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Critical sets in equiorthogonal frequency squares

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Abstract

In this paper, we study critical sets in pairs of equiorthogonal frequency squares. Using this stronger definition of orthogonality, a pair of equiorthogonal frequency squares is classified into one of three classes depending on the isomorphism or orthogonality of the corresponding rows and columns. We provide a general theorem determining the size of the critical set of a pair of equiorthogonal squares in which the corresponding rows and columns are isomorphic. For the other possible combinations of corresponding rows and columns, we make a few general observations with a detailed investigation into the conditions for the existence of an equiorthogonal mate and the size of a critical set for a pair of squares of order 8 based on 2 symbols.

1 Introduction

Critical sets in latin squares and frequency squares have been studied in papers such as Nelder [18], Smetaniuk [22], Curran and van Rees [5], Cooper, Donovan and Seberry [3], Cooper, McDonough and Mavron [4], Donovan, Cooper, Nott and Seberry [6], Donovan and Cooper [7], Fu, Fu and Rodger [11], Donovan and Howse [8], Fitina, Seberry and Sarvate [10], Bate and van Rees [1], Cavenagh [2], Keedwell [14], and SahaRay and Morgan [21]. Critical sets in pairs of orthogonal latin squares have been investigated in SahaRay, Adhikari and Seberry [19, 20], but little work has been done studying critical sets in sets of orthogonal frequency squares. Part of the reason for that might be the fact that the presence of orthogonality using the usual definition provides little, if any, reduction of the size of a critical set below that which would be obtained by considering the squares separately.

If we use a stronger definition of orthogonality, called equiorthogonality as considered in Morgan [15, 16, 17] we find that the size of a critical set for a pair of equiorthogonal frequency squares is often reduced significantly compared to the size of the union of the individual critical sets. We classify a pair of equiorthogonal frequency squares into one of three classes depending on the isomorphism or orthogonality of the corresponding rows and columns. We focus on the determination of the size of a critical set for a pair of squares for each of the three types of equiorthogonality, which require somewhat different approaches.

Before discussing the main results, some background information is needed which is presented in Section 2. In Section 3, we investigate the case of order 4 based on 2 symbols in detail; this case provides a good introduction to the general case of order n based on m symbols. The case of a pair of squares having isomorphic corresponding rows and columns is dealt with in detail in Section 4. For the case in which the corresponding rows are isomorphic and the corresponding columns are orthogonal, we make general observations on the size of a critical set in Section 5 and also provide a detailed study for the case of order 8 based on 2 symbols. The conditions for the existence of an orthogonal mate for such a square are explored in Section 6. Finally, the case of a pair of squares in which the corresponding rows and columns are orthogonal is dealt with in Section 7.

2 Preliminary Definitions and Notations

A frequency square, or F-square, $F = F(n; \alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ of order n is an $n \times n$ array with entries chosen from the set $N = \{0, 1, 2, \ldots, m-1\}$ such that each element i occurs α_i times in each row and in each column, where each α_i is a natural number and $\sum_{i=0}^{m-1} \alpha_i = n$. If all of the α_i are equal, they will be equal to n/m. The reader is referred to Hedayat and Seiden [12], Hedayat, Raghavarao and Seiden [13] and Morgan [15, 16, 17] for more information about F-squares. When there is no potential for ambiguity, we may refer to F-squares as squares. Following the notation from Morgan [15, 16, 17], we will use the notation $F^{(2)}(n; n/m)$ for a single frequency square of order n based on m symbols with constant frequency n/m.

Two $F^{(2)}(n; n/m)$ frequency squares are said to be *isotopic* if there exist permutations of the rows, columns, and set of symbols that transform one to the other. This condition is weaker than *isomorphism*, which only permits a permutation or relabeling of the symbols.

The usual definition of orthogonality for frequency squares requires only condition (a) listed below; see, for example, Hedayat and Seiden [12] or Hedayat, Raghavarao and Seiden [13]. Equiorthogonality is a strengthened form of orthogonality that makes more use of the inherent structure of the frequency squares.

Definition 2.1 In the case of $F^{(2)}(n; n/m)$ frequency squares, two such squares are considered to be equiorthogonal if the following conditions hold:

(a) each ordered pair appears exactly n^2/m^2 times in the superimposition of the squares;

(b) pairs of corresponding rows and columns are isomorphic (one is a relabeling of the other) or orthogonal (with each ordered pair appearing exactly n/m^2 times); and

(c) if one pair of corresponding rows is isomorphic (resp. orthogonal), then the same is true for all pairs of corresponding rows; likewise for the columns.

This results in three possible relationships between pairs of equiorthogonal frequency squares, which we will refer to as equiorthogonality Type I, Type II, and Type III.

Type I: corresponding rows and columns are both isomorphic.

Type II: without loss of generality, corresponding rows are isomorphic and corresponding columns are orthogonal. If the corresponding rows are orthogonal and the columns are isomorphic, the transpose can be taken of both squares. Note that a necessary condition for Type II equiorthogonality is that m^2 divide n. If m^2 does not divide n, it is impossible for corresponding rows or columns to be orthogonal, so the only possible orthogonality is Type I.

Type III: corresponding rows and columns are both orthogonal. Note that the condition that m^2 divide n is necessary here as well.

If we think of the frequency square as a set of ordered triples $F = \{(i, j; k)\}$ where element k occurs in position (i, j), a nonempty subset S of F is defined to be a *defining set* of F if F is the only frequency square with the appropriate parameters which has element k in position (i, j) for each $(i, j; k) \in S$ (i.e., F is *uniquely completable* from S). The subset S is a *critical set* of F if, in addition, every proper subset of S is contained in at least two frequency squares with the appropriate parameters. This definition is equivalent to that used by Cavenagh in [2] using slightly different language and is consistent with the definition of critical sets for Latin squares and other combinatorial designs (see, for example, [9] and [14]).

A critical set for a pair of equiorthogonal frequency squares may be defined analogously. Note that the type of equiorthogonality is not assumed in advance; the information contained in the critical set will determine the equiorthogonality type. It should also be noted that, if an isotopism is applied to a square or a pair of squares, the same isotopism, applied to a critical set, will yield a critical set for the new square or pair of squares.

We will see that, when m^2 does divide n, there are several different possibilities for the nature of the squares, but we still get a fair amount of information from Type II equiorthogonality. In Type III, however, there are a large number of possibilities for the squares and the relationship between them. Consequently, critical sets are noticeably larger in Type III than in Type I or Type II.

Beginning in Section 4, we will consider the individual cases in a bit more detail. First, we will explore some small cases, both to illustrate the concepts and to make note of some of the difficulties we will encounter in the general case.

