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Spherical and planar folding tessellations by kites and equilateral triangles*

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Abstract

We prove that there is a unique folding tessellation of the sphere and an infinite family of folding tessellations of the plane with prototiles a kite and an equilateral triangle. Each tiling of this family is obtained by successive gluing of two patterns composed of triangles and kites, respectively. The combinatorial structure and the symmetry group is achieved.

1 Introduction

By a *folding tessellation* or *folding tiling* of the sphere S^2 (plane $\mathbb{R}^2$) we mean an edge-to-edge pattern of spherical (planar) geodesic polygons that fills the sphere (plane) with no gaps and no overlaps, and such that the "underlying graph" has even valency at any vertex and the sums of alternate angles around each vertex are π.

For simplicity, in what follows it is assumed that the text refers to both spherical and planar contexts, except in the cases indicated.

A tiling τ is said *k-isohedral* or *k-tile-transitive* if there are exactly k transitivity classes of tiles, with respect to the symmetry group of τ. Spherical tilings with two types of polygons are k-isohedral, with $k \geq 2$; in the planar case we may have an infinite number of isohedrality classes (aperiodic tilings). If the edges of τ form k transitivity classes, the tiling is called *k-homeotoxal*. A study of a special class of 2-homeotoxal tilings is presented in [7]. Besides, τ is *k-isogonal* if the group of symmetries acts k-transitively on the vertices of the tiling. Renault [9] shows that under some restrictions every 1-isogonal planar tiling is combinatorially isomorphic to a tiling by regular polygons.

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Folding tilings are strongly related to the theory of isometric foldings on Riemannian manifolds. In fact, the set of singularities of any isometric folding corresponds to a folding tiling, see [10] for the foundations of this subject.

The study of these special class of tessellations was initiated in [1] with a complete classification of all spherical monohedral folding tilings. Ten years latter Ueno and Agaoka [12] have established the complete classification of all triangular spherical monohedral tilings (without any restriction on angles).

Dawson has also been interested in special classes of spherical tilings, see [4], [5] and [6], for instance.

The complete classification of all spherical folding tilings by rhombi and triangles was obtained in 2005 [2]. A detailed study of the triangular spherical folding tilings by equilateral and isosceles triangles is presented in [3].

In this paper our interest is focused in folding tilings whose prototiles are a kite and a equilateral triangle in both spherical and planar contexts. It can be seen [2] that there are not any spherical folding tiling with prototiles a parallelogram (square, rectangle, rhombus or a parallelogram itself) and an equilateral triangle. However, we shall show that the set of all folding tilings of the sphere by kites and equilateral triangles is reduced to an element. In the planar case, an infinite family of folding tessellations with prototiles a kite and an equilateral triangle completely determined are achieved. Each tiling of this family is obtained by successive gluing of two patterns composed of triangles and kites, respectively.

For additional and detailed information on tilings, see [8].

2 Folding Tessellations by Kites and Equilateral Triangles

Here we discuss folding tessellations by kites and equilateral triangles. Statements for the spherical and planar cases are often analogous, and so we only distinguish the arguments throughout the text in the appropriate places.

A kite K (Figure 1-I) is a quadrangle with two congruent pairs of adjacent sides, distinct from each other. Let us denote by $(\alpha_1, \alpha_2, \alpha_1, \alpha_3)$, $\alpha_2 > \alpha_3$, the internal angles of K in cyclic order. The side lengths are denoted by a and b, with $a < b$. T denotes an equilateral triangle with internal angle α and side length c, see Figure 1-II.

We shall denote by $\Omega(K, T)$ the set, up to isomorphism and deformation, of all folding tilings of S^2 $(\mathbb{R}^2)$ whose prototiles are K and T.

Taking into account the nature of the prototiles K and T, we have

$$2\alpha_1 + \alpha_2 + \alpha_3 > 2\pi \quad \text{and} \quad \alpha > \frac{\pi}{3}, \text{ in the spherical case, and}$$

$$2\alpha_1 + \alpha_2 + \alpha_3 = 2\pi \quad \text{and} \quad \alpha = \frac{\pi}{3}, \text{ in the planar case.}$$

As $\alpha_2 > \alpha_3$ we also have

$$\alpha_1 + \alpha_2 > \pi.$$

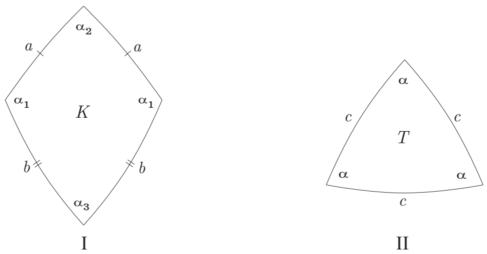


Figure 1: A kite and an equilateral triangle (spherical or planar).

