $\vec{P_m}$ and the directed cycle $\vec{C_n}$ by $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively, and $A(\vec{P_m}) = \{u_1u_2, u_2u_3, \ldots, u_{m-1}u_m\}$ and $A(\vec{C_n}) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Moreover, in the Cartesian product $\vec{P_m} \Box \vec{C_n}$, let $X_j = \bigcup_{i=1}^{n} \{(u_j, v_i)\}$ for $1 \leq j \leq m$ and let $Y_i = \bigcup_{j=1}^{m} \{(u_j, v_i)\}$ for $1 \leq i \leq n$. Throughout this paper, for Y_i, the subscript i is taken modulo n. Thus, if $i \leq 0$, then $Y_i = Y_{n+i}$, and if $i > n$, then $Y_i = Y_{i-n}$.

A vertex u dominates a vertex v if $u = v$ or $uv \in A(G)$. A set $D \subseteq V(G)$ is a dominating set of G if any vertex of $V(G)$ is dominated by at least one vertex of D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. In other words, a function $f : V(G) \to \{0, 1\}$ is called a *dominating function* if $\sum_{u \in N^-[v]} f(u) \geq 1$ holds for each vertex $v \in V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum weight of a dominating function on G. Jacobson and Kinch [6] first studied the domination number of Cartesian products of two undirected graphs and this work was then continued by many others. In particular the domination number of the Cartesian product of paths and cycles has been studied in [1, 3, 5, 11], etc. Recently, there are some research articles on the domination number of Cartesian products of directed paths and cycles ([7, 8, 15], etc.). In general, the determination of the domination number of a directed or undirected graph is a difficult question in graph theory. This problem has connections with information theory. For example the domination number of hyper cubes is linked to error-correcting codes.

We also consider the signed 2-independence number of directed graphs. The signed 2-independence number of undirected graphs has been studied in [4, 10] and elsewhere. Recently, Volkmann [12] began to investigate this parameter in digraphs. Formally, a function $f : V(G) \to \{-1, 1\}$ is called a *signed 2-independence function* (abbreviated S2IF) if $\sum_{u \in N^-[v]} f(u) \leq 1$ for each vertex $v \in V(G)$. The *signed 2-independence number* of G, denoted by $\alpha_s^2(G)$, is the maximum weight of a S2IF on G. We call a S2IF of weight $\alpha_s^2(G)$ an $\alpha_s^2(G)$-function on G. Volkmann [12] presented some upper bounds on $\alpha_s^2(G)$ for general digraphs G, Wang and Kim [13, 14] determined the exact values for the signed 2-independence number of Cartesian products $\vec{P_m}\Box\vec{P_n}$ $(1 \le m \le 5, n \ge 1)$ and $\vec{C_m}\Box\vec{P_n}$ $(2 \le m \le 5, n \ge 2)$. Throughout this paper, if f is a S2IF of G, then we let P and M denote the sets of those vertices in G which are assigned under f the value 1 and -1, respectively. Therefore $|V(G)| = |P| + |M|$ and $\alpha_s^2(G) = \max(|P| - |M|)$.

In this paper, we give the exact value of the domination number of the Cartesian product of the directed path $\vec{P_m}$ and the directed cycle $\vec{C_n}$, for any integers m, n. We also show that for a directed graph G with minimum indegree 1 and maximum indegree 2, the signed 2-independence number $\alpha_s^2 = |V(G)| - 2\gamma(G)$, where $\gamma(G)$ is the domination number of G. Consequently, we obtain the signed 2-independence number from the domination number of $\vec{P_m}\Box\vec{C_n}$ for all integers m and n.

2 Domination number of $\vec{P_m}\Box\vec{C_n}$ for $m \le 6$

Lemma 2.1. *Given a sequence S of n numbers, s_i, where the indices are considered modulo n. There exists an index a such that $\sum_{i=a}^{a+k} s_i/(k+1) \le \sum_{i=0}^{n-1} s_i/n$ for all k. In other words, there exists an index a in the sequence such that any subsequence starting from a has an average which is at most the average of the entire sequence.*

Proof. Without loss of generality we can assume the average of the sequence is 0, otherwise we just subtract the average from every element in the sequence.

Now let S_m be a subsequence of S with minimum sum. Then a is the index of the first element in S_m. Every sequence starting from a and ending inside S_m must have a sum less than or equal to 0, otherwise the remaining elements in S_m form a sequence with smaller sum. Every sequence starting from a and ending outside of S_m must have a sum less than or equal to 0, otherwise the compliment of this sequence with S_m appended to it will have a smaller sum than S_m.

□

In the following proofs we will apply Lemma 2.1 to prove lower bounds for $\gamma(\vec{P_m}\Box\vec{C_n})$ for some fixed m. We do this by considering a sequence S_D, where D is a dominating set for a directed cylinder $\vec{P_m}\Box\vec{C_n}$. Elements in this sequence are such that $s_i = |Y_{n-i} \cap D|$ is the i-th element of S_D.

Upper bounds will be proved by construction. We will give parts of dominating sets, A and B_i, such that a minimum dominating set can be constructed by taking the appropriate B_i, if any, and filling the remainder of the cylinder with copies of A.

This first proof is easy enough without Lemma 2.1 even though it can be proved in that way as well.

Theorem 2.2. $\gamma(\vec{P_1}\Box\vec{C_n}) = \lceil \frac{n}{2} \rceil$.

Proof. Any vertex of the cycle can dominate at most 2 vertices (itself and its outgoing neighbor).

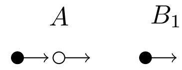

Figure 1: Upper bound for $\vec{P}_1 \Box \vec{C}_n$

An upper bound can be found in Figure 1. □

Now we will get a first example of how Lemma 2.1 will be used in the proofs.

Theorem 2.3. $\gamma(\vec{P}_2 \Box \vec{C}_n) = n$.

Proof. Assume there exists a dominating set D on $\vec{P}_2 \Box \vec{C}_n$ with less than n vertices, then, according to Lemma 2.1, there is an index a such that $\sum_{i=a}^{a+k} s_i/(k+1) < 1$, for all k. So $s_a = 0$, which means Y_{n-a} has no dominating vertices and $s_{a+1} < 2$. Therefore not all vertices in Y_{n-a} will be dominated, a contradiction.