3 The case $F^{(2)}(4; 2)$ and introduction to the case $F^{(2)}(n; n/2)$

Even in the case $m = 2$ and $n = 8$, there are a large number of nonisotopic frequency squares and an even larger number of possible relationships among them. The case $m = 2$ and $n = 4$ is somewhat more manageable. First of all, there are only two possible frequency squares (up to isotopism and relabeling),

$$\begin{array}{cccc} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array} \quad \text{and} \quad \begin{array}{cccc} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}.$$

It will be useful to have notations to describe the different structures of individual squares. We will use the term **pattern**, with the notation (row pattern)\(column pattern), where the row (resp. column) pattern refers to the number of rows (resp. columns) in each isomorphism class. We will use the pattern $4^1\backslash 4^1$ to describe the first square and $2^2\backslash 2^2$ to describe the second. Note that, in the first square, all of the rows are isomorphic and the same is true of the columns. In the second square, rows 0 and 3 are isomorphic and rows 1 and 2 are isomorphic with (coincidentally) the same being true of the columns.

It is not hard to show that the smallest possible critical set for each of these squares, considered individually, has size 4. Table 1 gives the possible equiorthogonality relationships among $F^{(2)}(4; 2)$ frequency squares, where elements of the critical sets are denoted in bold.

We can already make some observations from this table. First, there are no Type I pairs listed; we will see the reason for this in Section 4. Also note that Type III equiorthogonality may or may not reduce the size of the critical set below the amount that result from considering the squares individually. However, Type II reduces the size of the critical set of the second square. Note that the presence of zeroes in locations $(0, 0)$ and $(0, 1)$ of both squares in the first Type II pair listed guarantees that corresponding rows are isomorphic; it will be noted in the next section that, when $m = 2$, Type II equiorthogonality is guaranteed. The other zero is necessary to distinguish the last two rows of the second square; interestingly, a third entry is not necessary in the second square of the other Type II pair listed. Note that the $(1, 0)$ entry must be 1, or else the squares will be identical, hence not equiorthogonal. Isomorphism of corresponding rows allows us to complete the second row, after which columns 1 and 2 are determined and the rest of the second square can be completed.

Square				Mate				Pattern	Orth. Type	Size of Critical Set
0	**0**	1	1	**0**	**0**	1	1			
0	**0**	1	1	1	1	0	0	$4^1\backslash 4^1$,	II	4+3
1	1	0	0	**0**	0	1	1	$4^1\backslash 4^1$		
1	1	0	0	1	1	0	0			
0	**0**	1	1	**0**	1	1	**0**			
0	**0**	1	1	1	0	0	1	$4^1\backslash 4^1$,	III	4+4
1	1	0	0	**0**	1	1	**0**	$4^1\backslash 4^1$		
1	1	0	0	1	0	0	1			
0	1	1	**0**	**0**	**0**	1	1			
1	0	0	1	**0**	0	1	1	$2^2\backslash 2^2$,	III	4+3
0	1	0	1	1	1	0	0	$4^1\backslash 4^1$		
1	0	**1**	0	1	1	0	0			
0	**0**	1	1	**0**	**0**	1	1			
0	1	0	1	1	0	1	0	$2^2\backslash 2^2$,	II	4+2
1	0	**1**	0	0	1	0	1	$2^2\backslash 2^2$		
1	1	0	0	1	1	0	0			
0	**0**	1	1	**0**	1	**0**	1			
0	1	0	1	1	1	0	0	$2^2\backslash 2^2$,	III	4+3
1	0	**1**	0	**0**	0	1	1	$2^2\backslash 2^2$		
1	1	0	0	1	0	1	0			

Table 1

In the case of $F^{(2)}(8;4)$ frequency squares, many more patterns are possible, and even determining the size of a critical set for an individual square is not so straightforward. There are, however, some commonalities in all of the cases where $m = 2$.

One useful result follows directly from Theorem 5 of [2]:

Lemma 3.1 *A critical set for one $F^{(2)}(n;n/2)$ frequency square must have at least $n^2/4$ entries.*

We now provide several $F^{(2)}(n;n/2)$ frequency squares with different row and column patterns that satisfy this lower bound.

Proposition 3.2 *A $F^{(2)}(n;n/2)$ frequency square with any of the patterns (i) $n^1\backslash n^1$, (ii) $(n/2)^2\backslash(n/2)^2$, (iii) $(n/2)^2\backslash(3n/4)^1(n/4)^1$, (iv) $(3n/4)^1(n/4)^1\backslash(n/2)^2$, or (v) $(3n/4)^1(n/4)^1\backslash(3n/4)^1(n/4)^1$ has a critical set of size $n^2/4$.*

Proof: Without loss of generality, we can consider the $n^1\backslash n^1$ square to be in the block pattern with zeroes in the upper left and lower right; it is clear that the $n^2/4$

zeroes in the upper left form a critical set. For case (ii), note that there exist two nonisomorphic rows which, without loss of generality, can be written as

$$\begin{array}{cccc} 0\ldots0 & 0\ldots0 & 1\ldots1 & 1\ldots1 \\ 0\ldots0 & 1\ldots1 & 0\ldots0 & 1\ldots1. \end{array}$$

If we let p denote the size of the first grouping, then the other groupings must have sizes $n/2-p$, $n/2-p$, and p, respectively. In order for the column pattern to be as stated, the columns in the first and fourth groupings must all be isomorphic and $2p$ must equal $n/2$, from which we can conclude that 4 divides n. A similar argument involving the row pattern shows that the square must be isotopic to

$$\begin{array}{cccc} \mathbf{0} & \mathbf{0} & 1 & 1 \\ \mathbf{0} & 1 & 0 & 1 \\ 1 & 0 & \mathbf{1} & 0 \\ 1 & 1 & 0 & 0 \end{array}$$

with each entry representing a square block of size $n^2/16$. The entries in bold form the desired critical set.

For the other three cases, an argument analogous to that given for case (ii) will show that $3n/4$ and $n/4$ must be even, so 8 must divide n for the pattern to be possible. For case (iii), the square must be isotopic to

$$\begin{array}{cccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \mathbf{1} & 0 & 0 & 0 & 0 \end{array}$$

with each entry representing a square block of size $n^2/64$. The entries in bold form the desired critical set. A critical set for case (iv) may be obtained by taking the transpose of the square just considered for the third listed case.

