CONSTRUCTING DESIGNS USING THE UNION METHOD

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ABSTRACT. Given a design $\mathcal{D}$, other designs can be constructed by using unions of various sets of blocks as new blocks. This method of constructing designs is referred to as "the union method". In general, the constructed design has neither constant blocksize nor balance properties. Various situations in which balanced incomplete block designs (BIBD's) and group divisible designs (GDD's) can be constructed from BIBD's and GDD's using the union method are presented. Generally speaking, it is necessary to assume that the original BIBD or GDD has a "dual property". In the cases where a BIBD $\mathcal{D}_u$ is constructed, the question as to whether $\mathcal{D}_u$ is simple is considered. Simple BIBD's with less than thirty points are constructed for the following six parameter sets for which no such design was previously known: 2-(27, 12, 132), 2-(25, 7, 385), 2-(27, 7, 462), 2-(27, 5, 60), 2-(21, 6, 15), 2-(28, 8, 14).

1. INTRODUCTION

Morgan [11] has shown that the unions of pairs of blocks of a symmetric BIBD form a BIBD. Recently, Mahmoodian and Shirdarreh [7] have shown that the BIBD constructed in this way from a symmetric $2-(v, k, \lambda)$ design is simple, if $v \geq 2k$. In this paper we exhibit some situations in which taking the unions of carefully selected sets of blocks of BIBD's or group divisible designs yields a BIBD or a group divisible design. We refer to this method of constructing designs as "the union method". For the union method to yield designs with balance properties, it is generally necessary to assume that the design $\mathcal{D}$ we start with possesses a "dual property" (for example, that $\mathcal{D}$ possesses an affine resolution).

In Section 3 we apply the union method to group divisible designs with certain dual properties to obtain group divisible designs. In Section 4 we show that taking unions of pairs of blocks in different classes of the affine resolution of an affine resolvable BIBD yields a BIBD. If the number t of blocks in such a class is greater than two, then the constructed BIBD is simple. In the case where $t = 2$, the constructed BIBD is a 3-design. We give necessary and sufficient conditions that such a 3-design be simple. In Section 5 we investigate two situations in which taking unions of carefully selected sets of two

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or more blocks of a BIBD yields a BIBD. We give sufficient conditions for the BIBD's arising from the constructions of Section 5 to be simple.

The methods of this paper yield many infinite classes of BIBD's and group divisible designs. They also yield simple BIBD's with $v \leq 30$ for at least six parameter sets for which no such design was previously known.

2. PRELIMINARIES

We shall assume that the reader is acquainted with basic notions concerning designs. In particular, we shall assume familiarity with BIBD's and resolutions of designs. For information concerning these topics Street and Street [14] might be consulted.

A tactical configuration ([5], p.4) with v points, b blocks, r blocks containing each point and k points contained in each block is called (v, b, r, k)-*configuration*. The *replication number* of a (v, b, r, k)-configuration is r. The *connection number* $\lambda(P, Q)$ of a pair P, Q of distinct points of a (v, b, r, k)-configuration $\mathcal{C}$ is the number of blocks of $\mathcal{C}$ which contain both P and Q. A (v, b, r, k)-configuration $\mathcal{G}$ is said to be *group divisible design* (GDD) if there is a partition of its set of points into "groups" $\mathcal{P}_1, \ldots, \mathcal{P}_{m_2}$, where $m_2 \geq 2$, such that there are integers $m_1 \geq 2$ and λ_1 and λ_2 such that

(a) $|\mathcal{P}_i| = m_1$ for all $i = 1, \ldots, m_2$,
(b) any two points common to a group are contained in λ_1 blocks of $\mathcal{G}$,
(c) any two points in different groups are contained in λ_2 blocks of $\mathcal{G}$, and
(d) $\lambda_1 \neq \lambda_2$.

We say that such a GDD " has parameters $v, b, r, k; m_1, m_2; \lambda_1, \lambda_2$". We also say that $\mathcal{P}_1, \ldots, \mathcal{P}_{m_2}$ form a *group division* of $\mathcal{G}$. Clearly, for a GDD, the connection number of a pair of points depends only on whether the points are in the same group or not.

The parameters of a GDD satisfy $vr = bk$, $v = m_1 m_2$ and

$$(m_1 - 1)\lambda_1 + m_1(m_2 - 1)\lambda_2 = r(k - 1). \tag{1}$$

Also, group divisions can be exhaustively classified into the following mutually exclusive types:

1. Singular for which $r = \lambda_1$.
2. Semiregular for which $r > \lambda_1$ and $rk = v\lambda_2$.
3. Regular for which $r > \lambda_1$ and $rk > v\lambda_2$.

Since a GDD has a unique group division, we can apply the terms "singular", "semiregular" and "regular" to GDD's as well as to group divisions. Singular, semiregular and regular GDD's will be referred to as SGDD's, SRGDD's and RGDD's respectively.

A point P of a (v, b, r, k)-configuration $\mathcal{C}$ is said to be *repeated* if there is another point Q of $\mathcal{C}$ such that the set of blocks containing Q is equal to the set of blocks containing P. That an SGDD always possesses repeated points is shown in [2]. Also, if $\mathcal{G}$ is an SRGDD or RGDD, then $r > \lambda_1$, and $r = \lambda_2$ implies (using (1)) that $k > v$, which is impossible. So we have that a GDD is singular if and only if it possesses repeated points.

A GDD $\mathcal{G}$ is semiregular if and only if each block of $\mathcal{G}$ meets each group of $\mathcal{G}$ in the same numbers of points. For an SRGDD with parameters v, b, r, $k; m_1$, $m_2; \lambda_1$, λ_2 we have $v = m_1 m_2$ and

$$b = \left(\frac{m_1 m_2}{k}\right)^2 \lambda_2$$
$$r = \left(\frac{m_1 m_2}{k}\right) \lambda_2$$

and

$$\lambda_1 = \frac{m_1 \lambda_2 (k - m_2)}{k(m_1 - 1)}.$$

If $\lambda_1 = 0$, then $k = m_2$ and $\mathcal{G}$ has parameters

$$m_1 m_2,\ \lambda_2 m_1^2,\ \lambda_2 m_1,\ m_2; m_1,\ m_2; 0,\ \lambda_2. \tag{2}$$

An SRGDD with parameters (2) is called a *transversal design*. The groups of a transversal design $\mathcal{G}$ form an affine resolution of the dual $\mathcal{G}^d$ of $\mathcal{G}$.