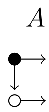

Figure 2: Upper bound for $\vec{P}_2 \Box \vec{C}_n$

An upper bound can be found in Figure 2. □

For the following proof we need to look at a slightly longer part of the sequence before we reach a contradiction.

Theorem 2.4. $\gamma(\vec{P}_3 \Box \vec{C}_n) = \lceil \frac{5n}{4} \rceil$.

Proof. Assume there exists a dominating set D on $\vec{P}_3 \Box \vec{C}_n$ with less than $5n/4$ vertices, then, according to Lemma 2.1, there is an index a such that $\sum_{i=a}^{a+k} s_i/(k+1) < 5/4$, for all k. This means $s_a \in \{0, 1\}$. If $s_a = 0$, then $s_{a+1} < 5/2$ but to dominate Y_{n-a} we need $s_{a+1} = 3$. So $s_a = 1$ and $s_{a+1} < 3/2$, because Y_{n-a} needs to be dominated we get $s_{a+1} = 1$. Similarly we have $s_{a+2} = 1$ and $s_{a+3} = 1$. However, it can be easily seen that 4 consecutive Y_i with only one dominating vertex are not possible in a dominating set, a contradiction. For $n < 4$ the lower bound is easily found by brute force and applying Lemma 2.1.

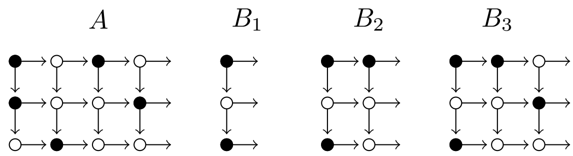

Figure 3: Upper bound for $\vec{P}_3 \Box \vec{C}_n$

An upper bound can be found in Figure 3. □

The next proof is slightly more complicated and we will use a more algorithmic approach to see what dominating sets are still possible. So as we get restrictions on the number of dominating vertices, we will build all possible corresponding dominating sets. We will use Lemma 2.5 to further reduce the number of dominating sets we have to consider. A proof for this lemma can be found in [5].

Lemma 2.5. *Given a graph G and two vertex sets, A and B, with $|A| \geq |B|$ and $(\cup_{v\in A} N^+_G[v]) \subseteq (\cup_{v\in B} N^+_G[v])$. If S_A, with $A \subseteq S_A$, is a minimum dominating set for G, then there exists a minimum dominating set for G, S_B, with $B \subseteq S_B$.*

Therefore we do not have to consider minimum dominating sets containing A, when searching for a minimum dominating set for G, with A and G according to Lemma 2.5.

Theorem 2.6. $\gamma(\vec{P}_4 \Box \vec{C}_n) = \lceil \frac{5n}{3} \rceil$.

Proof. Assume there exists a dominating set D on $\vec{P}_4 \Box \vec{C}_n$ with less than $5n/3$ vertices, then, according to Lemma 2.1, there is an index a such that $\sum_{i=a}^{a+k} s_i/(k+1) < 5/3$, for all k. This means $s_a \in \{0, 1\}$. If $s_a = 0$, we need $s_{a+1} \geq 4$ to dominate Y_{n-a}, but $s_a + s_{a+1} < 10/3$. So $s_a = 1$ and $s_{a+1} = 2$, this gives us three possible situations (Figure 4).

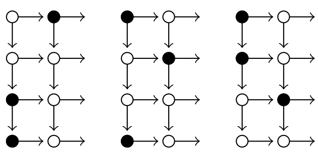

Figure 4: Possible dominating vertices in $\vec{P}_4 \Box \vec{C}_n$ for $k = 1$

Now we will always build all possible dominating sets for the first $k + 1$ paths from the possible dominating sets for the first k paths. While we do this we use Lemmas 2.1 and 2.5 to keep the possible dominating sets to a minimum.

In Figures 5 to 12 this is done for $k < 10$. We notice that there is a bijection between possible dominating sets, S, for $k = 6$ and possible dominating sets, T, for $k = 9$ such that $Y_{n-a} \cap S = Y_{n-a} \cap T$, $(u_j, v_{n-a-6}) \in Y_{n-a-6} \cap S$ if and only if $(u_j, v_{n-a-9}) \in Y_{n-a-9} \cap T$, and $|T| = |S| + 5$. For $k = 6$ we then have that $5(k+1)/3 - |S| = 5(k+4)/3 - |T| = c$, which means that we can add at most $c + 5/3$ dominating vertices to $Y_{n-a-k-1}$ or $Y_{n-a-k-4}$. It follows that for all $k \geq 6$ there is a similar bijection between possible dominating sets for k and $k + 3$ and Lemma 2.1 will give the same restrictions.

However, in none of these sets is Y_{n-a-k} dominated by Y_{n-a}. This means we need more vertices to create a dominating set for $\vec{P}_4 \Box \vec{C}_n$. This is a contradiction with the assumed existence of a smaller dominating set.

An upper bound can be found in Figure 13. □

For $\gamma(\vec{P}_5 \Box \vec{C}_n)$ we have an easy proof once again.

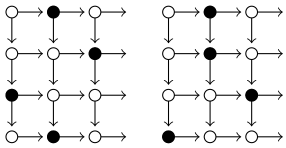

Figure 5: Possible dominating vertices in $\vec{P}_4 \square \vec{C}_n$ for $k = 2$

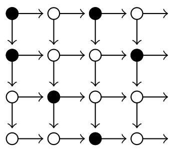

Figure 6: Possible dominating vertices in $\vec{P}_4 \square \vec{C}_n$ for $k = 3$

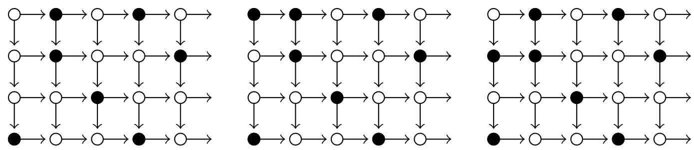

Figure 7: Possible dominating vertices in $\vec{P}_4 \square \vec{C}_n$ for $k = 4$

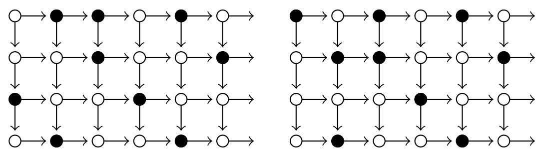

Figure 8: Possible dominating vertices in $\vec{P}_4 \square \vec{C}_n$ for $k = 5$