Finally, for case (v), the square must be isotopic to

$$\begin{array}{cccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & \mathbf{0} & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

with each entry representing a square block of size $n^2/64$. The entries in bold form the desired critical set. ∎

It is not hard to see that a row pattern of n^1 implies the same column pattern, but things can get very complicated very quickly when the number of nonisomorphic rows or columns increases. For $F^{(2)}(8;4)$ frequency squares, the possible row (or column) patterns are 8^1, $6^1 2^1$, $5^1 1^3$, 4^2, $4^1 2^2$, $4^1 1^4$, $3^2 1^2$, $3^1 2^1 1^3$, $3^1 1^5$, 2^4, $2^3 1^2$, $2^2 1^4$, $2^1 1^6$, and 1^8. Some patterns, such as $7^1 1^1$, are impossible in a frequency square. In addition, some combinations of row patterns and column patterns turn out not to be possible.

Determining the size of a critical set even for a single general $F^{(2)}(8;4)$ frequency square turns out to be a difficult problem. We conjecture that every such square has a critical set with no more than 20 entries. In Appendix A, we provide a set of examples that is not intended to be exhaustive.

4 Equiorthogonality Type I

The fact that corresponding rows and columns are both isomorphic gives a large amount of information about the nature of the squares and the relationship between them.

Lemma 4.1 *A pair of Type I equiorthogonal frequency squares must have $m > 2$.*

Proof: If $m = 2$ with corresponding rows and columns isomorphic, the resulting squares are isomorphic, not equiorthogonal; either the squares are identical, or one is a relabeling of the other with the 0's and 1's interchanged. ∎

In the $F^{(2)}(4;2)$ case, we have already seen that it is possible to have equiorthogonal frequency squares in which parallel rows or columns are not isomorphic to each other. However, any square with a Type I equiorthogonal mate satisfies a more stringent condition.

Proposition 4.2 *Any square with a Type I equiorthogonal mate must have all of its rows isomorphic and all of its columns isomorphic.*

Proof: Without loss of generality, assume that the rows and columns of a pair of Type I equiorthogonal squares have been permuted and the symbols have been relabeled so that the first row of both squares is $0,\ldots,0,1,\ldots,1,\ldots,(m-1),\ldots,(m-1)$ and the first column of the first square also is $0,\ldots,0,1,\ldots,1,\ldots,(m-1),\ldots,(m-1)$. Since corresponding columns are isomorphic, this means that the entries in rows 0 through $n/m-1$ of column 0 in the second square must be 0.

Let us now suppose that, out of rows 0 through $n/m-1$, there are k rows in which there are zeroes outside of the upper left block (i.e., the intersection of rows 0 through $n/m-1$ and columns 0 through $n/m-1$). In particular, in row ϵ_i let there be δ_i zeroes outside of the upper left block for $i = 1, \ldots, k$. By isomorphism of corresponding rows, the same will be true in the second square.

We find that the ordered pair $(0, 0)$ occurs $(n/m)^2 - \sum_{i=1}^{k} \delta_i$ times in the upper left block and $\sum_{i=1}^{k} \delta_i$ times in those rows but outside of the upper left block. This is a total of $(n/m)^2$, which means that there can be no others. If $k > 0$, however, there would have to be at least one column out of columns 0 to $n/m - 1$ with more zeroes below row $n/m - 1$, and isomorphism of corresponding columns would result in zeroes in the same location in the second square, yielding too many occurrences of $(0, 0)$. We conclude that $k = 0$ and both squares must have all zeroes in the upper left block. A similar argument can be used to show that the entire square organizes into such blocks. Since all rows consist of m blocks of n/m identical digits, they are all isomorphic; the same is true of the columns. ∎

Corollary 4.3 *Any square with a Type I equiorthogonal mate is isotopic to a square consisting of a square array of m^2 blocks, each of which is a square with n^2/m^2 elements. The converse is true if the block pattern corresponds to an $m \times m$ latin square with an orthogonal mate.*

Proof: The isotopism comes from the proof of Proposition 4.2. For the converse, consider the permuted original square to be an $m \times m$ latin square consisting of blocks. The second square can be constructed by taking a latin square, consisting of blocks, that is orthogonal to the permuted original square. Reversing the isotopism the same way in both squares will yield a Type I equiorthogonal mate for the first square. ∎

We now make an observation which is simple yet crucial.

Proposition 4.4 *If m^2 does not divide n, then a pair of equiorthogonal $F^{(2)}(n; n/m)$ frequency squares must be Type I equiorthogonal.*

Proof: It is impossible for corresponding rows or columns to be orthogonal if m^2 does not divide n. ∎

With that in mind, we now state the main theorem of this section. Recall that there is a simultaneous isotopism for Type I equiorthogonal squares that organizes them into a block pattern which can be viewed as an $m \times m$ latin square consisting of square blocks of n^2/m^2 elements. For brevity, we will refer to such a frequency square as being in the block pattern and call such a resulting latin square the corresponding latin square.

Theorem 4.5 *(a) If m^2 does not divide n, then there is a critical set for a pair of Type I equiorthogonal $F^{(2)}(n; n/m)$ frequency squares whose size is $2(n/m-1)(m-1)$ plus the size of a critical set for the first corresponding latin square plus the size of a critical set for the second corresponding latin square.*

(b) If m^2 divides n, then there is a critical set for a pair of Type I equiorthogonal $F^{(2)}(n; n/m)$ frequency squares whose size is $2(n/m - 1)(m - 1)$ plus the size of

a critical set for the first corresponding latin square plus $2n/m^2$ plus the size of a critical set for the second corresponding latin square.

Proof: (a) By Corollary 4.3, we know that the same permutation of rows and columns will take the frequency squares to the same block pattern; a critical set for the pair of corresponding orthogonal latin squares of order m then determines which symbol is in each block. To know exactly how to divide the square into blocks, it is necessary and sufficient to know $m-1$ of the column groupings (each of which has n/m columns) and $m-1$ of the row groupings (each of which has n/m rows). Once we have $m-1$ sets of identical columns (or rows), we know that the remaining n/m columns (or rows) must be identical. With less information about the groupings, the locations of the blocks will not be uniquely determined.

The size of the critical set for the first frequency square is determined as follows. We start with a critical set for the first corresponding latin square, which must have entries in at least $m-1$ rows and at least $m-1$ columns of the latin square. Let us consider any single entry with symbol j. All of the other $n/m-1$ rows identical to this row must have a j in the same column. Filling in those symbols will determine a set of n/m identical rows. We need to do that process for $m-1$ groups of columns and $m-1$ groups of rows (as we noted, the last column group and row group will then be known). So we need a total of $2(n/m-1)(m-1)$ plus the size of a critical set to uniquely determine all entries in the first frequency square. The critical set for the second latin square is then enough to determine the entire second frequency square.