Using (1), an RGDD with $\lambda_1 = 0$ satisfies

$$(v - m_1)\lambda_2 = r(k - 1). \tag{3}$$

A GDD $\mathcal{G}$ is said to be *self-dual* if $\mathcal{G}^d$ is a GDD with the same parameters as $\mathcal{G}$. Mitchell [**10**] has shown that, if $\mathcal{G}$ and $\mathcal{G}^d$ are GDD's without repeated points, then $\mathcal{G}$ and $\mathcal{G}^d$ are both semiregular or $\mathcal{G}$ is self-dual and $\mathcal{G}$ and $\mathcal{G}^d$ are both regular. He has also shown that the groups of $\mathcal{G}$ and $\mathcal{G}^d$ form a tactical decomposition ([**5**], p.7) in these cases.

The following result of Bose ([**1**], p.95) will be of use to us in Section 3.

Result 1. Let $\mathcal{G}$ be a self-dual RGDD with parameters $m_1m_2, m_1m_2, k, k; m_1, m_2; 0, \lambda_2$. An incidence matrix of the tactical decomposition ([**5**], p.17) formed by the groups of $\mathcal{G}$ and $\mathcal{G}^d$ is an incidence matrix of a (symmetric) $2 - (m_2,\ k,\ m_1\lambda_2)$ design.

Remark. That a $2 - (m_2,\ k,\ m_1\lambda_2)$ design, as in Result 1, is symmetric can be inferred from (3).

We illustrate Result 1 using the self-dual RGDD $\mathcal{G}$ with parameters $8, 8, 3, 3; 2, 4; 0, 1$ given next. This RGDD has points 1, ..., 8 and blocks $\overline{1}$, ..., $\overline{8}$.

$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{8}$
1	5	2	6	3	7	4	8
2	6	3	7	4	8	5	1
4	8	5	1	6	2	7	3

The groups of $\mathcal{G}$ are $g_i = \{i,\ i+4\}$, $i = 1, 2, 3, 4$, and the groups of $\mathcal{G}^d$ are $\overline{g}_i = \{\overline{i},\ \overline{i+1}\}$, $i = 1, 3, 5, 7$. The entry in the i^{th} row and j^{th} column of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

gives us the number of points of g_i in a block of $\overline{g}_j$. It is thus an incidence matrix of the tactical decomposition of $\mathcal{G}$ formed by the g_i and $\overline{g}_j$. It is also an incidence matrix of a $2-(4, 3, 2)$ design.

3. GDD's with a dual property

Consider a GDD $\mathcal{G}$ with parameters $v, b, r, k; m_1, m_2; \lambda_1, \lambda_2$ whose dual $\mathcal{G}^d$ is a GDD with parameters $b, v, k, r; \overline{m}_1, \overline{m}_2; 0, \rho_2$. In this situation we shall apply the union method in the following way: Take as new blocks the unions of all the pairs of blocks in different groups of $\mathcal{G}^d$. We shall show (with one exception) that, if $\mathcal{G}$ and $\mathcal{G}^d$ are both semiregular or both regular, then these new blocks form a GDD with the same groups as $\mathcal{G}$. We shall denote by $\mathcal{G}_u$ the design whose points are those of $\mathcal{G}$ and whose blocks are those obtained by applying the union method as above. In showing that $\mathcal{G}_u$ is almost always a GDD we split the analysis into two cases.

Case 1. $\mathcal{G}$ and $\mathcal{G}^d$ are both semiregular.

It is sufficient to show that each block of $\mathcal{G}_u$ contains the same number of points and that the connection numbers $\lambda(P, Q)$ of $\mathcal{G}_u$ depend only on whether P and Q belong to the same group of $\mathcal{G}$ or not.

Each block of $\mathcal{G}_u$ has $2k - \rho_2$ points in it.

Consider a pair P and Q of distinct points of $\mathcal{G}$.

Subcase 1.1. P and Q belong to the same group of $\mathcal{G}$.

There are λ_1 blocks of $\mathcal{G}$ containing both P and Q. The blocks of $\mathcal{G}_u$ obtained from these blocks are $\lambda_1(b - \overline{m}_1) - \frac{\lambda_1(\lambda_1-1)}{2}$ in number. The other blocks of $\mathcal{G}_u$ containing P and Q arise from taking the unions of blocks of $\mathcal{G}$ containing P, but not Q, with the blocks of $\mathcal{G}$ containing Q, but not P. Since there is a block in the group of $\mathcal{G}^d$ of a block of $\mathcal{G}$ containing P, but not Q, which contains Q, but not P, we have that $(r-\lambda_1)(r-\lambda_1-1)$ blocks of $\mathcal{G}_u$ containing P and Q arise in this way. We thus have that there are precisely

$$\lambda_1(b - \overline{m}_1) - \frac{\lambda_1(\lambda_1 - 1)}{2} + (r - \lambda_1)(r - \lambda_1 - 1) \tag{4}$$

blocks of $\mathcal{G}_u$ containing P and Q.

Subcase 1.2. P and Q are in different groups of $\mathcal{G}$.

A similar argument to that which dealt with Subcase 1.1 yields that there are precisely

$$\lambda_2(b - \overline{m}_1) - \frac{\lambda_2(\lambda_2 - 1)}{2} + (r - \lambda_2)(r - \lambda_2 - 1) \tag{5}$$

blocks of $\mathcal{G}_u$ containing P and Q in this case.

As an example, let $\mathcal{G}$ be a self-dual SRGDD with parameters $q^2, q^2, q, q; q, q; 0, 1$, where q is a prime power. (Such designs are known to exist for all such q.) If $q > 2$, then $\mathcal{G}_u$ is an RGDD with parameters

$$q^2, \frac{q^3(q-1)}{2}, \frac{q(q-1)(2q-1)}{2}, 2q-1; q, q; q(q-1), 2(q-1)^2.$$

If $q = 2$, then $\mathcal{G}_u$ is a $2 - (4, 3, 2)$ design.

Remark. Many classes of examples of SRGDD's whose duals are transversal designs can be obtained by the methods of [**12**] and [**13**]. Thus there is much scope for applying the union method to such designs.

As earlier pointed out a self-dual GDD with parameters $4,4,2,2;2,2;0,1$ yields a $2-(4,3,2)$ design $\mathcal{G}_u$. In fact, for $\mathcal{G}$ semiregular, no other $\mathcal{G}_u$ is a BIBD, as we now show.

