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Some strongly regular graphs with the parameters of Paley graphs

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Abstract

We construct two-intersection sets in PG$(5, q)$, q odd, admitting $PSL(2, q^2)$, whose associated strongly regular graphs have the same parameters as, but are not isomorphic to, Paley graphs.

1 Introduction

A **k-set** K **of type** (m, n) **in** PG(d, q) is a set K of k points such that every hyperplane of the projective space contains either m or n points of K, $m < n$. These sets are also called **two-intersection sets**, as they have two intersection numbers (with respect to hyperplanes), m and n. A hyperplane is called **i-secant** to K if $|H \cap K| = i$.

A graph Γ that is simple, undirected, and loopless of order v is a **strongly regular graph** with parameters v, k, λ, μ whenever Γ is not complete or edgeless and (i) each vertex is adjacent to k vertices, (ii) for each pair of adjacent vertices, there are λ vertices adjacent to both, and (iii) for each pair of non-adjacent vertices, there are μ vertices adjacent to both.

Let X be a t-set of type (m, n) in PG$(d-1, q)$ that spans PG$(d-1, q)$. Embed PG$(d-1, q)$ in PG(d, q) as a hyperplane H, and define $\Gamma(X)$ to be the graph with vertices the points of PG(d, q) not in H and two vertices P and Q adjacent if and only if $PQ \cap H \in X$. Then by [2, Theorem 3.2], $\Gamma(X)$ is a strongly regular graph with parameters $v = q^d, k = t(q-1), \lambda = t^2 + tq - 3t + qm + qn - tqm - tqn + mnq^2$, and $\mu = \frac{q^2}{q^d}(t-m)(t-n)$, which motivates the study of two-intersection sets. If two-intersection sets are equivalent (under $P\Gamma L(d, q)$), then the associated strongly regular graphs are isomorphic.

The history of two-intersection sets in finite projective spaces stretches back to at least two 1966 papers by Tallini Scafati [9, 10]. Calderbank and Kantor surveyed these sets in 1986 [2] in what has become the standard reference. Postdating their survey, a number of two-intersection sets have been constructed, and in turn, strongly regular graphs (as well as projective two-weight linear codes). However, the main motivation for this paper came from [4] which was surveyed in [2].

2 Parameters of sets of type (m,n)

Let t_m and t_n denote the number of m-secants and n-secants to a k-set K of type (m, n) in $\mathrm{PG}(d, q)$. Then elementary counting leads to the fundamental equations

$$t_m + t_n = \frac{q^{d+1} - 1}{q - 1}, \tag{1}$$

$$mt_m + nt_n = t\frac{q^d - 1}{q - 1}, \tag{2}$$

$$m(m-1)t_m + n(n-1)t_n = t(t-1)\frac{q^{d-1} - 1}{q - 1}. \tag{3}$$

By taking the linear combination $mn(1) + (1 - m - n)(2) + (3)$, we obtain a quadratic in k:

$$k^2\frac{q^{d-1} - 1}{q - 1} + k(1 - m - n)\frac{q^d - 1}{q - 1} - k\frac{q^{d-1} - 1}{q - 1} + mn\frac{q^{d+1} - 1}{q - 1} = 0 \tag{4}$$

Now, let r_n and r_m denote the number of n-secants and m-secants, respectively, through a point $P \in K$. Let s_n and s_m be the number of n-secants and m-secants through a point $Q \notin K$. Then simple counting yields

$$\begin{aligned} s_n + s_m &= \tfrac{q^d-1}{q-1}, \\ ns_n + ms_m &= k\tfrac{q^{d-1}-1}{q-1}, \\ r_n + r_m &= \tfrac{q^d-1}{q-1}, \\ (n-1)r_n + (m-1)r_m &= (k-1)\tfrac{q^{d-1}-1}{q-1}. \end{aligned}$$

We can then solve for r_m, r_n, s_m, s_n, obtaining

$$r_m = \frac{(n-1)\frac{q^d-1}{q-1} - (k-1)\frac{q^{d-1}-1}{q-1}}{n-m},\ r_n = \frac{(k-1)\frac{q^{d-1}-1}{q-1} - (m-1)\frac{q^d-1}{q-1}}{n-m},$$
$$s_m = \frac{n\frac{q^d-1}{q-1} - k\frac{q^{d-1}-1}{q-1}}{n-m},\ s_n = \frac{k\frac{q^{d-1}-1}{q-1} - m\frac{q^d-1}{q-1}}{n-m}.$$

An important consequence of these equations is that $n - m$ is a divisor of q^{d-1} (as $r_m - s_m \in \mathbb{Z}$).

From each set of type (m, n), we can form three related sets [5].

Lemma 1. *Let K be a k-set of type (m, n) in $\mathrm{PG}(d, q)$. Then*

a.) The complement of K is a $(\frac{q^{d+1}-1}{q-1} - k)$-set of type $(\frac{q^d-1}{q-1} - n, \frac{q^d-1}{q-1} - m)$ in $\mathrm{PG}(d, q)$.

b.) The m-secants of K form a t_m-set of type (r_m, s_m) in the dual of $\mathrm{PG}(d, q)$.

c.) The n-secants of K form a t_n-set of type (s_n, r_n) in the dual of $\mathrm{PG}(d, q)$.

These equations allow having a solution corresponding to a possible $\frac{q^{(d+1)/2}+1}{q^{(d-1)/2}+1}(m + q^{\frac{d-1}{2}})$-set of type $(m, m + q^{\frac{d-1}{2}})$ in $\mathrm{PG}(d, q)$ (where q is a square if d is even). We investigate such sets below.