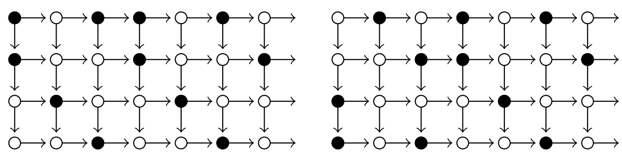

Figure 9: Possible dominating vertices in $\vec{P}_4 \square \vec{C}_n$ for $k = 6$

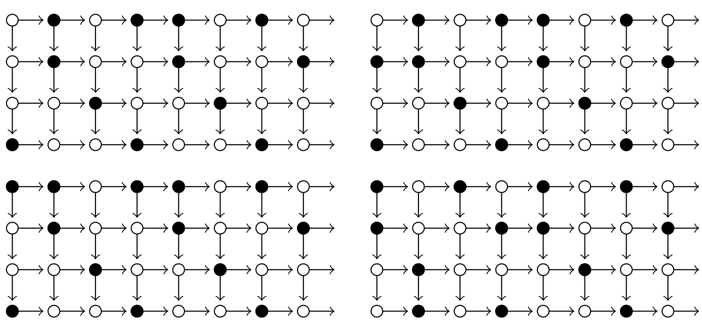

Figure 10: Possible dominating vertices in $\vec{P}_4 \square \vec{C}_n$ for $k = 7$

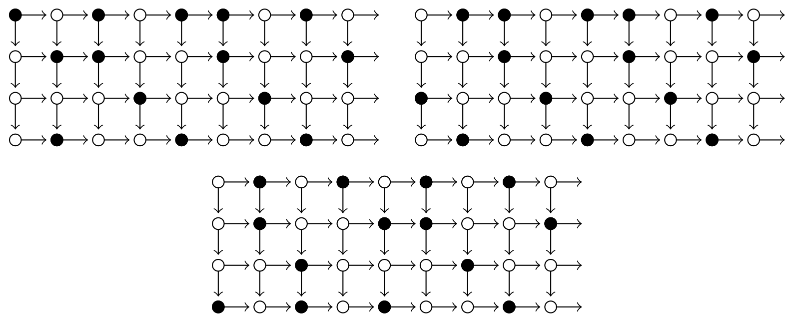

Figure 11: Possible dominating vertices in $\vec{P}_4 \square \vec{C}_n$ for $k = 8$

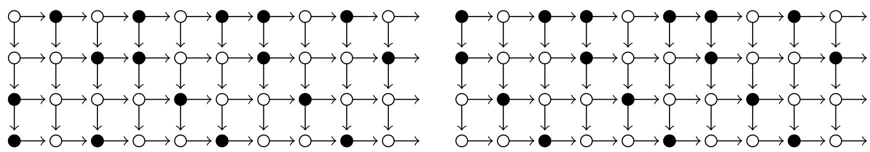

Figure 12: Possible dominating vertices in $\vec{P}_4 \square \vec{C}_n$ for $k = 9$

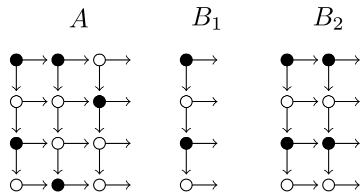

Figure 13: Upper bound for $\vec{P}_4 \square \vec{C}_n$

Theorem 2.7. *$\gamma(\vec{P}_5 \Box \vec{C}_n) = 2n$ if $n \notin \{1, 2, 5\}$.*

Proof. Assume there exists a dominating set D on $\vec{P}_5 \Box \vec{C}_n$ with less than $2n$ vertices, then, according to Lemma 2.1, there is an index a such that $\sum_{i=a}^{a+k} s_i/(k+1) < 2$, for all k. So $s_a < 2$, which means Y_{n-a} has at most 1 dominating vertex. Since Y_{n-a} has at least $5 - 2s_a$ undominated vertices remaining we have that $s_{a+1} \geq 5 - 2s_a$, a contradiction with $s_a + s_{a+1} < 4$.

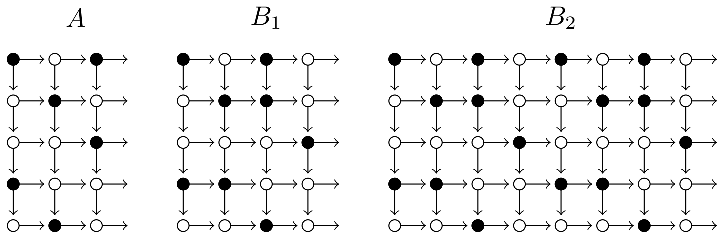

Figure 14: Upper bound for $\vec{P}_5 \Box \vec{C}_n$

An upper bound can be found in Figure 14. □

The last proof in this section will be similar to the proof for $m = 4$, just slightly longer.

Theorem 2.8. *$\gamma(\vec{P}_6 \Box \vec{C}_n) = \frac{7n+4}{3}$ if $n \equiv 2 (\mathrm{mod}\ 3)$ and $n > 2$.*

Proof. Assume there exists a dominating set D on $\vec{P}_6 \Box \vec{C}_n$, $n \equiv 2\,(\mathrm{mod}\ 3)$ with at most $(7n+1)/3$ vertices, then, according to Lemma 2.1, there is an index a such that $\sum_{i=a}^{a+k} s_i/(k+1) \leq 7/3 + 1/3n$, for all k.

Assume for a moment that $k + 1 < n$ we can rewrite the sum as $\sum_{i=a}^{a+k} s_i < (7k+8)/3$, for all k. Of course we also have to consider $k + 1 = n$, when $n \equiv 2 \pmod 3$, but only when trying to close the cylinder, not while generating a strict part of it.

Because $s_a \leq 2$, $s_a + s_{a+1} \leq 4$ and $s_{a+1} \geq 6 - s_a$, we have that $s_a = s_{a+1} = 2$. In Figure 15 all possibilities for this are shown.

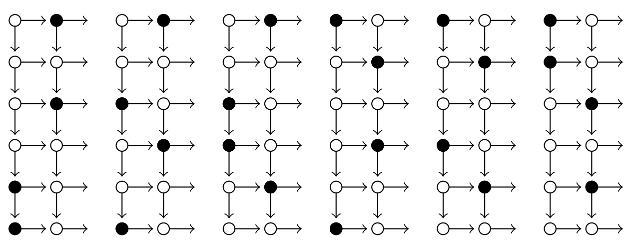

Figure 15: Possible dominating vertices in $\vec{P}_6 \Box \vec{C}_n$ for $k = 1$

Again we will build all possible dominating sets for the first $k+1$ paths from the possible dominating sets for the first k paths.