Suppose $\mathcal{G}_u$ is a BIBD. Equating the expressions (4) and (5) yields $\lambda_1 = \lambda_2$ (which is impossible since $\mathcal{G}$ is a GDD, not a BIBD) or $\lambda_1 + \lambda_2 = 4r - 2\overline{m}_1(r-1) - 3$, using $b = m_1 r$. Now, if $\overline{m}_1 \geq 3$, then we have (in the latter case)

$$\lambda_1 + \lambda_2 \leq 4r - 6(r-1) - 3 = 3 - 2r < 0,$$

which is absurd. Clearly $\overline{m}_1 = 2$ and so $\lambda_1 + \lambda_2 = 1$. Thus $\lambda_1 = 0$ and $\lambda_2 = 1$, since $\lambda_2 > \lambda_1$ for SRGDDs. (For an SRGDD, (1) yields $m_1(\lambda_2 - \lambda_1) = r - \lambda_1 > 0$.) We thus have that $\mathcal{G}$ and $\mathcal{G}^d$ are each a transversal design. So $\mathcal{G}$ is self-dual (see, for example, [**12**] p.164) whence $m_1 = \overline{m}_1 = 2$. Therefore $\mathcal{G}$ is a symmetric transversal design with $m_1 = 2$ and $\lambda_2 = 1$. So $\mathcal{G}$ has parameters $4,4,2,2;2,2;0,1$ (see (2) in Section 2).

Before turning to the case where $\mathcal{G}$ is regular we shall show that $\mathcal{G}_u$ is never singular when $\mathcal{G}$ is semiregular.

If $\mathcal{G}_u$ is singular, then points in the same group of $\mathcal{G}$ (equivalently, of $\mathcal{G}_u$) occur in the same blocks of $\mathcal{G}_u$. Consider distinct points P and Q in the same group of $\mathcal{G}$. Since $r > \lambda_1$, there is a block B of $\mathcal{G}$ containing P, but not Q. If $\lambda_1 > 0$, then there is a block D of $\mathcal{G}$ containing P and Q. In the group of D there is a block F containing neither P nor Q. $B \cup F$ is a block of $\mathcal{G}_u$ containing P, but not Q. If $\lambda_1 = 0$, then there are blocks B and D in different groups containing P, but not Q. $B \cup D$ is a block of $\mathcal{G}_u$ containing P, but not Q. In either case it follows that $\mathcal{G}_u$ is not singular.

Remarks. (a) $\mathcal{G}_u$ is semiregular if and only if each block of $\mathcal{G}_u$ meets each group of $\mathcal{G}$ in the same number of points. Since each block of $\mathcal{G}$ meets each group of $\mathcal{G}$ in the same number of points, we have that $\mathcal{G}_u$ is semiregular if and only if the intersection of two blocks in different groups of $\mathcal{G}^d$ meets each group of $\mathcal{G}$ in $\frac{\rho_2}{m_2}$ points. Clearly $\mathcal{G}_u$ is semiregular only if m_2 is a divisor of ρ_2.

(b) From (2) the parameters of a symmetric transversal design have the form $\lambda_2 m_1^2$, $\lambda_2 m_1^2$, $\lambda_2 m_1$, $\lambda_2 m_1; m_1, \lambda_2 m_1; 0, \lambda_2$. From Part (a) of these remarks, $\mathcal{G}_u$ is not semiregular when $\mathcal{G}$ is symmetric.

Case 2. $\mathcal{G}$ is regular.

As in Case 1 each block of $\mathcal{G}_u$ contains $2k - \rho_2$ points. Also, by Mitchell [**10**], $\mathcal{G}$ must be self-dual, with parameters $v, v, k, k; m_1, m_2; 0, \lambda_2$ say.

Consider a pair of distinct points P, Q of $\mathcal{G}$.

Subcase 2.1. P and Q belong to the same group of $\mathcal{G}$.

Since $\lambda_1 = 0$, there are no blocks containing both P and Q. So all the blocks of $\mathcal{G}_u$ containing P and Q come from the union of a block of $\mathcal{G}$ containing P, but not Q, with a block of $\mathcal{G}$ containing Q, but not P. For each of the k blocks B containing P, there are $k-1$ blocks containing Q and which are not in the group of B. (There must be precisely one block in the group of B containing Q since the groups form a tactical decomposition of $\mathcal{G}$ and the blocks of $\mathcal{G}$ in a group of $\mathcal{G}^d$ are disjoint.) It follows that $\mathcal{G}_u$ has $k(k-1)$ blocks containing P and Q.

Subcase 2.2. P and Q belong to different groups of $\mathcal{G}$.

There are λ_2 blocks of $\mathcal{G}$ containing both P and Q. Taking the union of such blocks with the blocks outside their groups yields $\lambda_2(v-m_1)-\frac{\lambda_2(\lambda_2-1)}{2}$ blocks of $\mathcal{G}_u$ containing P and Q. The remaining blocks of $\mathcal{G}_u$ containing P and Q arise from taking the union of a block containing P, but not Q, with a block containing Q, but not P. By Result 1, there are $(m_1-1)\lambda_2$ groups of $\mathcal{G}^d$ containing a block with P but not Q in it and a block with Q but not P in it. Also, there are $k-m_1\lambda_2$ groups of $\mathcal{G}^d$ containing a block containing P (resp. Q) and no block containing $Q\,(P)$. It follows that $\mathcal{G}_u$ has

$$\lambda_2(v-m_1)-\frac{\lambda_2(\lambda_2-1)}{2}+(m_1-1)\lambda_2((m_1-1)\lambda_2-1)+2(m_1-1)\lambda_2(k-m_1\lambda_2)+(k-m_1\lambda_2)^2$$

blocks containing P and Q.

If $\mathcal{G}$ is regular, then $\mathcal{G}_u$ is never a BIBD. To see this, note that from (3) we have

$$k(k-1)=\lambda_2(v-m_1). \tag{6}$$

So, if $\mathcal{G}_u$ is a BIBD, we must have

$$-\frac{\lambda_2(\lambda_2-1)}{2}+(m_1-1)\lambda_2((m_1-1)\lambda_2-1)+2(m_1-1)\lambda_2(k-m_1\lambda_2)+(k-m_1\lambda_2)^2=0. \tag{7}$$

Now, since $m_1\geq 2$, we have

$$1\geq\frac{m_1-\frac{3}{2}}{m_1^2-2m_1+\frac{1}{2}}.$$

Then, since $\lambda_2\geq 1$, we can obtain

$$(m_1-1)\lambda_2((m_1-1)\lambda_2-1)-\frac{\lambda_2(\lambda_2-1)}{2}\geq 0. \tag{8}$$

So the left side of (7) is a sum of three non-negative terms. Each of these terms must be zero. But equality occurs in (8) if and only if $m_1=2$ and $\lambda_2=1$. So we have $m_1=2$, $\lambda_2=1$ and $k=m_1\lambda_2=2$. But then (6) yields $v=4$. Hence, $\mathcal{G}$ has parameters $4,4,2,2;2,2;0,1$ and so is semiregular, a contradiction.