(b) In this case, we need additional entries in the second frequency square to ensure that the squares are Type I equiorthogonal. We accomplish this by making sure that there is one ordered pair that occurs more than n/m^2 times in a row and in a column. This requires the size of a critical set for the second latin square, plus an additional n/m^2 to guarantee that corresponding columns are isomorphic and an additional n/m^2 to guarantee that corresponding rows are isomorphic.

Once we know that the frequency squares are Type I equiorthogonal, we can permute the rows and columns of the two frequency squares to obtain the same block pattern as described in part (a). As before, the first frequency square requires $2(n/m-1)(m-1)$ plus the size of a critical set for the first corresponding latin square. ∎

Here are examples illustrating each of these cases.

$$
\begin{array}{cccccc}
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & 2 & 2 \\
\mathbf{0} & 0 & 1 & 1 & 2 & 2 \\
\mathbf{1} & 1 & 2 & 2 & 0 & 0 \\
\mathbf{1} & 1 & 2 & 2 & 0 & 0 \\
2 & 2 & 0 & 0 & 1 & 1 \\
2 & 2 & 0 & 0 & 1 & 1
\end{array}
\quad
\begin{array}{cccccc}
\mathbf{0} & 0 & \mathbf{1} & 1 & 2 & 2 \\
0 & 0 & 1 & 1 & 2 & 2 \\
2 & 2 & 0 & 0 & 1 & 1 \\
2 & 2 & 0 & 0 & 1 & 1 \\
1 & 1 & 2 & 2 & 0 & 0 \\
1 & 1 & 2 & 2 & 0 & 0
\end{array}.
$$

A critical set for this pair requires 9 elements, 7 in the first square and 2 in the second. Since we have a pair of equiorthogonal frequency squares in which $m^2 = 9$ does not divide $n = 6$, Theorem 4.5 (a) applies. We know we have Type I and the squares are known to be in the block pattern; therefore, 3 entries plus $2(2-1)(3-1)$ for a total of 7 entries are enough to determine the first frequency square. Since the second frequency square must be in the same block pattern by the proof of Proposition 4.2, all we need is one entry inside each block of the corresponding latin square.

$$\begin{array}{ccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 2 & 2 & 2 \\ \mathbf{0} & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ \mathbf{0} & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ \mathbf{1} & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ \mathbf{1} & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ \mathbf{1} & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \quad \begin{array}{ccccccccc} \mathbf{0} & \mathbf{0} & 0 & \mathbf{1} & 1 & 1 & 2 & 2 & 2 \\ \mathbf{0} & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \end{array}.$$

In this pair, two entries are not enough for unique completability of the second square because we cannot tell if we have Type I or Type II. The two additional entries are enough to guarantee that corresponding rows and columns are isomorphic, therefore we have Type I.

5 Equiorthogonality Type II

As we have seen, in Type I equiorthogonal frequency squares, all of the rows of each square must be isomorphic and all of the columns of each square must be isomorphic. In Type II, there are many more possibilities. When the general case becomes unwieldy, we will use the case $F^{(2)}(8;4)$ for illustration.

Note that Type II includes cases where corresponding rows are isomorphic and corresponding columns are orthogonal as well as cases where corresponding rows are orthogonal and corresponding columns are isomorphic. Without loss of generality, we will confine our attention to isomorphic corresponding rows and orthogonal corresponding columns; the other case can be obtained by taking the transpose of both squares. However, in considering critical sets, we will never assume this *a priori*, just as we never assume Type I, II, or III.

As we will see, enumerating all of the possible cases even with relatively small parameters such as $n = 8$ and $m = 2$ is not an easy task. Here is one observation that is helpful in the case $m = 2$.

Proposition 5.1 *A pair of Type II equiorthogonal $F^{(2)}(n; n/2)$ frequency squares must have the same pattern (as defined in Section 3).*

Proof: From the definition of Type II equiorthogonality, corresponding rows are isomorphic, so isomorphism of two rows in the first square will automatically make the corresponding rows isomorphic in the second square.

Now let us assume that columns i and j are isomorphic in the first square. First suppose that both columns have a 0 in row 0. By isomorphism of corresponding rows, both columns must have the same symbol in row 0 of the second square (it may be a 0 or a 1).

Now consider row 1. Since columns i and j are isomorphic, locations $(1, i)$ and $(1, j)$ in the first square must have the same symbol. By isomorphism of corresponding rows, the same must be true in the second square. We will find that columns i and j are the same in both squares, therefore isomorphic.

If both columns have a 1 in row 0, the argument is exactly the same. If the symbols in locations $(0, i)$ and $(0, j)$ of the first square are different, a similar argument shows that the symbols in locations (k, i) and (k, j) of the second square must be different for all $k = 1, \ldots, n-1$, so the fact that $m = 2$ guarantees that each column will have a 0 whenever the other has a 1, therefore they are isomorphic. ∎

This is not necessarily true if $m > 2$. For example, Figure 1 illustrates a pair of Type II equiorthogonal $F^{(2)}(16; 4)$ frequency squares in which the pattern of the first square is $16^1\backslash 16^1$ while the pattern of the second square is $16^1\backslash 4^4$.

We will now investigate critical sets for pairs of Type II equiorthogonal frequency squares. As noted in Lemma 3.1, a critical set for one $F^{(2)}(n; n/2)$ frequency square must have at least $n^2/4$ entries. If $n = 8$, this results in 16 entries in a minimal critical set. As we noted at the end of Section 3, there are some squares that seem to need as many as 20 elements.

Once the first square is known, we will see that a critical set for the second $F^{(2)}(8; 4)$ equiorthogonal square needs 3, 4, 5, or 6 entries. The fact that no more than 6 are needed follows from a more general result:

Lemma 5.2 *In a pair of Type II equiorthogonal $F^{(2)}(n; n/m)$ frequency squares whose first square is known, the number of elements in a critical set for the second square is less than or equal to $(n/m)(m-1)^2 + n/m^2$.*

Proof: Without loss of generality, we may assume that row 0 and column 0 are arranged as $0, \ldots, 0, 1, \ldots, 1, \ldots, (m-1), \ldots, (m-1)$ in the first square. If we situate all of the symbols from 0 to $m-2$ in column 0 of the second square, we will be able to fill in the remaining n/m cells with symbol $m-1$. So $n - n/m = (m-1)n/m$ entries suffice in the first column. We will situate these entries so that column 0 of the second square is orthogonal to column 0 of the first square. The symbol 0 is then entered into n/m^2 additional cells in row 0 of the second square, guaranteeing that row 0 of the second square is isomorphic to row 0 of the first square. We can now be certain that the squares are Type II equiorthogonal.