In Figures 15 to 21 this is done for $k < 8$. We notice that there is a bijection between possible dominating sets, S, for $k = 4$ and possible dominating sets, T, for $k = 7$ such that $Y_{n-a} \cap S = Y_{n-a} \cap T$, $(u_j, v_{n-a-4}) \in Y_{n-a-4} \cap S$ if and only if $(u_j, v_{n-a-7}) \in Y_{n-a-7} \cap T$, and $|T| = |S| + 7$. For $k = 4$ we then have that $(7k + 8)/3 - |S| = (7(k + 3) + 8)/3 - |T|$, which means that we can add at most $c + 7/3$ dominating vertices to $Y_{n-a-k-1}$ or $Y_{n-a-k-4}$. Therefore, for all $k \geq 4$ there is a similar bijection between possible dominating sets for k and $k + 3$ and Lemma 2.1 will give the same restrictions.

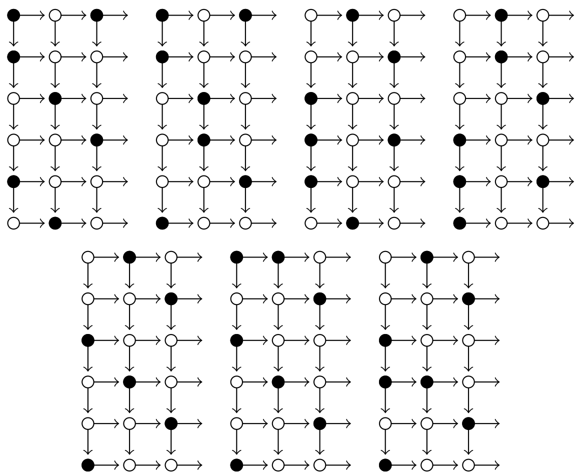

Figure 16: Possible dominating vertices in $\vec{P}_6 \square \vec{C}_n$ for $k = 2$

When $n \equiv 2 \pmod 3$ and $n = k + 1$, we try to close the cylinder. We can use at most $(7n + 1)/3$ dominating vertices in the entire cylinder. This means we can add one more dominating vertex in Y_{n-a-k} than was possible in $Y_{n-a-k+3}$, because using the same bound as in previous steps, we would only be allowed to use strictly less than $(7k + 8)/3 = (7(n - 1) + 8)/3 = (7n + 1)/3$ dominating vertices. Now Y_{n-a-k} should be dominated by the dominating vertices from $Y_{n-a-k} = Y_{n-a+1}$ and Y_{n-a}. However, we can see that this is not possible, so we have a contradiction.

An upper bound can be found in Figure 22. □

3 General formula

Here we will prove the general formula for $\gamma(\vec{P}_m \square \vec{C}_n)$ for each $n \mod 3$ and $m \geq 4$. All proofs for the lower bounds will be done by induction on m.

Upper bounds for the formulae can be constructed by starting from the minimum dominating sets for $\vec{P}_5 \square \vec{C}_n$ as shown in Figure 14 and adding (u_j, v_i) to the dominating set for $j > 5$ if and only if (u_{j-2}, v_{i+1}) is in the dominating set. Since we also have that (u_5, v_i) is in the dominating set if and only if (u_3, v_{i+1}) is in the dominating set, we know that the constructed set will be dominating. Furthermore we have that $\gamma(\vec{P}_{m+2} \square \vec{C}_n)$ has $2n/3$, $(2n + 1)/3$, or $(2n + 2)/3$ more dominating vertices than $\gamma(\vec{P}_m \square \vec{C}_n)$, when n is 0, 1, or 2 modulo 3, respectively.

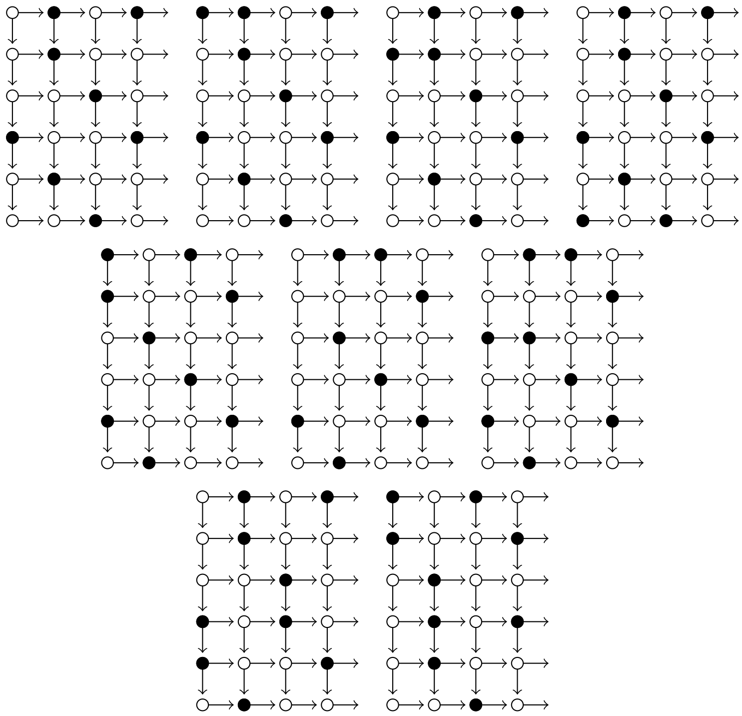

Figure 17: Possible dominating vertices in $\vec{P}_6 \Box \vec{C}_n$ for $k = 3$