If $\mathcal{G}$ is regular, then $\mathcal{G}_u$ is regular. To establish this we show that $\mathcal{G}_u$ is not singular and then that $\mathcal{G}_u$ is not semiregular.

There are $k\geq 2$ block groups with a block containing a chosen point P of point group g_i and no other point of g_i. The union of two such blocks contains P, but no other point of g_i. Hence, $\mathcal{G}_u$ is not singular.

If the blocks in two groups $\bar{g}_h$ and $\bar{g}_j$ of $\mathcal{G}^d$ do not meet a group g_i of $\mathcal{G}$, then the union of a block from $\bar{g}_h$ and a block from $\bar{g}_j$ contains no point of g_i and so $\mathcal{G}_u$ is not semiregular in this case. Since $\mathcal{G}$ is not semiregular, the only other possibility is that there is precisely one group $\bar{g}_j$ whose blocks do not contain a point of g_i. In this case $k = m_2 - 1$. Since $k \geq 2$, we have $m_2 \geq 3$. But then some blocks of $\mathcal{G}_u$ contain one point of g_i and some contain two points of g_i. So $\mathcal{G}_u$ is again not semiregular.

Finite geometries provide examples of self-dual RGDD's with $\lambda_1 = 0$. For example, the external structure ([5], p.3) at a chosen point of the BIBD formed by the points and hyperlanes of a h-dimensional affine geometry ($h \geq 2$) over $GF(q)$ is a self-dual RGDD with parameters

$$q^h - 1,\, q^h - 1,\, q^{h-1},\, q^{h-1}\,;\, q-1,\, \frac{q^h-1}{q-1}\,;\, 0,\, q^{h-2}$$

(see Mitchell [9]). Other examples arise from the existence of "Baer subplanes" of finite projective planes, that is, from the existence of projective subplanes of order m of projective planes of order m^2. The substructure of such a plane formed by the points and lines outside a Baer subplane is a self-dual RGDD with parameters

$$m^4 - m,\, m^4 - m,\, m^2,\, m^2\,;\, m^2 - m,\, m^2 + m + 1\,;\, 0,\, 1.$$

We summarize the results of this section in the following theorem.

Theorem 1. *Let $\mathcal{G}$ be a GDD with parameters v, b, r, k; m_1, m_2; λ_1, λ_2 whose dual $\mathcal{G}^d$ is a GDD with parameters b, v, k, r; $\overline{m}_1$, $\overline{m}_2$; 0, ρ_2 and $\mathcal{G}_u$ be the design whose points are those of $\mathcal{G}$ and whose blocks are the unions of all the pairs of blocks in different groups of $\mathcal{G}^d$.*

(a) $\mathcal{G}_u$ is a GDD unless $\mathcal{G}$ has parameters $4,4,2,2;2,2;0,1$ in which case $\mathcal{G}_u$ is a $2-(4,3,2)$ design.

(b) $\mathcal{G}_u$ is never an SGDD.

(c) If $\mathcal{G}$ is an SRGDD, then $\mathcal{G}_u$ is an SRGDD if and only if each intersection of two blocks in different groups of $\mathcal{G}^d$ contain $\frac{\rho_2}{m_2}$ points of each group of $\mathcal{G}$. In consequence, $\mathcal{G}_u$ is an SRGDD only if m_2 is a divisor of ρ_2. Furthermore $\mathcal{G}_u$ is an RGDD if $\mathcal{G}$ is a self-dual transversal design.

(d) If $\mathcal{G}$ is an RGDD, then $\mathcal{G}_u$ is an RGDD.

4. Affine BIBD's

Let $\mathcal{A}$ be a BIBD which possesses an affine resolution with t blocks in each affine resolution class and each pair of non-parallel blocks meeting in μ (> 0) points. We call such a BIBD an ARD(μ, t). $\mathcal{A}$ has parameters $v = \mu t^2$, $b = \frac{t(\mu t^2-1)}{t-1}$, $r = \frac{\mu t^2 - 1}{t-1}$, $k = \mu t$ and $\lambda = \frac{\mu t - 1}{t-1}$ ([5], pp.72-3). If $t = 2$, then $\mathcal{A}$ is a $3-(4\mu, 2\mu, \mu-1)$ design. We also note that $t-1$ is a divisor of $\mu - 1$.

We can construct a BIBD $\mathcal{D}_u$ from $\mathcal{A}$ by employing the union method as follows: Take as new blocks the union of all the pairs of blocks in different affine resolution classes of $\mathcal{A}$. Clearly, each block of $\mathcal{D}_u$ has $\mu(2t-1)$ points in it. The connection number for a pair of distinct points of $\mathcal{D}_u$ is

$$\lambda t(r-1) - \frac{\lambda(\lambda-1)}{2} + (r-\lambda)(r-\lambda-1)$$

which can be shown to be equal to

$$\frac{t}{2}\left(\frac{\mu t-1}{t-1}\right)\left(2\left(\frac{\mu t-1}{t-1}\right) - \frac{\mu-1}{t-1}\right)(2t-1)$$

after some algebra. We thus have

Theorem 2. *$\mathcal{D}_u$ is a* $2-\left(\mu t^2,\ \mu(2t-1),\ \frac{t}{2}\left(\frac{\mu t-1}{t-1}\right)\left(2\left(\frac{\mu t-1}{t-1}\right)-\frac{\mu-1}{t-1}\right)(2t-1)\right)$ *design.*

Remark. The number of blocks of $\mathcal{D}_u$ is $\frac{t^3(\mu t^2-1)(\mu t-1)}{2(t-1)^2}$ and the replication number of $\mathcal{D}_u$ is $\frac{t(\mu t^2-1)(\mu t-1)(2t-1)}{2(t-1)^2}$.

In the situation where $t=2$ we can say more.