Theorem 2. *cf.[4] The union of a $\frac{q^{(d+1)/2}+1}{q^{(d-1)/2}+1}(m_1 + q^{(d-1)/2})$-set K_1 of type $(m_1, m_1 + q^{(d-1)/2})$ in $\mathrm{PG}(d, q)$ and a $\frac{q^{(d+1)/2}+1}{q^{(d-1)/2}+1}(m_2 + q^{(d-1)/2})$-set K_2 of type $(m_2, m_2 + q^{(d-1)/2})$ in $\mathrm{PG}(d, q)$, disjoint from K_1, is a $\frac{q^{(d+1)/2}+1}{q^{(d-1)/2}+1}(m_1 + m_2 + 2q^{(d-1)/2})$-set K of type $(m_1 + m_2 + q^{(d-1)/2}, m_1 + m_2 + 2q^{(d-1)/2})$ in $\mathrm{PG}(d, q)$ for q odd.*

Proof. Any hyperplane intersects K in either m_1+m_2, $m_1+m_2+q^{(d-1)/2}$, or $m_1+m_2+2q^{(d-1)/2}$ points. We must show that there are no hyperplanes in the first class. Consider the fundamental equations (1), (2), and (3) for the set K. These are three linear equations in the unknowns $t_{m_1+m_2}$, $t_{m_1+m_2+q^{(d-1)/2}}$, and $t_{m_1+m_2+2q^{(d-1)/2}}$. It is easy to see that the coefficient matrix has determinant $2q^{(3/2)(q-1)}$. Thus, it is non-singular, and hence these equations have a unique solution. As an $\frac{q^{(d+1)/2}+1}{q^{(d-1)/2}+1}(m_1+m_2+2q^{(d-1)/2})$-set of type $(m_1+m_2+q^{(d-1)/2}, m_1+m_2+2q^{(d-1)/2})$ is arithmetically feasible, we can find a solution to these equations with $t_{m_1+m_2}=0$. Hence, this must be the unique solution. □

More specifically, we will consider two infinite families of two-intersection sets in $\mathrm{PG}(5,q)$. In [2, Example FE1, page 112], Calderbank and Kantor constructed the following family.

Theorem 3. *There exists a $(q^5+q^2)/2$-set K_1 of type $((q^4-q^2)/2,(q^4+q^2)/2)$ in $\mathrm{PG}(5,q)$ admitting $\mathrm{P}\Omega^-(5,q)$ for q odd. Namely, if Q is a nondegenerate quadratic form on $\mathrm{GF}(q)^6$ of minus type, then $K_1=\{\langle v\rangle : Q(v)$ is a non-square $\}$.*

In [3, Remark 3.3(4)], Cossidente and Penttila constructed another infinite family of two-intersection sets in $\mathrm{PG}(5,q)$.

Theorem 4. *There exists a $(q^4+q^3+q+1)/2$-set K_2 of type $((q^3-q^2+q+1)/2,(q^3+q^2+q+1)/2)$ in $\mathrm{PG}(5,q)$ admitting $\mathrm{PSL}(2,q^2)$, for q odd. Moreover, there is a nondegenerate quadratic form Q on $\mathrm{GF}(q)^6$ of minus type, with $Q(v)=0$ for all $\langle v\rangle\in K_1$.*

It should be noted that K_1 is disjoint from K_2, if we fix the quadratic form Q on $\mathrm{GF}(q)^6$ of minus type. Thus

Theorem 5. *There exists a $(q^5+q^4+q^3+q^2+q+1)/2$-set K of type $((q^4+q^3+q+1)/2,(q^4+q^3+2q^2+q+1)/2)$ in $\mathrm{PG}(5,q)$ admitting the group $\mathrm{PSL}(2,q^2)$ for q odd.*

Proof. This follows from the observation immediately before this theorem and from Theorem 2, with $K=K_1\cup K_2$. □

This set has the same parameters as some previously constructed sets. First we consider the Paley set [8], which is an orbit of the group generated by the square of a Singer cycle.

Theorem 6. *The sets K from Theorem 5 are inequivalent to Paley sets.*

Proof. Let K be a set from Theorem 5, and let P be a Paley set in $\mathrm{PG}(5,q)$ for q odd. If K and P are equivalent under an element of $\mathrm{P\Gamma L}(6,q)$, then $\Gamma(K)$ and $\Gamma(P)$ are isomorphic, and thus have isomorphic automorphism groups. By [6, Corollary 8.2] or [7], the automorphism group of $\Gamma(P)$ (the Paley graph) is a subgroup of $\mathrm{A\Gamma L}(1,q^6)$, and so is solvable. But $\mathrm{PSL}(2,q^2)$ is a subgroup of $\mathrm{Aut}(\Gamma(K))$, so $\mathrm{Aut}(\Gamma(K))$ is not solvable. Therefore, K and P are not equivalent. □

Corollary 7. *The strongly regular graphs arising from the sets of Theorem 5 have the same parameters as, but are not isomorphic to, Paley graphs. Namely, the parameters are $(q^6,\frac{q^6-1}{2},\frac{q^6-5}{4},\frac{q^6-1}{4})$.*

Remark The same proof shows that $\Gamma(K)$ is not isomorphic to the strongly regular graphs constructed from commutative semifields of order q^6 in Theorem 1.3 of Chen and Polhill (2011)[1], by [1, Theorem 1.4]. Also, K contains no plane, so cannot be equivalent to the union of a partial 2-spread of size $\frac{q^3+1}{2}$.

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