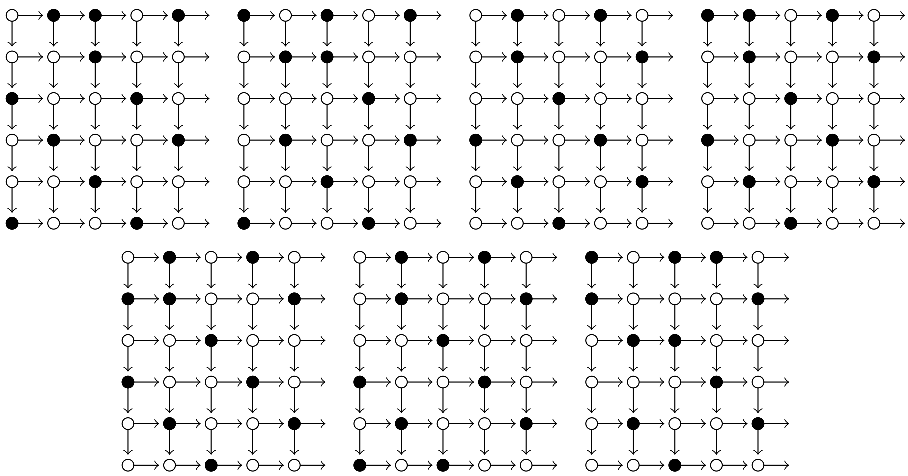

Figure 18: Possible dominating vertices in $\vec{P}_6 \Box \vec{C}_n$ for $k = 4$

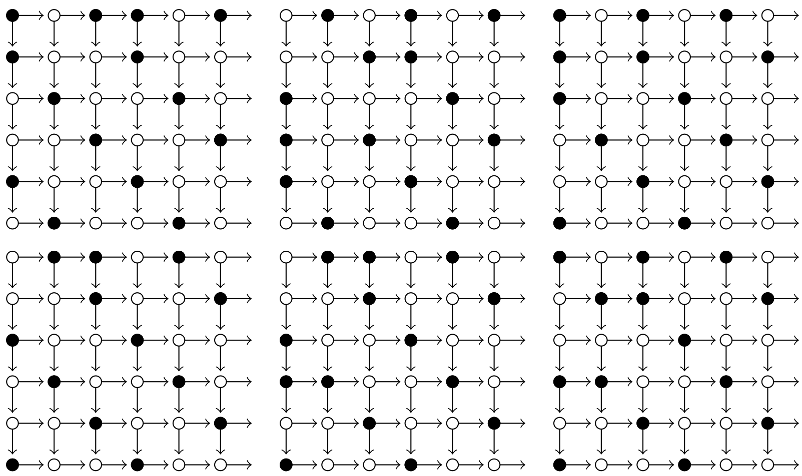

Figure 19: Possible dominating vertices in $\vec{P}_6 \square \vec{C}_n$ for $k = 5$

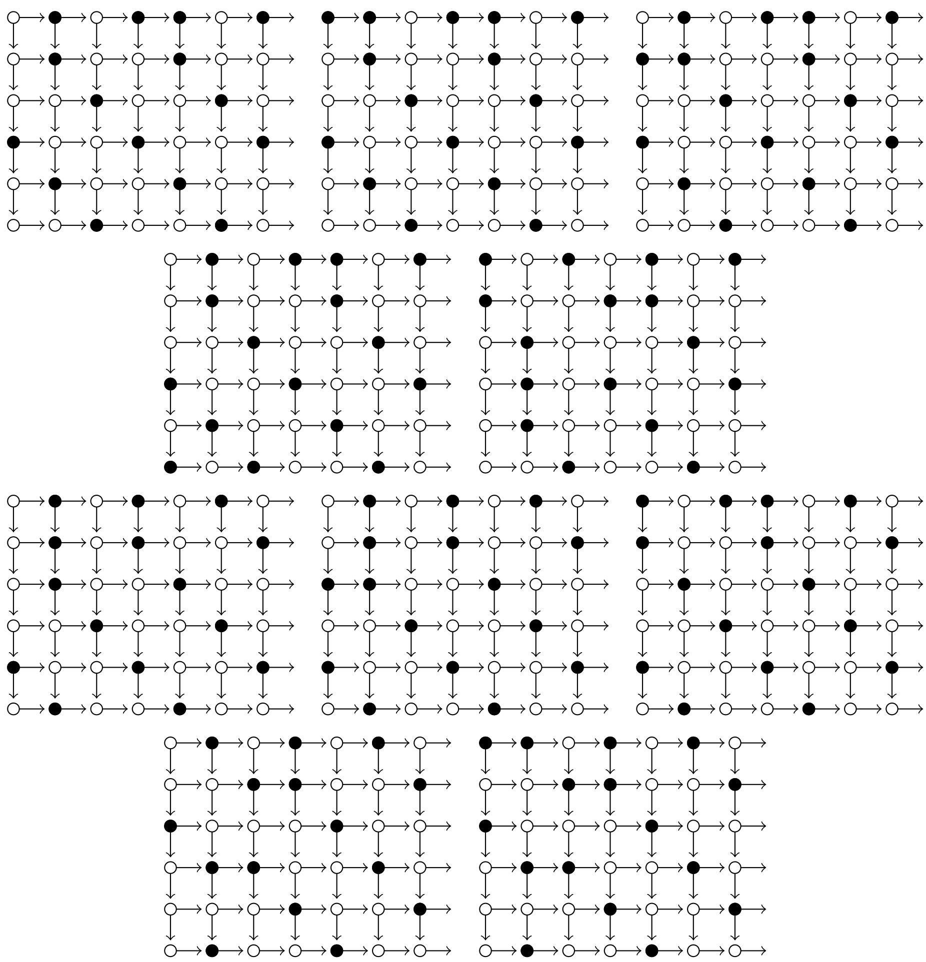

Figure 20: Possible dominating vertices in $\vec{P}_6 \square \vec{C}_n$ for $k = 6$