Theorem 3. *If $t=2$, then $\mathcal{D}_u$ is a* $3-(4\mu,\ 3\mu,\ \frac{3(3\mu-1)(3\mu-2)}{2})$ *design.*

Proof. Consider distinct points P, Q, R of $\mathcal{A}$. There are $\mu-1$ blocks of $\mathcal{A}$ containing all three of these points. Unions with such blocks involved yield $(\mu-1)(8\mu-4)-\frac{(\mu-1)(\mu-2)}{2}$ blocks of $\mathcal{D}_u$ containing P, Q and R.

Also, there are μ blocks of $\mathcal{A}$

(i) containing P and Q, but not R, and
(ii) containing Q and R, but not P.

Unions, where one block is of Type (i) and one is of Type (ii), yield μ^2 blocks of $\mathcal{D}_u$ containing P, Q and R. Cyclicly rotating the roles of P, Q and R yields $2\mu^2$ further blocks of $\mathcal{D}_u$ containing P, Q and R.

There are μ blocks of $\mathcal{A}$

(iii) containing R, but not P nor Q.

Unions, where one block is of Type (i) and one is of Type (iii), yield $\mu(\mu-1)$ blocks of $\mathcal{D}_u$ containing P, Q and R. (For a given block of Type (i), one of the blocks of Type (iii) is in its affine resolution class.) Rotating the roles of P, Q and R yields $2\mu(\mu-1)$ further blocks of $\mathcal{D}_u$ containing P, Q and R.

The number of blocks of $\mathcal{D}_u$ containing P, Q and R is

$$(\mu-1)(8\mu-4) - \frac{(\mu-1)(\mu-2)}{2} + 3\mu^2 + 3\mu(\mu-1) = \frac{3(3\mu-1)(3\mu-2)}{2}. \qquad \blacksquare$$

The question as to when we obtain simple BIBD's by applying the union method, as in this section, is answered in the next theorem. Before proceeding to it some preliminary discussion is needed.

Result 2. (Kimberley, [6]) Let A and B be two blocks of an ARD(μ, t) $\mathcal{A}$ which meet in μ points.

(a) The number of blocks of $\mathcal{A}$ which contain $A \cap B$ is less than or equal to $t+1$.

(b) If the number of blocks of $\mathcal{A}$ which contain $A \cap B$ is $t+1$, then the blocks of $\mathcal{A}$ which contain $A \cap B$ partition $(A \cap B)^c$ (= the complement of $A \cap B$).

We say that $A \cap B$ is a *good intersection* if the number of blocks of $\mathcal{A}$ containing $A \cap B$ is $t+1$. A block, each of whose non-empty intersections is good, is called a *good block*.

Result 3. (Kimberley, [6]) $\mathcal{A}$ is an ARD (μ, t), all of whose blocks are good, if and only if $\mathcal{A}$ is isomorphic to the BIBD formed by the points and hyperplanes of a finite affine space.

Consider an ARD(μ, 2) $\mathcal{A}$. Suppose $A \cap B$ is a good intersection of $\mathcal{A}$. Let c_A and c_B be the affine resolution classes of $\mathcal{A}$ containing A and B, respectively. Also, let C be the third block of $\mathcal{A}$ containing $A \cap B$ and c_C be the affine resolution class of $\mathcal{A}$ containing C. It is not difficult to show that $X \cap Y$ is a good intersection of $\mathcal{A}$ for any two blocks of $\mathcal{A}$ chosen from different classes among c_A, c_B and c_C. Such a triple of affine resolution classes is said to be *closed*.

Lemma 1. *Let A and B be blocks in different affine resolution classes c_A and c_B, respectively, of an ARD (μ, 2) $\mathcal{A}$. Then $A^c \cap B^c$ is a good intersection if and only if there is a block D of $\mathcal{A}$ such that $D \notin c_A \cup c_B$ and $A \cap B \cap D = \emptyset$.*

Proof. Suppose $A^c \cap B^c$ is a good intersection. Then there is a block $D^c \notin c_A \cup c_B$ such that $A^c \cap B^c = A^c \cap D^c = B^c \cap D^c$. By Result 1(b), $A^c \cup B^c \cup D^c = (A \cap B \cap D)^c$ is the point set of $\mathcal{A}$, whence $A \cap B \cap D = \emptyset$. Since D is in the affine resolution class of D^c we have $D \notin c_A \cup c_B$.

Suppose there is $D \notin c_A \cup c_B$ such that $A \cap B \cap D = \emptyset$. Then $|A \cup B \cup D| = 6\mu - 3\mu + 0 = 3\mu$. So $|A^c \cap B^c \cap D^c| = \mu = |A^c \cap B^c|$. Clearly $A^c \cap B^c \cap D^c = A^c \cap B^c$. Similarly, $B^c \cap D^c = A^c \cap B^c \cap D^c = A^c \cap D^c$. So $A^c \cap B^c$ is a good intersection. ∎

A *line* of BIBD $\mathcal{D}$ is the intersection of all the blocks of $\mathcal{D}$ containing two points of $\mathcal{D}$. For all lines L of a (v, b, r, k, λ)-design, we have $|L| \leq \frac{b-\lambda}{r-\lambda}$ ([5], p.65). For a $2-(4\mu-1, 2\mu-1, \mu-1)$ design $\mathcal{H}$, we have $|L| = 2$ or 3 for all lines L of $\mathcal{H}$.

Let $\mathcal{A}$ be an ARD(μ, 2), P a point of $\mathcal{A}$ and c_1, c_2 and c_3 be a closed triple of affine resolution classes of $\mathcal{A}$. Consider the interior structure $\mathcal{H}$ of $\mathcal{A}$ at P. $\mathcal{H}$ is a $2-(4\mu-1, 2\mu-1, \mu-1)$ design. For $i = 1, 2, 3$, let B_i be the block of c_i containing P. The intersection of the blocks of $\mathcal{H}$ corresponding to B_1, B_2, B_3 contain $\mu - 1$ points of $\mathcal{H}$. In the dual $\mathcal{H}^d$ of $\mathcal{H}$ these points (in their role as blocks of $\mathcal{H}^d$) have B_1, B_2 and B_3 in their intersection. The line L of $\mathcal{H}^d$ defined by B_1 and B_2 thus satisfies $\{B_1, B_2, B_3\} \subseteq L$. But $|L| \leq 3$ and so we have $|L| = 3$. Conversely, it is not difficult to show that a line of order three in a $2-(4\mu-1, 2\mu-1, \mu-1)$ design $\mathcal{H}$ corresponds to a closed triple of affine resolution classes in the 1-point extension of $\mathcal{H}^d$.