By isomorphism of corresponding rows, columns 1 through $n/m - 1$ are identical to column 0. We now repeat the process used for column 0 to enter $(m-1)n/m$ entries in column n/m, and the next $n/m - 1$ columns are identical to it. This only needs to be done a total of $m-1$ times because the last n/m columns will be forced. Specificiation of these $(m-1)^2 n/m + n/m^2$ entries in the second square guarantees unique completability. ∎

0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
1	1	1	1	0	0	0	0	3	3	3	3	2	2	2	2
1	1	1	1	0	0	0	0	3	3	3	3	2	2	2	2
1	1	1	1	0	0	0	0	3	3	3	3	2	2	2	2
1	1	1	1	0	0	0	0	3	3	3	3	2	2	2	2
2	2	2	2	3	3	3	3	0	0	0	0	1	1	1	1
2	2	2	2	3	3	3	3	0	0	0	0	1	1	1	1
2	2	2	2	3	3	3	3	0	0	0	0	1	1	1	1
2	2	2	2	3	3	3	3	0	0	0	0	1	1	1	1
3	3	3	3	2	2	2	2	1	1	1	1	0	0	0	0
3	3	3	3	2	2	2	2	1	1	1	1	0	0	0	0
3	3	3	3	2	2	2	2	1	1	1	1	0	0	0	0
3	3	3	3	2	2	2	2	1	1	1	1	0	0	0	0

0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
1	1	1	1	2	2	2	2	3	3	3	3	0	0	0	0
2	2	2	2	3	3	3	3	0	0	0	0	1	1	1	1
3	3	3	3	0	0	0	0	1	1	1	1	2	2	2	2
0	0	0	0	2	2	2	2	3	3	3	3	1	1	1	1
1	1	1	1	0	0	0	0	2	2	2	2	3	3	3	3
2	2	2	2	3	3	3	3	1	1	1	1	0	0	0	0
3	3	3	3	1	1	1	1	0	0	0	0	2	2	2	2
0	0	0	0	3	3	3	3	1	1	1	1	2	2	2	2
1	1	1	1	2	2	2	2	0	0	0	0	3	3	3	3
2	2	2	2	1	1	1	1	3	3	3	3	0	0	0	0
3	3	3	3	0	0	0	0	2	2	2	2	1	1	1	1
0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
1	1	1	1	0	0	0	0	3	3	3	3	2	2	2	2
2	2	2	2	3	3	3	3	0	0	0	0	1	1	1	1
3	3	3	3	2	2	2	2	1	1	1	1	0	0	0	0

Figure 1: Two Type II equiorthogonal frequency squares with different patterns

Corollary 5.3 *In a pair of Type II equiorthogonal $F^{(2)}(8;4)$ frequency squares whose first square is known, the number of elements in a critical set for the second square is less than or equal to 6.*

Proposition 5.4 *In a pair of Type II equiorthogonal $F^{(2)}(8;4)$ frequency squares whose first square is known, the number of elements in a critical set for the second square is greater than or equal to 3, and the lower bound can be attained.*

Proof: There needs to be one row with at least three entries to guarantee that corresponding rows are isomorphic, establishing the lower bound.

Here is a pair of squares in which three entries form a critical set for the second square, illustrating that the lower bound can be attained:

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}
\quad \text{and} \quad
\begin{array}{cccccccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}.$$

First, note that a 1 in location $(1,0)$ yields the following partial completion:

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}
\quad \text{and} \quad
\begin{array}{cccccccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & &
\end{array}.$$

The only two ways to fill in rows 2 and 3 are as follows:

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}
\quad \text{and} \quad
\begin{array}{cccccccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & &
\end{array}$$

or

$$
\begin{array}{cccccccc}
0&0&0&0&1&1&1&1\\
0&0&1&1&0&0&1&1\\
0&1&0&1&0&1&0&1\\
0&1&1&0&0&1&1&0\\
1&0&0&1&1&0&1&0\\
1&0&1&0&1&0&0&1\\
1&1&0&0&1&1&0&0\\
1&1&1&1&0&0&0&0
\end{array}
\quad \text{and} \quad
\begin{array}{cccccccc}
\mathbf{0}&\mathbf{0}&\mathbf{0}&0&1&1&1&1\\
1&1&0&0&1&1&0&0\\
1&0&1&0&1&0&1&0\\
0&1&1&0&0&1&1&0
\end{array}.
$$

In both cases, the partial column with all zeroes forces completion of the rest of the square using isomorphism of corresponding rows, but the completions are not legal frequency squares. Therefore, there must be a 0 in location $(1,0)$ and also in location $(1,1)$.

By orthogonality of corresponding columns, entries $(2,0)$ and $(3,0)$ of the second square must be 1. Then, by isomorphism of corresponding rows, entries $(2,1)$ and $(3,1)$ must be 0, and the whole square can then be completed. ∎

Proposition 5.5 *The upper bound established in Corollary 5.3 can be attained.*

Proof: Here is an example of a pair where the upper bound is attained:

$$
\begin{array}{cccccccc}
0&0&0&0&1&1&1&1\\
0&0&0&0&1&1&1&1\\
0&0&0&0&1&1&1&1\\
0&0&0&0&1&1&1&1\\
1&1&1&1&0&0&0&0\\
1&1&1&1&0&0&0&0\\
1&1&1&1&0&0&0&0\\
1&1&1&1&0&0&0&0
\end{array}
\quad \text{and} \quad
\begin{array}{cccccccc}
\mathbf{0}&\mathbf{0}&\mathbf{0}&0&1&1&1&1\\
\mathbf{0}&0&0&0&1&1&1&1\\
1&1&1&1&0&0&0&0\\
1&1&1&1&0&0&0&0\\
\mathbf{0}&0&0&0&1&1&1&1\\
\mathbf{0}&0&0&0&1&1&1&1\\
1&1&1&1&0&0&0&0\\
1&1&1&1&0&0&0&0
\end{array}.
$$

Note that removal of any of the zeroes in the first row opens up the possibility of Type III equiorthogonality, and removal of any of the zeroes in the first column leaves ambiguities in the positioning of the rows. ∎

We would like to have conditions to determine whether there exists a critical set for the second square of size 3, 4, or 5. We now provide some sufficient conditions. Our convention for this series of results is that isotopic configurations will be considered to be the same but we will not relabel any entries. All entries of the first square are considered to be known; however, in the second square, only elements of the critical set are known until others are deduced.