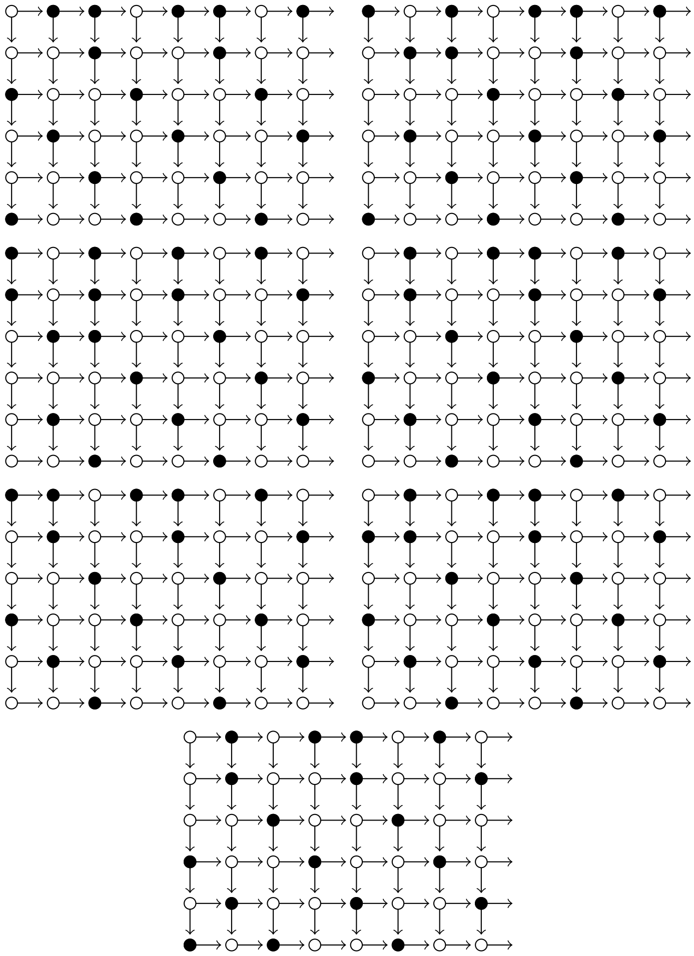

Figure 21: Possible dominating vertices in $\vec{P}_6 \Box \vec{C}_n$ for $k = 7$

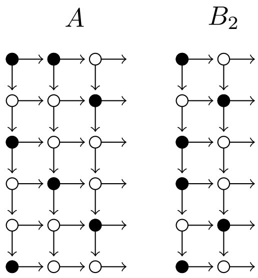

Figure 22: Upper bound for $\vec{P}_6 \Box \vec{C}_n$

Theorem 3.1. *$\gamma(\vec{P_m}\Box\vec{C_n}) = \frac{(m+1)n}{3}$ if $n \equiv 0\,(\text{mod } 3)$.*

Proof. We know the formula is true for $m = 4$ and $m = 5$. So assume it is true for all $4 \leq m < k$.

Let D be a minimum dominating set on $\vec{P_k}\Box\vec{C_n}$, with $n \equiv 0(\text{mod } 3)$. By induction we know that $|D \cap (\bigcup_{i=1}^{k-1} X_i)| \geq kn/3$ and that $|D \cap (\bigcup_{i=1}^{k-2} X_i)| \geq (k-1)n/3$. From the upper bound we can see that $|D \cap (\bigcup_{i=1}^{k} X_i)| \leq (k+1)n/3$. By subtracting these first two inequalities from the last one we get $|D \cap X_k| \leq n/3$ and $|D \cap X_{k-1}| + |D \cap X_k| \leq 2n/3$. And by looking at what vertices can dominate X_k we get $|D \cap X_{k-1}| + 2|D \cap X_k| \geq n$. Now it follows that $|D \cap X_k| = n/3$ and thus that $|D \cap (\bigcup_{i=1}^{k} X_i)| \geq (k+1)n/3$.

Combined with the upper bound this completes the proof. □

Theorem 3.2. *$\gamma(\vec{P_m}\Box\vec{C_n}) = \left\lceil \frac{(m+1)(2n+1)}{6} \right\rceil - 1$ if $m \geq 4$ and $n \equiv 1(\text{mod } 3)$.*

Proof. When m is odd we can write the formula as $\gamma(\vec{P_m}\Box\vec{C_n}) = \frac{(m+1)(2n+1)}{6} - 1$ and when m is even as $\gamma(\vec{P_m}\Box\vec{C_n}) = \frac{(m+1)(2n+1)+3}{6} - 1$.

We know the formula is true for $m = 4$ and $m = 5$. So assume it is true for all $4 \leq m < k$.

Let D be a minimum dominating set on $\vec{P_k}\Box\vec{C_n}$, with $n \equiv 1(\text{mod } 3)$. By induction we know that $|D \cap (\bigcup_{i=1}^{k-1} X_i)| \geq \left\lceil \frac{k(2n+1)}{6} \right\rceil - 1$ and that $|D \cap (\bigcup_{i=1}^{k-2} X_i)| \geq \left\lceil \frac{(k-1)(2n+1)}{6} \right\rceil - 1$. From the upper bound we can see that

$$|D \cap (\bigcup_{i=1}^{k} X_i)| \leq \left\lceil \frac{(k+1)(2n+1)}{6} \right\rceil - 1.$$

By subtracting these first two inequalities from the last one we get $|D \cap X_k| \leq (n-1)/3$ when k is odd or $|D \cap X_k| \leq (n+2)/3$ when k is even and $|D \cap X_{k-1}| + |D \cap X_k| \leq (2n+1)/3$, respectively. And by looking at what vertices can dominate X_k we get $|D \cap X_{k-1}| + 2|D \cap X_k| \geq n$. Now it follows that $|D \cap X_k| = (n-1)/3$ and $|D \cap X_{k-1}| = (n+2)/3$ when k is odd and either $|D \cap X_k| = (n-1)/3$ and $|D \cap X_{k-1}| = (n+2)/3$ or $|D \cap X_k| = (n+2)/3$ and $|D \cap X_{k-1}| = (n-1)/3$ when k is even. Summing $|D \cap X_k|$, $|D \cap X_{k-1}|$ and $|D \cap (\bigcup_{i=1}^{k-2} X_i)|$ gives us that $|D \cap (\bigcup_{i=1}^{k} X_i)| \geq \left\lceil \frac{(k+1)(2n+1)}{6} \right\rceil - 1$. □

Theorem 3.3. *$\gamma(\vec{P_m}\Box\vec{C_2}) = m$.*

Proof. It is trivial to see that $\gamma(\vec{P_1}\Box\vec{C_2}) = 1$ and $\gamma(\vec{P_2}\Box\vec{C_2}) = 2$. Now assume $\gamma(\vec{P_m}\Box\vec{C_2}) = m$ for all $m < k$.

There is no reason to have more than one dominating vertex in X_k. If there is one dominating vertex in X_k then we have at least k dominating vertices in total, since we need at least $k-1$ for $\vec{P_{k-1}}\Box\vec{C_2}$. If there is no dominating vertex in X_k, then we need 2 dominating vertices in X_{k-1} and since we need at least $k-2$ dominating vertices for $\vec{P}_{k-2}\Box\vec{C}_2$ there are at least k dominating vertices in total.