Let $\mathcal{D}_u$ be the BIBD of Theorem 2.

Theorem 4. *(a) If $t > 2$, then $\mathcal{D}_u$ is simple.*

(b) *If* $t=2$, *then* $\mathcal{D}_u$ *is a simple* 3*-design if and only if* $\mathcal{A}$ *possesses no good intersections.*

Proof. Suppose two blocks of $\mathcal{D}_u$ are equal, say $A \cup B = C \cup D$, where $A \neq B$, $C \neq D$. Without loss of generality, we can assume $B \neq C, D$. Now $A \cup B = A \cup C \cup D$ and so

$$|A| + |B| - |A \cap B| = |A| + |C| + |D| - |A \cap C| - |A \cap D| - |C \cap D| + |A \cap C \cap D|$$

which yields

$$|A \cap C \cap D| = |A \cap C| + |A \cap D| - \mu t. \tag{9}$$

We split our analysis into two cases.

Case 1. A, C and D are in different affine resolution classes.

In this case (9) gives us $|A \cap C \cap D| = 2\mu - \mu t \geq 0$, whence $t = 2$. Note here that we have $A \cap C \cap D = \emptyset$.

Case 2. A and C or A and D are in the same affine resolution class.

Without loss of generality, assume A and C are in the same class. If $A \neq C$, then (9) yields $|A \cap C \cap D| = \mu - \mu t = 0$ and so $t = 1$, which is absurd. If $A = C$, then $A \cup B = A \cup D$ and so $B \setminus ((A \cap B) \cup (D \cap B)) = \emptyset$. Thus $|B| - |A \cap B| - |D \cap B| + |A \cap B \cap D| = 0$, from which we obtain $|A \cap B \cap D| = 2\mu - \mu t \geq 0$. Again we infer $t = 2$. Here we then have $A \cap B \cap D = \emptyset$.

Clearly, if $t > 2$, then we have a contradiction and so Part (a) is established.

We now proceed to establish Part (b). From now on let $t = 2$.

Suppose $\mathcal{A}$ possesses a good intersection, say $A \cap B$. Let D be the third block of $\mathcal{A}$ containing $A \cap B$. Then A^c, B^c and D^c are blocks in different affine resolution classes of $\mathcal{A}$ and $A^c \cup B^c = (A \cap B)^c = (A \cap D)^c = A^c \cup D^c$. Therefore $\mathcal{D}_u$ is not simple.

Suppose $\mathcal{D}_u$ is not simple. Let $A \cup B = C \cup D$, where $A \neq B$ and $C \neq D$. If A, B, C and D are in different affine resolution classes, then our previous analysis yields $A \cap C \cap D = A \cap B \cap C = B \cap C \cap D = A \cap B \cap D = \emptyset$. So $|A \cup B \cup C \cup D| = 8\mu - 6\mu + 0 - 0 = 2\mu$. But $A \cup B = C \cup D$ and so $|A \cup B \cup C \cup D| = |A \cup B| = 3\mu$. We infer that A, B, C and D cannot be in different affine resolution classes.

As in our earlier analysis we can assume without loss of generality that $B \neq C, D$. That analysis shows that, either A, C and D are in different affine resolution classes and, in consequence, $A \cap C \cap D = \emptyset$ or (without loss of generality) $A = C$ and, in consequence, $A \cap B \cap D = \emptyset$. In the former case we can assume, without loss of generality, that B is in the affine resolution class of C, whence $B \cap C = \emptyset$. Then A, B and D are in different affine resolution classes and so $A \cap B \cap D = \emptyset$. As in the previous paragraph, the Principle of Inclusion - exclusion leads to a contradiction. In the latter case B and D must be in different affine resolution classes, since $A \cup B = C \cup D$. So $A^c \cap B^c$ is a good intersection, by Lemma 1. ∎

Corollary 1. *Let* $t = 2$. *The multiplicity of a block of* $\mathcal{D}_u$ *is* 1 *or* 3.

Proof. Suppose the multiplicity of $A \cup B$ is greater than one, say $A \cup B = C \cup D$, where $B \neq C, D$. From the proof of Theorem 4, (without loss of generality) $A = C$, A, B and

D are in different affine resolution classes and $A \cap B \cap D = \emptyset$. From the proof of Lemma 1, we have $A^c \cap B^c = B^c \cap D^c = D^c \cap A^c$ and so $A \cup B = B \cup D = D \cup A$. Therefore the multiplicity of $A \cup B$ is at least three.

Suppose $A \cup B = C \cup D = E \cup F$, where $\{A, B\} \neq \{C, D\}, \{E, F\}$. Without loss of generality $A = C$ or $B = C$ and A, B and D are distinct. Also, we have

(i) $B = E$ and A, B and F are distinct, or
(ii) $B = F$ and A, B and E are distinct, or
(iii) $A = E$ and A, B and F are distinct, or
(iv) $A = F$ and A, B and E are distinct.

In Case (i), $A^c \cap C^c$ is contained in A^c, B^c, D^c and F^c. By Result 2(a), we have $F^c = A^c$ or B^c or D^c. Clearly $F^c = D^c$ and so $F = D$. Case (ii) leads similarly to $E = D$, Case (iii) to $F = D$ and Case (iv) to $E = D$. We thus have $\{C, D\} = \{E, F\}$, or $\{C, D\} = \{A, D\}$ and $\{E, F\} = \{B, D\}$, or $\{C, D\} = \{B, D\}$ and $\{E, F\} = \{A, D\}$. It follows that the multiplicity of $A \cup B$ is at most three and so is exactly three. ■

Corollary 2. *Let $t = 2$ and A, B be blocks of $\mathcal{A}$ in different affine resolution classes. The multiplicity of the block $A \cup B$ of $\mathcal{D}_u$ is three if and only if $A^c \cap B^c$ is a good intersection.*

Proof. Suppose $A \cup B$ has multiplicity three. Then, from the proof of Corollary 1, there is $D \neq A, B$ such that $A \cup B = A \cup D = B \cup D$. But then $A^c \cap B^c = A^c \cap D^c = B^c \cap D^c$ and so $A^c \cap B^c$ is good.