We begin by pointing out that any folding tessellation with such prototiles has at least two cells congruent, respectively, to K and T, such that they are in adjacent positions in one and only one of the situations illustrated in Figure 2.

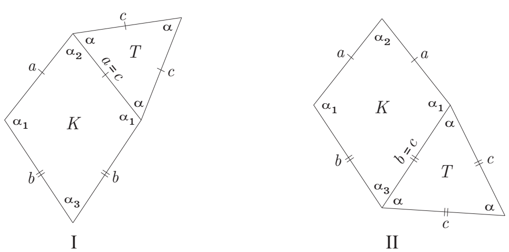


Figure 2: Distinct cases of adjacency.

After certain initial assumptions are made, it is usually possible to deduce sequentially the nature and orientation of most of the other tiles. Eventually either a complete tiling, or an impossible configuration proving that the hypothetical tiling fails to exist, is reached. In the diagrams that follows the order in which this deductions can be made is indicated by the numbering of the tiles. For $j \geq 2$, the location of tiling j can be deduced directly from the configurations of tiles $(1, 2, \ldots, j-1)$ and from the hypothesis that the configuration is part of a complete tiling, except where otherwise indicated.

Concerning the angles of the kite K we have necessarily one and only one of the following situations:

$$\alpha_1 \geq \alpha_2 > \alpha_3 \quad \text{or} \quad \alpha_2 > \alpha_1 \geq \alpha_3 \quad \text{or} \quad \alpha_2 > \alpha_3 > \alpha_1.$$

In the following propositions we consider separately each one of these cases.

Proposition 2.1. *If* $\alpha_1 \geq \alpha_2 > \alpha_3$, *then* *(i) in the spherical case,* $\Omega(K,T)$ *is composed of a single folding tiling, such that* $\alpha_2+\alpha_2=\pi$ *and* $\alpha_1+\alpha_3=\pi=\alpha_1+\alpha$*, with* $\alpha\approx 65.2°$*. A planar representation and a 3D representation of a such tiling are illustrated in Figures 7 and 9, respectively.*

(ii) in the planar case, $\Omega(K,T)$ *is the empty set.*

Proof. We consider separately the cases of adjacency illustrated in Figure 2.

1. Suppose firstly the case of adjacency of type I, performed by the condition $a=c$ (Figure 2-I).

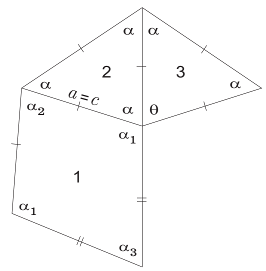


Figure 3: Planar representation.

With the labeling of Figure 3, one has $\theta\neq\alpha_1$ and $\theta\neq\alpha_2$, since $2\alpha_1\geq\alpha_1+\alpha_2>\pi$. On the other hand, taking into account the side lengths, we also have $\theta\neq\alpha_3$. And so $\theta=\alpha$. Now, it follows that $\alpha_1+\alpha=\pi$ or $\alpha_1+\alpha<\pi$.

1.1. Firstly, suppose that $\alpha_1+\alpha=\pi$. Then $\alpha_2>\alpha$ and the planar configuration illustrated in Figure 3 is extended as follows (Figure 4-I).

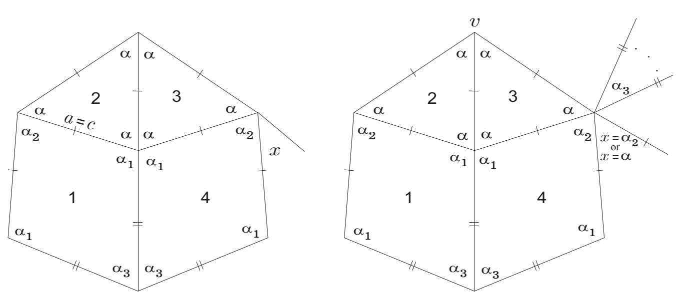


Figure 4: Planar representations.

Now, with the labeling of this figure, one has $x \neq \alpha_3$ (see side lengths) and $x \neq \alpha_1$ (observe that if $x = \alpha_1$, then such a vertex must have valency four, since $\alpha + \alpha_1 = \pi$; by analysis of the edge lengths, the other sum of alternate angles is $\alpha_2 + \alpha_1$; however this is a contradiction since $\alpha_2 + \alpha_1 > \pi$). Therefore $x = \alpha_2$ or $x = \alpha$.