If we take Y_1 as a dominating set we have an upper bound. □

Theorem 3.4. $\gamma(\vec{P}_m\Box\vec{C}_5) = 2m+1$.

Proof. In Section 2 we have proves this for $m \leq 4$. Now assume $\gamma(\vec{P}_m\Box\vec{C}_5) = 2m+1$ for $m < k$.

If we have 2 or more dominating vertices in X_k we have a dominating set of at least $2k+1$ dominating vertices for $\vec{P}_k\Box\vec{C}_5$ because $\gamma(\vec{P}_{k-1}\Box\vec{C}_5) = 2k-1$.

If we only have one dominating vertex in X_k we need 3 dominating vertices in X_{k-1}. We know $\gamma(\vec{P}_{k-2}\Box\vec{C}_5) = 2k-3$. So we need at least $2k+1$ dominating vertices in the dominating set for $\vec{P}_k\Box\vec{C}_5)$.

Similarly if we have no dominating vertices in X_k we have at least $2k+2$ dominating vertices in total.

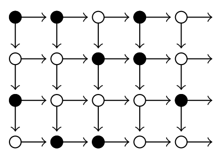

Figure 23: Minimum dominating set for $\vec{P}_4\Box\vec{C}_5$

An upper bound can easily be constructed by having 3 dominating vertices in X_1 and 2 adjacent dominating vertices in X_i, $i > 1$. An example can be found in Figure 23. □

Theorem 3.5. $\gamma(\vec{P}_m\Box\vec{C}_n) = \begin{cases} \frac{(m+1)(n+1)}{3} - 1, & \text{if } m \equiv 0 \,(\text{mod } 2) \\ \frac{(m+1)(n+1)}{3} - 2, & \text{if } m \equiv 1 \,(\text{mod } 2) \end{cases}$
if $m \geq 4$, $n \geq 8$, and $n \equiv 2 \,(\text{mod } 3)$.

Proof. We know the formula is true for $m = 4$, $m = 5$, and $m = 6$. So assume it is true for all $4 \leq m < k$. We will handle m odd and m even separately.

Let D be a minimum dominating set on $\vec{P}_k\Box\vec{C}_n$, with $n \equiv 2 (\text{mod } 3)$ and $k \equiv 1$ (mod 2). By induction we know that $|D\cap(\bigcup_{i=1}^{k-2} X_i)| \geq (k-1)(n+1)/3-2$. From the upper bound we can see that $|D \cap (\bigcup_{i=1}^{k} X_i)| \leq (k+1)(n+1)/3-2$. By subtracting these two inequalities we get $|D\cap X_{k-1}| + |D\cap X_k| \leq 2(n+1)/3$. And by looking at what vertices can dominate X_k we get $|D \cap X_{k-1}| + 2|D \cap X_k| \geq n$. Now it follows that $|D\cap X_k| \geq (n-2)/3$ and thus that $|D\cap(\bigcup_{i=1}^{k} X_i)| \geq (k+1)(n+1)/3-2$. This concludes the odd case.

Let D be a minimum dominating set on $\vec{P}_k\Box\vec{C}_n$, with $n \equiv 2 (\text{mod } 3)$ and $k \equiv 0$ (mod 2). By induction we know that $|D\cap(\bigcup_{i=1}^{k-2} X_i)| \geq (k-1)(n+1)/3-1$. From the upper bound we can see that $|D \cap (\bigcup_{i=1}^{k} X_i)| \leq (k+1)(n+1)/3-1$. By subtracting these two inequalities we get $|D\cap X_{k-1}| + |D\cap X_k| \leq 2(n+1)/3$. And by looking at what vertices can dominate X_k we get $|D \cap X_{k-1}| + 2|D \cap X_k| \geq n$. Now it follows that $|D \cap X_k| \geq (n-2)/3$.

If $|D \cap X_k| = (n-2)/3$, then $|D \cap X_{k-1}| \geq (n+4)/3$, and $|D \cap (\bigcup_{i=1}^{k} X_i)| = |D \cap (\bigcup_{i=1}^{k-2} X_i)| + |D \cap X_{k-1}| + |D \cap X_k| \geq (k+1)(n+1)/3 - 1$.

If $|D \cap X_k| = (n+1)/3$, then $|D \cap X_{k-1}| \geq (n-2)/3$. If $|D \cap X_{k-1}| \geq (n+1)/3$ we have our lower bound again. So assume $|D| \leq (k+1)(n+1)/3 - 2$ and thus $|D \cap X_{k-1}| = (n-2)/3$. Then we also have $|D \cap X_{k-2}| = (n+4)/3$. From this it is clear that every vertex in X_k and X_{k-1} is dominated by exactly one vertex. Therefore there is only one possible way to put the dominating vertices in X_k (taking rotational symmetry into account), since no two dominating vertices and no three not-dominating vertices may be adjacent. Without loss of generalization we can say the vertices in Y_0 and Y_i, with $i \equiv 2 \pmod 3$ and $i < n$, are the dominating vertices of X_k. This means the vertices in Y_i, with $i \equiv 1 \pmod 3$ and $4 \leq i < n$ are the dominating vertices in X_{k-1} and the vertices in Y_1, Y_2, and Y_i, with $i \equiv 0 \pmod 3$ and $3 \leq i < n$ are the dominating vertices in X_{k-2}. Since $|D| = |D \cap (\bigcup_{i=1}^{k} X_i)| = |D \cap (\bigcup_{i=1}^{k-3} X_i)| + |D \cap X_{k-2}| + |D \cap X_{k-1}| + |D \cap X_k| \leq (k+1)(n+1)/3 - 2$, we have that $|D \cap (\bigcup_{i=1}^{k-3} X_i)| \leq (k+1)(n+1)/3 - 2 - (n+1) = (k-2)(n+1)/3 - 2$. Because of the lower bounds for $|D \cap (\bigcup_{i=1}^{k-3} X_i)|$ and $|D \cap (\bigcup_{i=1}^{k-4} X_i)|$, we know that $|D \cap X_{k-3}| = (n-2)/3$ — the smallest k for which we need this proof is $k = 8$ and the formulae holds from $k = 4$, so we can go back as far as $k - 4$ in the proof. But the dominating vertices of X_{k-3} now have to be in Y_i, with with $i \equiv 2 \pmod 3$ and $5 \leq i \leq n$. However, this would mean we can keep the dominating vertices from X_i, $i \leq k-3$ and have the vertices in Y_1 and Y_i, with $i \equiv 0 \pmod 3$ and $3 \leq i < n$, as dominating vertices in X_{k-2} and this would give us a dominating set for $\vec{P}_{k-2} \Box \vec{C}_n$ with $(k-1)(n+1)/3 - 2$. This is a contradiction with the induction hypothesis. So also in this case we have $|D \cap (\bigcup_{i=1}^{k} X_i)| \geq (k+1)(n+1)/3 - 1$.