The converse is easily shown. ■

Corollary 3. *Let $t = 2$. $\mathcal{D}_u$ is a 3-multiple of a $3 - \left(4\mu,\, 3\mu,\, \frac{(3\mu-1)(3\mu-2)}{2}\right)$ design $\mathcal{D}^*$ if and only if μ is a power of two and $\mathcal{A}$ is isomorphic to the BIBD formed by the points and hyperplanes of a finite affine space over $GF(2)$.*

Proof. Suppose $\mathcal{D}_u$ is a 3-multiple. Consider A and B in different affine resolution classes of $\mathcal{A}$. Then $A^c \cup B^c$ has multiplicity equal to three. So $A \cap B$ is a good intersection, by Corollary 2. By Result 3, $\mathcal{A}$ is isomorphic to the BIBD formed by the points and hyperplanes of a finite affine space over $GF(2)$.

Suppose μ is a power of two and $\mathcal{A}$ is isomorphic to the BIBD formed by the points and hyperplanes of a finite affine space over $GF(2)$. By Result 3, every intersection of $\mathcal{A}$ is good. Consider A and B in different affine resolution classes. Then $A^c \cap B^c$ is a good intersection. By Corollary 2, $A \cup B$ has multiplicity three. ■

Corollary 4. *If there is a $2 - (4\mu - 1,\, 2\mu - 1,\, \mu - 1)$ design all of whose lines are of cardinality two, then there is a simple $3 - (4\mu,\, 3\mu,\, \frac{3(3\mu-1)(3\mu-2)}{2})$ design.*

Proof. Suppose there is a $2 - (4\mu - 1,\, 2\mu - 1,\, \mu - 1)$ design $\mathcal{H}$ all of whose lines are of order two. Then the 1-point extension $\mathcal{A}$ of $\mathcal{H}^d$ has no good intersections. So the BIBD $\mathcal{D}_u$ constructible from $\mathcal{A}$ using the union method is simple, by Corollaries 1 and 2. ■

ARD(q^{h-2}, q)'s are known to exist for each prime power q and integer $h \geq 2$. Theorems 2 and 4 yield that there exist simple $2-(q^h, q^{h-2}(2q-1), \frac{q(q^{h-1}-1)(2q^{h-1}-q^{h-2}-1)(2q-1)}{2(q-1)^2})$ designs $\mathcal{D}_u$ for each prime power $q \geq 3$ and integer $h \geq 2$. For $q = h = 3$, the complement of $\mathcal{D}_u$ is a simple $2-(27, 12, 132)$ design. Such a design is listed as unknown in [**3**].

The complement of the BIBD $\mathcal{D}^*$ in the statement of Corollary 3 to Theorem 4 is a simple $3-(4\mu, \mu, \frac{(\mu-1)(\mu-2)}{6})$ design, provided $\mu > 2$. This corollary shows the existence of a class of simple $3-(2^{d+2}, 2^d, \frac{(2^d-1)(2^{d-1}-1)}{3})$ designs for all $d \geq 2$. It is, however, not difficult to verify that these simple 3-designs are isomorphic to those formed by the points and d-dimensional subspaces of AG$(d+2, 2)$.

5. Further applications of the union method

In certain situations unions of more than two blocks can be used to construct BIBD's. In this section we give two illustrations of this.

Let $\mathcal{D}$ be a $2-(v, k, 1)$ design with replication number r, n be an integer such that $2 \leq n \leq r-1$ and P be any point of $\mathcal{D}$. Take the union of any n blocks containing P. The unions so obtained as P ranges over the set of points of $\mathcal{D}$ are the blocks of a BIBD $\mathcal{D}_u(n)$ with the same set of points as $\mathcal{D}$. In fact, we have

Theorem 5. *$\mathcal{D}_u(n)$ is a $2-(v, nk-n+1, \binom{r-1}{n-1}(nk-n+1))$ design.*

This result has been established by Morgan [**11**] in the cases where $\mathcal{D}$ is a finite projective plane or a finite affine plane. The proof of Theorem 5 is analogous to the argument given in [**11**] for $\mathcal{D}$ a finite projective plane and so is omitted.

The question as to whether the BIBD's $D_u(n)$ are simple is, in general, difficult to answer. However, in certain cases they can be shown to be simple quite easily.

Let B and C be blocks of $\mathcal{D}_u(n)$ obtained by taking the union of n blocks containing P and Q, respectively. If $P = Q$, then different choices of the n blocks yield different unions. The same is obviously true if $P \neq Q$ and the block of $\mathcal{D}$ containing P and Q is in at most one of the two unions. Thus, for $B = C$ to arise from different unions, we must have $P \neq Q$ and the block (A say) of $\mathcal{D}$ containing P and Q is used in the union that produces B and in the union that produces C. Now, suppose $B = C$ in this situation. There are $r-n$ blocks of $\mathcal{D}$ containing P which do not meet a block of $\mathcal{D}$ ($\neq A$) in the union that produces C. But the number of blocks containing P and not meeting a block containing Q is $r-k$. Clearly we have $r-n \leq r-k$ and so $n \geq k$. Also there are $n-1$ blocks containing Q which do not meet a block on P not used in the union that produces B. So we have $r-k \geq n-1$. From this analysis we have the following proposition:

Proposition 1. *If $n < k$ or $r-k+1 < n$, then $\mathcal{D}_u(n)$ is simple. In consequence, if $r+2 \leq 2k$, then $\mathcal{D}_u(n)$ is simple for all n such that $2 \leq n \leq r-1$.*

Theorem 5, Proposition 1 and known results concerning the existence of finite projective and affine planes and Steiner triple systems allow us to establish the following results:

1. If $\mathcal{D}$ is
   (i) a finite projective plane, or
   (ii) a finite affine plane of order greater than two,

then $\mathcal{D}_u(n)$ is simple for all n such that $2 \leq n \leq r-1$. In particular, for every prime power q and integer n such that $2 \leq n \leq q$, there is a simple $2-(q^2+q+1,\ nq+1,\ \binom{q}{n-1}(nq+1))$ design and, for every prime power $q > 2$ and integer n such that $2 \leq n \leq q$, there is a simple $2-(q^2,\ nq-n+1,\ \binom{q}{n-1}(nq-n+1))$ design. (These designs were constructed in [11]. There (see p.348) it was noted that they are simple.)