We would like to note that, although we have usually arranged our equiorthogonal pairs with column 0 of the first square having all of the zeroes at the top, the configurations in the next two propositions can be presented most compactly with column 0 of the second square having all of the zeroes at the top.

Proposition 5.6 *In a pair of Type II equiorthogonal $F^{(2)}(8;4)$ frequency squares whose first square is known, if either of the following configurations is present, a* *critical set for the second square will need no more than four entries. The configurations are left justified in the applicable squares.*

$$\text{Configuration 1: First square: } \begin{array}{llll} 0 & 0 & 0 \\ 0 & 0 & \\ 1 & 0 & \\ 1 & 0 & \\ 0 & & \\ 0 & & \\ 1 & & \\ 1 & & \end{array} \text{; Second square: } \begin{array}{lll} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \\ 0 & 1 & \\ 0 & 1 & \\ 1 & & \\ 1 & & \\ 1 & & \\ 1 & & \end{array}$$

$$\text{Configuration 2: First square: } \begin{array}{llll} 0 & 1 & 0 & 0 \\ 0 & 1 & & \\ 1 & 1 & & \\ 1 & 1 & & \\ 0 & & & \\ 0 & & & \\ 1 & & & \\ 1 & & & \end{array} \text{; Second square: } \begin{array}{llll} \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & & \\ 0 & 0 & & \\ 0 & 0 & & \\ 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{array}$$

Proof: In Configuration 1, the three zeroes in row 0 of the second square guarantee isomorphic corresponding rows, after which we can conclude that location (1,1) has a 0. Orthogonality of corresponding columns gives the ones in locations (2,1) and (3,1), after which isomorphism of corresponding rows gives the zeroes in locations (2,0) and (3,0). With all zeroes in column 0 known, the second square can now be completed.

In Configuration 2, we again get isomorphism of corresponding rows from row 0. This forces the ones in locations (0,1) and (1,1). The completion of the second square proceeds similarly to Configuration 1.

In both cases, the four entries may form a critical set, or the critical set may have size 3 as demonstrated in Proposition 5.4, depending on the composition of the undefined entries in the pair. ∎

Proposition 5.7 *In a pair of Type II equiorthogonal $F^{(2)}(8;4)$ frequency squares whose first square is known, if either of the following configurations is present, a critical set for the second square will need no more than five entries. The configurations are left justified in the applicable squares.*

$$\text{Configuration 3: First square: } \begin{array}{llll} 0 & 1 & 0 & 0 \\ 0 & 1 & & \\ 1 & 1 & & \\ 1 & & & \\ 0 & & & \\ 0 & & & \\ 1 & & & \\ 1 & & & \end{array} \text{; Second square: } \begin{array}{llll} \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & & \\ 0 & 0 & & \\ \mathbf{0} & & & \\ 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{array}$$

$$\text{Configuration 4: First square: } \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \\ 1 & 0 & \\ 1 & & \\ 0 & & \\ 0 & & \\ 1 & & \\ 1 & & \end{array} \quad ; \text{ Second square: } \begin{array}{ccc} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \\ 0 & 1 & \\ \mathbf{0} & 0 & \\ 1 & & \\ 1 & & \\ 1 & & \\ 1 & & \end{array}$$

Proof: Isomorphism of corresponding rows is established immediately. In Configuration 3, we thereby deduce ones in locations (0,1) and (1,1). Orthogonality of corresponding columns forces a 0 in location (2,1), therefore also in location (2,0). With all of the zeroes in column 0 known, the square can be completed.

In Configuration 4, isomorphism of corresponding rows gives a 0 in location (1,1), orthogonality of corresponding columns gives a 1 in location (2,1), and isomorphism of corresponding rows gives a 0 in location (2,0).

As before, it is possible that a critical set may have fewer than these five entries. ∎

6 Conditions for the existence of a Type II equiorthogonal mate

Existence of an orthogonal mate for a $F^{(2)}(n; n/m)$ frequency square is straightforward in Type I but poses a complicated question in the other types. We now provide a result that addresses whether a $F^{(2)}(8; 4)$ may have an equiorthogonal mate with isomorphic corresponding rows and orthogonal corresponding columns.

Theorem 6.1 *Consider the following configuration:*

$$\begin{array}{cccccccc} . & . & 1 & 1 & 1 & . & . & . \\ . & . & 1 & 1 & 1 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & 1 & 1 & 1 & . & . & . & . \\ 1 & . & 1 & . & . & . & . & . \\ 1 & . & . & 1 & . & . & . & . \\ 1 & 1 & . & . & 1 & . & . & . \\ 1 & 1 & . & . & . & . & . & . \end{array}$$

A $F^{(2)}(8; 4)$ *frequency square with this configuration fails to have a Type II (with isomorphic corresponding rows) equiorthogonal mate. However, any proper subset of the configuration can be completed to a frequency square that admits such a Type II equiorthogonal mate.*

Proof: First, let us show that any square with the given configuration cannot have an equiorthogonal mate.

Without loss of generality, we may assume that the entry in location (0,0) of the second square is a 0. Because of isomorphism of corresponding rows, we may begin as follows:

0	.	1	1	1	.	.	.		0	.	1	1	1	.	.	.
0	.	1	1	1	.	.	.									
0	.	0	0	.	.	.	.									
0	1	1	1	.	.	.	.									
1	.	1	0	.	.	.	.									
1	.	0	1	.	.	.	.									
1	1	0	0	1	.	.	.									
1	1	0	0	.	.	.	.									

By orthogonality of corresponding columns, exactly one of locations (1,0), (2,0), or (3,0) in the second square is a 0. We will rule out all three possibilities.

If location (1,0) has a 0, we get the following:

0	.	1	1	1	.	.	.		0	.	1	1	1	.	.	.
0	.	1	1	1	.	.	.		0	.	1	1	1	.	.	.
0	.	0	0	.	.	.	.		1	.	1	1	.	.	.	.
0	1	1	1	.	.	.	.		1	0	0	0	.	.	.	.
1	.	1	0	.	.	.	.									
1	.	0	1	.	.	.	.									
1	1	0	0	1	.	.	.									
1	1	0	0	.	.	.	.									

Orthogonality of corresponding columns gives 0 in location (4,2) and location (5,3) of the second square:

0	.	1	1	1	.	.	.		0	.	1	1	1	.	.	.
0	.	1	1	1	.	.	.		0	.	1	1	1	.	.	.
0	.	0	0	.	.	.	.		1	.	1	1	.	.	.	.
0	1	1	1	.	.	.	.		1	0	0	0	.	.	.	.
1	.	1	0	.	.	.	.		0	.	0	1				
1	.	0	1	.	.	.	.		0	.	1	0				
1	1	0	0	1	.	.	.									
1	1	0	0	.	.	.	.									

and we will have to violate orthogonality of column 4 in the next step.