If $|D \cap X_k| \geq (n+4)/3$, then $|D \cap (\bigcup_{i=1}^{k} X_i)| = |D \cap (\bigcup_{i=1}^{k-1} X_i)| + |D \cap X_k| \geq (k+1)(n+1)/3 - 1$.

Combined with the upper bound this completes the proof. □

These results are summarized in Table 1.

4 Signed 2-independence number of $\vec{P}_m \Box \vec{C}_n$

Let G be a directed graph G with minimum indegree 1 and maximum indegree 2, and let f be a S2IF on G. For every vertex v, we have that $N_G^-[v] \cap M \neq \emptyset$. It follows that M is a dominating set for G. Therefore the signed 2-independence number $\alpha_s^2 = \max(|P| - |M|) = \max(|V(G)| - 2|M|) = |V(G)| - 2\gamma(G)$, where $\gamma(G)$ is the domination number of G. Hence we can obtain the signed 2-independence number of the Cartesian product of directed path $\vec{P}_m$ and directed cycle $\vec{C}_n$ directly from $\gamma(\vec{P}_m \Box \vec{C}_n)$.

According to Theorem 2.2, Theorem 2.3, and Theorem 2.4, the following results are trivial.

m	n	$\gamma(\overrightarrow{P}_m \Box \overrightarrow{C}_n)$
1	ALL	$\lceil \frac{n}{2} \rceil$
2	ALL	n
3	ALL	$\lceil \frac{5n}{4} \rceil$
4	ALL	$\lceil \frac{5n}{3} \rceil$
5	$n = 1$	$\lceil \frac{m+1}{2} \rceil - 1$
	$n = 2$	5
	$n = 5$	11
	$n \notin \{1, 2, 5\}$	$2n$
6	$n = 2$	6
	$n \equiv 2 (\text{mod } 3)$, $n > 2$	$\frac{7n+4}{3}$
$m \geq 6$	$n \equiv 0 (\text{mod } 3)$	$\frac{(m+1)n}{3}$
$m \geq 6$	$n \equiv 1 (\text{mod } 3)$	$\lceil \frac{(m+1)(2n+1)}{6} \rceil - 1$
$m \geq 7$, $m \equiv 0$ (mod 2)	$n \equiv 2 (\text{mod } 3)$, $n \geq 8$	$\frac{(m+1)(n+1)}{3} - 1$
$m \geq 7$, $m \equiv 1$ (mod 2)		$\frac{(m+1)(n+1)}{3} - 2$
ALL	2	m
ALL	5	$2m + 1$

Table 1: Domination number of $\vec{P_m} \Box \vec{C_n}$

Corollary 4.1. *For any integer $n \geq 2$,*

$$\alpha_s^2(\vec{P_1} \Box \vec{C_n}) = \begin{cases} 0 & \textit{for } n \textit{ even,} \\ -1 & \textit{for } n \textit{ odd.} \end{cases}$$

Corollary 4.2. *For any integer $n \geq 2$, $\alpha_s^2(\vec{P_2} \Box \vec{C_n}) = 0$.*

Corollary 4.3. *For any integer $n \geq 2$, $\alpha_s^2(\vec{P_3} \Box \vec{C_n}) = 3n - 2 \left\lceil \frac{5n}{4} \right\rceil$.*

Applying Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4, and Theorem 3.5, the following results can be easily obtained.

Corollary 4.4. $\alpha_s^2(\vec{P_m} \Box \vec{C_n}) = \frac{(m-2)n}{3}$ *if $m \geq 4$ and $n \equiv 0 (\text{mod } 3)$.*

Corollary 4.5. $\alpha_s^2(\vec{P_m} \Box \vec{C_n}) = mn - 2 \left\lceil \frac{(m+1)(2n+1)}{6} \right\rceil - 2$ *if $m \geq 4$ and $n \equiv 1 (\text{mod } 3)$.*

Corollary 4.6. $\alpha_s^2(\vec{P_m} \Box \vec{C_2}) = 0$.

Corollary 4.7. $\alpha_s^2(\vec{P_m} \Box \vec{C_5}) = m - 2$.

m	n	$\alpha_s^2(\overrightarrow{P}_m \Box \overrightarrow{C}_n)$
1	$n \geq 2$, n : even	0
	$n \geq 2$, n : odd	-1
2	$n \geq 2$	0
3	$n \geq 2$	$3n - 2\lceil \frac{5n}{4} \rceil$
$m \geq 4$, m : even	$n \geq 8$, $n \equiv 2 (\text{mod } 3)$	$\frac{(m-2)(n-2)}{3}$
$m \geq 4$, m : odd	$n \geq 8$, $n \equiv 2 (\text{mod } 3)$	$\frac{(m-2)(n-2)}{3} + 2$
$m \geq 4$	$n \equiv 0 (\text{mod } 3)$	$\frac{(m-1)n}{3}$
$m \geq 4$	$n \equiv 1 (\text{mod } 3)$	$mn - 2\lceil \frac{(m+1)(2n+1)}{6} \rceil - 2$
ALL	2	0
ALL	5	$m - 2$

Table 2: Signed 2-independence number of $\vec{P}_m \Box \vec{C}_n$

Corollary 4.8. $\alpha_s^2(\vec{P}_m \Box \vec{C}_n) = \begin{cases} \frac{(m-2)(n-2)}{3}, & \text{if } m \equiv 0 \,(\text{mod } 2) \\ \frac{(m-2)(n-2)}{3} + 2, & \text{if } m \equiv 1 \,(\text{mod } 2) \end{cases}$ *if* $m \geq 4$, $n \geq 8$ *and* $n \equiv 2 \,(\text{mod } 3)$.

These results are summarized in Table 2.

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