(i) In the spherical context, in any of these two cases, as $\alpha_1 \geq \alpha_2 > \alpha > \frac{\pi}{3}$, the sum of alternate angles containing x must be of the form $x + \alpha + k\alpha_3 = \pi$, for some $k \geq 1$, Figure 4-II (observe that if $k = 0$, then $\alpha_2 + \alpha = \pi$, and so $\alpha_2 = \alpha_1 > \frac{\pi}{2}$; it follows that vertex v is forced to be surrounded by five consecutive angles α, which leads to an impossibility). Taking into account the side lengths, the remaining sum of alternate angles around a such vertex is $\alpha_2 + \alpha_1 + (k-1)\alpha_3 + \alpha_1$, which is greater than π, leading us to a contradiction.

(ii) In the planar context, beyond an analogous situation to the spherical case, we also have that if $x + \alpha = \pi$, with $x = \alpha_2$, then $\alpha_1 = \alpha_2 = \frac{2\pi}{3}$. As the sum of the internal angles of K is 2π, it follows that $\alpha_3 = 0$, which is not possible. Additionally, we also have the possibility of $x = \alpha$ and the sum of alternate angles containing x be of the form $x + \alpha + \alpha = \pi$; as the remaining sum of alternate angles must be $\alpha_2 + \rho + \rho$, with $\rho \in \{\alpha, \alpha_2\}$, which is greater than π, we also reach a contradiction in this case.

1.2. Suppose now that $\alpha_1 + \alpha < \pi$ (Figure 3). As $\alpha_1 \geq \alpha_2 > \alpha$, with $\alpha > \frac{\pi}{3}$ in the spherical case and $\alpha = \frac{\pi}{3}$ in the planar case, then we must have $\alpha_1 + \alpha + k\alpha_3 = \pi$, $k \geq 1$, as illustrated in Figure 5.

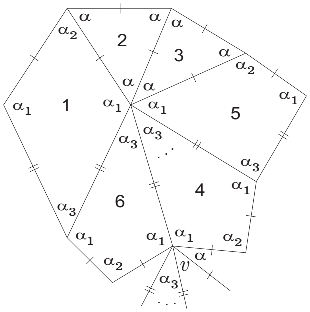


Figure 5: Planar representation.

Since $\alpha_1 + \alpha_2 > \pi$ and analyzing the edge lengths of the prototiles, one of the sums of alternate angles at vertex v (Figure 5) is also $\alpha_1 + \alpha + k\alpha_3 = \pi$. However at the same vertex we obtain $\alpha_1 + \alpha_1 + (k-1)\alpha_3 + \alpha_1 > \pi$, forced by the side lengths, which is an impossibility.

2. Here we consider the case of adjacency of type II, i.e., $b = c$ (Figure 6-I). With the labeling of this figure, we have necessarily $x = \alpha_3$ or $x = \alpha$.

2.1. Firstly suppose that $x = \alpha_3$, as indicated in Figure 6-II. It follows that $\alpha_1 + \alpha_3 = \pi$ or $\alpha_1 + \alpha_3 < \pi$.

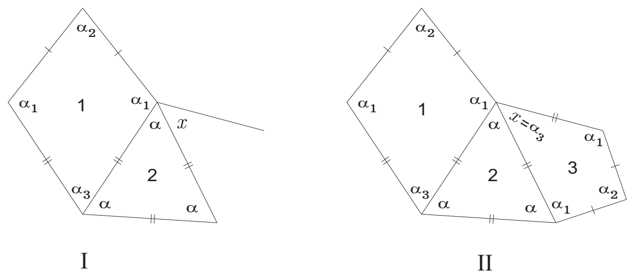


Figure 6: Planar representations.

2.1.1. Considering that $\alpha_1 + \alpha_3 = \pi$ we also obtain $\alpha_1 + \alpha = \pi$, and so $\alpha = \alpha_3$. Then, we have $\alpha_1 \geq \alpha_2 > \alpha = \alpha_3$, with $\alpha > \frac{\pi}{3}$ in the spherical case and $\alpha = \frac{\pi}{3}$ in the planar case. We reach an impossibility in the planar case, as $2\alpha_1 + \alpha_2 + \alpha_3 > 2\pi$. In the spherical context only vertices of valency four are allowed and the planar configuration extends in a unique way to obtain a closed planar representation as illustrated in Figure 7. Observe that the positioning of the first tiles forces the appearance of vertices with two angles α_2 in adjacent positions. By the relation between angles and attending to the side lengths, such vertices must be surrounded by four angles α_2, i.e., $\alpha_2 = \frac{\pi}{2}$.