If location (2,0) in the second square is a 0, then locations (6,0) and (7,0) will have to have the other two zeroes in column 0 and orthogonality of column 1 will be violated. If location (3,0) in the second square is a 0, locations (4,0) and (5,0) must have the other two zeroes in column 0 and orthogonality of column 1 will again be violated. Therefore, no square with the given configuration can have an equiorthogonal mate.

The remainder of the proof proceeds by constructing a completion and an equiorthogonal mate for the configuration with any single entry removed. For example, if the 1 in location (6,4) is removed, here is one possible completion:

0	0	**1**	**1**	**1**	1	0	0
0	0	**1**	**1**	**1**	1	0	0
0	0	0	0	1	1	1	1
0	**1**	**1**	**1**	1	0	0	0
1	1	**1**	0	0	1	0	0
1	0	0	**1**	0	0	1	1
1	**1**	0	0	*0*	0	1	1
1	**1**	0	0	0	0	1	1

0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
1	1	1	1	0	0	0	0
1	0	0	0	0	1	1	1
0	0	0	1	1	0	1	1
0	1	1	0	1	1	0	0
1	1	0	0	0	0	1	1
1	1	0	0	0	0	1	1

For brevity, the other cases are listed in Appendix B. ∎

7 Equiorthogonality Type III and Topics for Further Research

When two frequency squares are Type III equiorthogonal, the patterns of the two squares do not need to be the same, and the sizes of the critical sets are noticeably larger. In some cases, the fact of equiorthogonality does not seem to reduce the size of the critical set much if at all; most of those gains seem to be in Type I and Type II. For now, we will undertake a brief examination of Type III.

Theorem 7.1 *Suppose an $F^{(2)}(8;4)$ frequency square has pattern $8^1\backslash 8^1$. Then a critical set for a Type III equiorthogonal mate consists of exactly 16 entries.*

Proof: Without loss of generality, we can consider the first square to be organized into the block pattern. Because corresponding rows and columns are orthogonal, the second square will be of the form

$$\begin{array}{cc} F_1 & F_2 \\ F_3 & F_4 \end{array}$$

where each F_i is a frequency square. We can permute the rows and columns of both squares so that the first row and column of the second square are

0	0	1	1	0	0	1	1
0							
1							
1							
0						.	
0							
1							
1							

A critical set for each F_i will have exactly 4 elements which cannot be reduced based on orthogonality considerations. A critical set for F_1 can be taken consisting of the entries in location (0,0), (0,1), (1,0), and one other. For F_2, we use locations (0,4) and three more, (4,0) and three more for F_3, and any critical set of size 4 for F_4. Note that the presence of the entries in locations (0,0), (0,1), (0,4), (1,0), and

(4,0) guarantee that neither corresponding rows nor corresponding columns can be isomorphic; therefore, we must have Type III. The union of those 16 entries forms a critical set for the second square. ∎

Much more investigation into Type III is possible. Other future research possibilities include generalizations of some of the results in this paper to larger m and n, specific conditions for a frequency square to have a minimal critical set of a specific size and for a pair of equiorthogonal frequency squares to have a combined critical set of a specific size, and determination of which patterns are possible for individual frequency squares and for pairs of equiorthogonal frequency squares.

Appendix A

Here we present critical sets for some frequency squares with a variety of row and column patterns. Note that the patterns $8^1\backslash 8^1$, $4^2\backslash 4^2$, $4^2\backslash 6^1 2^1$, $6^1 2^1\backslash 4^2$, and $6^1 2^1\backslash 6^1 2^1$ have critical sets of size 16 by Proposition 3.2.

Square								Square Type	Size of Critical Set
0	**0**	**0**	**0**	1	1	1	1		
0	**0**	**0**	**0**	1	1	1	1		
0	1	0	**0**	**1**	0	**1**	**1**		
0	1	0	1	**1**	0	0	**1**	$4^1 2^2\backslash 4^1 2^2$	17
1	1	1	1	0	0	0	0		
1	1	1	1	0	0	0	0		
1	**0**	1	1	0	1	0	0		
1	**0**	1	**0**	0	1	1	0		
0	**0**	**0**	**0**	1	1	1	1		
0	**0**	**0**	**0**	1	1	1	1		
0	1	1	0	0	**1**	**1**	0		
0	1	**0**	1	1	0	0	1	$4^1 2^2\backslash 3^2 1^2$	17
1	1	1	1	0	0	0	0		
1	1	1	1	0	0	0	0		
1	**0**	1	0	0	**1**	**1**	0		
1	**0**	**0**	1	1	0	0	1		
0	0	0	0	1	1	1	1		
0	0	0	**1**	1	1	1	**0**		
0	0	0	**1**	1	1	0	1		
0	0	0	0	1	1	1	1	$4^1 2^2\backslash 5^1 1^3$	17
1	1	1	1	**0**	**0**	**0**	**0**		
1	1	1	1	**0**	**0**	**0**	**0**		
1	1	1	0	**0**	**0**	1	**0**		
1	1	1	0	**0**	**0**	**0**	1		

Square								Square Type	Size of Critical Set
0	0	0	**0**	**1**	1	**1**	1		
0	0	1	1	0	**0**	1	1		
1	**1**	0	**0**	0	**0**	1	1		
1	**1**	0	**0**	**1**	1	0	**0**	$4^1 2^2 \backslash 2^4$	20
1	1	**0**	**0**	0	**0**	1	1		
1	**1**	**1**	1	0	0	0	0		
0	0	**1**	1	1	1	0	0		
0	0	**1**	1	1	1	0	0		

Appendix B

Now we provide the details of the proof of Theorem 6.1, where we show that the removal of any of the 1's in the given configuration admits a completion to a frequency square with an equiorthogonal mate. We already considered the removal of the 1 in location (6,4). Elements of the configuration are in bold face, with the changed element in italics.

	0	0	**1**	**1**	**1**	0	0	1		0	0	1	1	1	0	0	1
	0	0	**1**	**1**	*0*	0	1	1		0	0	1	1	0	0	1	1
	0	0	0	0	1	1	1	1		1	1	1	1	0	0	0	0
Location (1,4):	0	**1**	**1**	**1**	0	1	0	0		1	0	0	0	1	0	1	1
	1	1	**1**	0	0	0	0	1		0	0	0	1	1	1	1	0
	1	0	0	**1**	1	0	1	0		0	1	1	0	0	1	0	1
	1	**1**	0	0	**1**	1	0	0		1	1	0	0	1	1	0	0
	1	**1**	0	0	0	1	1	0		1	1	0	0	0	1	1	0

Location (0,4): same as (1,4) with rows 0 and 1 interchanged.