2. For each $v \equiv 1$ or $3 \mod 6$, there exist simple $2-(v,\ 5,\ \frac{5(v-3)}{2})$ designs.

In order to appreciate the difficulties that arise when one further investigates the question as to whether $\mathcal{D}_u(n)$ is simple, let $\mathcal{D}$ be a Steiner triple system and $n = 3$. It is easy to show that $\mathcal{D}_u(3)$ is simple if and only if Pasch's configuration (Figure 1) does not occur in $\mathcal{D}$, that is, if and only if $\mathcal{D}$ is a "quadrilateral - free" Steiner triple system.

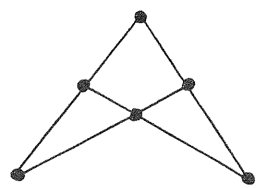

FIGURE 1

It is known that quadrilateral-free Steiner triple systems exist for infinitely many values of v ([4]). Brouwer has conjectured that there is a quadrilateral-free Steiner triple system for each possible v other than 7 and 13. This conjecture, however, is far from settled. In any event, quadrilateral-free Steiner triple systems exist for $v = 25$ and 27 and so there exists a simple $2-(25,\ 7,\ 385)$ design and a simple $2-(27,\ 7,\ 462)$ design. Also, $\mathcal{D}_u(2)$, where $\mathcal{D}$ is any Steiner triple system with 27 points, is a simple $2-(27,\ 5,\ 60)$ design. According to [3] such designs were previously unknown.

For further details about what is known about quadrilateral-free Steiner triple systems [4] should be consulted.

Let $\mathcal{D}$ be a $2-(v,\ k,\ \lambda)$ design with replication number r and a 1-resolution. Each 1-resolution class of $\mathcal{D}$ contains $t = \frac{v}{k}$ blocks of $\mathcal{D}$. Let c be any 1-resolution class of $\mathcal{D}$ and n be an integer such that $2 \leq n \leq t-1$. Take the union of any n blocks of c. The unions so obtained as c ranges over all the 1-resolution classes of $\mathcal{D}$ are the blocks of a BIBD $\mathcal{D}_p(n)$ with the same set of points as $\mathcal{D}$. In fact, we have

Theorem 6. *$\mathcal{D}_p(n)$ is a $2-(v,\ nk,\ \lambda\binom{t-1}{n-1}+(r-\lambda)\binom{t-2}{n-2})$ design.*

Proof. Clearly each block of $\mathcal{D}_p(n)$ contains nk points of $\mathcal{D}_p(n)$. Consider a pair P, Q of distinct points of $\mathcal{D}_p(n)$. There are λ 1-resolution classes containing a block joining P and Q. The union of such a block with any $n-1$ of the other $t-1$ blocks in its 1-resolution class is a block of $\mathcal{D}_p(n)$ containing P and Q. Also, there are $r-\lambda$ 1-resolution classes which contain a block containing P, but not Q, and a block containing Q, but not P. The union of such a pair of blocks with any $n-2$ other blocks from their 1-resolution class is a block of $\mathcal{D}_p(n)$ joining P and Q. Since $\mathcal{D}_p(n)$ has no further blocks joining P and Q, the connection number of P and Q in $\mathcal{D}_p(n)$ is $\lambda\binom{t-1}{n-1} + (r-\lambda)\binom{t-2}{n-2}$. ■

Remark. $\mathcal{D}_p(t-1)$ is the complement of $\mathcal{D}$. Also, $\mathcal{D}_p(t-n)$ is the complement of $\mathcal{D}_p(n)$ for each n such that $2 \leq n \leq t-2$.

Suppose $\lambda = 1$. Then, if $n < k$, then $\mathcal{D}_p(n)$ is simple. Also, if $n > t-k$, then $\mathcal{D}_p(t-n)$ is simple and so the complement $\mathcal{D}_p(n)$ of $\mathcal{D}_p(t-n)$ is simple. This leads to

Proposition 2. *If $\lambda = 1$ and $n < k$ or $n > t-k$, then $\mathcal{D}_p(n)$ is simple. In consequence, if $\lambda = 1$ and $2k \geq t+1$, then $\mathcal{D}_p(n)$ is simple for all n such that $2 \leq n \leq t-1$.*

Proposition 2 immediately yields that, if $\mathcal{D}$ is a finite affine plane, then $\mathcal{D}_p(n)$ is simple for all n such that $2 \leq n \leq t-1$.

There exist 1-resolvable $2-(21, 3, 1)$ designs and also 1-resolvable $2-(28, 4, 1)$ designs ([**8**], p.278). From this we can infer the existence of simple $2-(21, 6, 15)$ designs and $2-(28, 8, 14)$ designs. Such simple designs were not previously known to exist.

References

1. R.C. Bose, *Symmetric group divisible designs with the dual property*, J. Statist. Plann. Inference **1** (1977), 87–101.
2. R.C. Bose and W.S. Connor, *Combinatorial properties of group divisible incomplete block designs*, Ann. Math. Statist. **23** (1952), 367–383.
3. Y.M. Chee, C.J. Colbourn and D.L. Kreher, *Simple t-designs with $v \leq 30$*, Ars Combin. **29** (1990), 193–258.
4. C.J. Colbourn and A. Rosa, *Leaves, excesses and neighbourhoods in triple systems*, Australas. J. Combin. **4** (1991), 143–178.
5. P. Dembowski, "Finite Geometries", Springer, New York, 1968.
6. M. Kimberley, *On the construction of certain Hadamard designs*, Math. Z. **119** (1971), 41–59.
7. E.S. Mahmoodian and M. Shirdarreh, *Some new families of simple t-designs*, Australas. J. Combin. **5** (1992), 159–168.
8. R. Mathon and A. Rosa, *Tables of parameters of BIBDs with $r \leq 41$ including existence, enumeration and resolvability results*, Annals Discrete Math. **26** (1985), 275–307.
9. C.J. Mitchell, *An infinite family of symmetric designs*, Discrete Math. **26** (1979), 247–250.
10. C.J. Mitchell, *Group divisible designs with dual properties*, Mitt. Math. Sem. Giessen **165** (1984), 105–117.
11. E.J. Morgan, *Construction of balanced incomplete block designs*, J. Austral. Math. Soc. **23** (Ser. A) (1977), 348–353.

12. A.J. Rahilly, *Semiregular group divisible designs whose duals are semiregular group divisible*, Australas. J. Combin. **2** (1990), 157–179.
13. A.J. Rahilly, *Semiregular group divisible designs with dual properties*, Bull. Austral. Math. Soc. **45** (1992), 61–69.
14. A.P. Street and D.J. Street, "Combinatorics of Experimental Design", Oxford University Press, Oxford, 1987.

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