	0	0	**1**	**1**	**1**	1	0	0		0	0	1	1	1	1	0	0
	0	0	**1**	**1**	**1**	1	0	0		1	1	0	0	0	0	1	1
	0	0	0	0	1	1	1	1		0	0	0	0	1	1	1	1
Location (5,3):	0	**1**	**1**	**1**	0	0	0	1		1	0	0	0	1	1	1	0
	1	1	**1**	1	0	0	0	0		1	1	1	1	0	0	0	0
	1	0	0	*0*	0	1	1	1		0	1	1	1	1	0	0	0
	1	**1**	0	0	**1**	0	1	0		0	0	1	1	0	1	0	1
	1	**1**	0	0	0	0	1	1		1	1	0	0	0	0	1	1

	0	0	**1**	**1**	**1**	1	0	0	0	0	1	1	1	1	0	0
	0	0	**1**	**1**	**1**	1	0	0	1	1	0	0	0	0	1	1
	0	1	0	0	0	1	1	1	0	1	0	0	0	1	1	1
Location (3,3):	0	**1**	**1**	*0*	0	1	1	0	1	0	0	1	1	0	0	1
	1	0	**1**	0	0	0	1	1	1	0	1	0	0	0	1	1
	1	0	0	**1**	0	0	1	1	0	1	1	0	1	1	0	0
	1	**1**	0	1	**1**	0	0	0	1	1	0	1	1	0	0	0
	1	**1**	0	0	1	0	0	1	0	0	1	1	0	1	1	0

	0	0	**1**	**1**	**1**	1	0	0	0	0	1	1	1	1	0	0
	0	0	**1**	*0*	**1**	0	1	1	1	1	0	1	0	1	0	0
	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
Location (1,3):	0	**1**	**1**	**1**	0	0	0	1	1	0	0	0	1	1	1	0
	1	1	**1**	1	0	0	0	0	1	1	1	1	0	0	0	0
	1	0	0	**1**	0	1	1	0	0	1	1	0	1	0	0	1
	1	**1**	0	0	**1**	1	0	0	0	0	1	1	0	0	1	1
	1	**1**	0	0	0	0	1	1	1	1	0	0	0	0	1	1

Location (0,3): same as (1,3) with rows 0 and 1 interchanged.

Location (4,2): same as (5,3) with rows 4 and 5 and columns 2 and 3 interchanged.

Location (3,2): same as (3,3) with rows 4 and 5 and columns 2 and 3 interchanged.

Location (1,2): same as (1,3) with rows 4 and 5 and columns 2 and 3 interchanged.

Location (0,2): same as (1,2) with rows 0 and 1 interchanged.

	0	0	**1**	**1**	**1**	1	0	0	0	0	1	1	1	1	0	0
	0	0	**1**	**1**	**1**	1	0	0	1	1	0	0	0	0	1	1
	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
Location (7,1):	0	**1**	**1**	**1**	0	0	0	1	1	0	0	0	1	1	1	0
	1	1	**1**	0	0	0	0	1	1	1	1	0	0	0	0	1
	1	1	0	**1**	0	0	1	0	1	1	0	1	0	0	1	0
	1	**1**	0	0	**1**	0	1	0	0	0	1	1	0	1	0	1
	1	*0*	0	0	0	1	1	1	0	1	1	1	1	0	0	0

	0	0	**1**	**1**	**1**	1	0	0	0	0	1	1	1	1	0	0
	0	0	**1**	**1**	**1**	1	0	0	1	1	0	0	0	0	1	1
	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
Location (6,1):	0	**1**	**1**	**1**	0	1	0	0	1	0	0	0	1	0	1	1
	1	1	**1**	0	0	0	0	1	1	1	1	0	0	0	0	1
	1	1	0	**1**	0	0	1	0	1	1	0	1	0	0	1	0
	1	*0*	0	0	**1**	0	1	1	0	1	1	1	0	1	0	0
	1	**1**	0	0	0	0	1	1	0	0	1	1	1	1	0	0

	0	0	**1**	**1**	**1**	1	0	0	0	0	1	1	1	1	0	0
	0	0	**1**	**1**	**1**	1	0	0	1	1	0	0	0	0	1	1
	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
Location (3,1):	0	*0*	**1**	**1**	0	1	0	1	1	1	0	0	1	0	1	0
	1	1	**1**	0	0	0	0	1	1	1	1	0	0	0	0	1
	1	1	0	**1**	0	0	1	0	1	1	0	1	0	0	1	0
	1	**1**	0	0	**1**	0	1	0	0	0	1	1	0	1	0	1
	1	**1**	0	0	0	0	1	1	0	0	1	1	1	1	0	0

	0	0	**1**	**1**	**1**	1	0	0	0	0	1	1	1	1	0	0
	0	0	**1**	**1**	**1**	1	0	0	1	1	0	0	0	0	1	1
	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1
Location (7,0):	0	**1**	**1**	**1**	0	0	0	1	0	1	1	1	0	0	0	1
	1	0	**1**	0	1	0	0	1	0	1	0	1	0	1	1	0
	1	1	0	**1**	0	0	1	0	0	0	1	0	1	1	0	1
	1	**1**	0	0	**1**	0	1	0	1	1	0	0	1	0	1	0
	0	**1**	0	0	0	1	1	1	1	0	1	1	1	0	0	0

	0	0	**1**	**1**	**1**	1	0	0	0	0	1	1	1	1	0	0
	0	0	**1**	**1**	**1**	1	0	0	0	0	1	1	1	1	0	0
	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
Location (6,0):	0	**1**	**1**	**1**	1	0	0	0	1	0	0	0	0	1	1	1
	1	0	**1**	0	0	1	0	1	0	1	0	1	1	0	1	0
	1	0	0	**1**	0	1	1	0	0	1	1	0	1	0	0	1
	0	**1**	0	0	**1**	0	1	1	1	0	1	1	0	1	0	0
	1	**1**	0	0	0	0	1	1	1	1	0	0	0	0	1	1

$$\text{Location (5,0):}\quad \begin{array}{cccccccc} 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 0 & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 1 & 0 & 1 \\ \mathit{0} & 0 & 0 & \mathbf{1} & 0 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 1 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \qquad \begin{array}{cccccccc} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

Location (4,0): same as (5,0) with rows 4 and 5 and columns 2 and 3 interchanged.

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