As $\alpha_3 = \alpha$ and $b = c$, then the kite K is the union an equilateral triangle congruent to T with an isosceles triangle, say T', of angles $\frac{\pi}{2}$, θ, θ, for some θ, see Figure 8.

One has $\alpha + \theta = \alpha_1 = \pi - \alpha$, i.e., $\theta = \pi - 2\alpha$. Using the second spherical law of cosines, we obtain

$$\frac{\cos\alpha + \cos^2\alpha}{\sin^2\alpha} = \frac{\cos^2(2\alpha)}{\sin^2(2\alpha)}.$$

The unique solution of this equation in the interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ is $\alpha \approx 65.2°$. We also obtain $a \approx 31.8°$ and $b \approx 43.7°$. In Figure 9 a 3D representation of a such folding tiling is illustrated.

This tiling is generated from a spherical snub cube by adding a vertex in the center of each square, joining it to the square vertices and removing the edges of the square.

2.1.2. Now we shall suppose that $\alpha_1 + \alpha_3 < \pi$ (Figure 6-II). Taking into account the relation between angles, we must have $\alpha_1 + \alpha_3 + k_1\alpha + k_2\alpha_3 = \pi$, with $k_1 = 0$ or $k_1 = 1$, $k_2 \geq 0$ and $k_1 + k_2 \geq 1$, as represented in Figure 10-I (observe that if $k_1 \geq 2$, then $\alpha_1 + \alpha_3 + k_1\alpha + k_2\alpha_3 > \alpha_1 + 2\alpha > \frac{\pi}{2} + 2\frac{\pi}{3} > \pi$). The angle $y = \alpha_1$ is forced by the side lengths (note that α and α_3 are angles whose sides are equal).

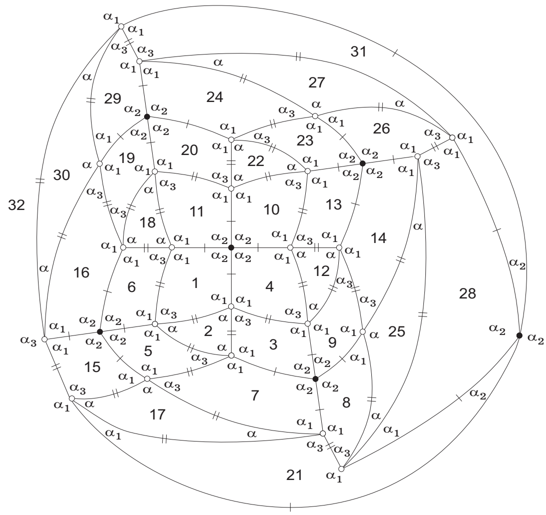


Figure 7: Extended planar representation.

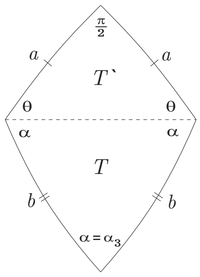


Figure 8: $K = T \cup T'$.

Such information leads to $k_1 = 1$ and tile 5 is also completely determined as indicated in Figure 10-II (observe that the sum of alternate angles containing $y = \alpha_1$ also contains α; as $\alpha_1 + \alpha$ must be less than π, we have $\alpha_1 + \alpha + k\alpha_3 = \pi$, $k \geq 1$, which determines tile 5). However we cannot avoid a contradiction at vertex w, since $2\alpha_1 > \alpha_1 + \alpha_2 > \pi$.

2.2. Finally we suppose that $x = \alpha$. Similarly to the previous case we shall consider separately the subcases $\alpha_1 + \alpha < \pi$ and $\alpha_1 + \alpha = \pi$.

2.2.1. If $\alpha_1 + \alpha < \pi$, then $\alpha_1 + \alpha + k\alpha_3 = \pi$, for some $k \geq 1$. This situation is illustrated in Figure 11. An analogous analysis to the one considered in 2.1.2. lead us to an impossibility at vertex w'.

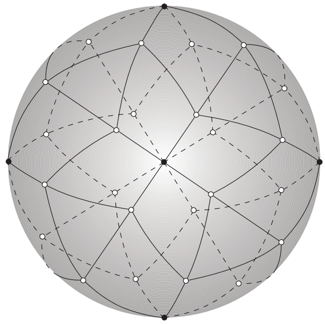


Figure 9: 3D representation.

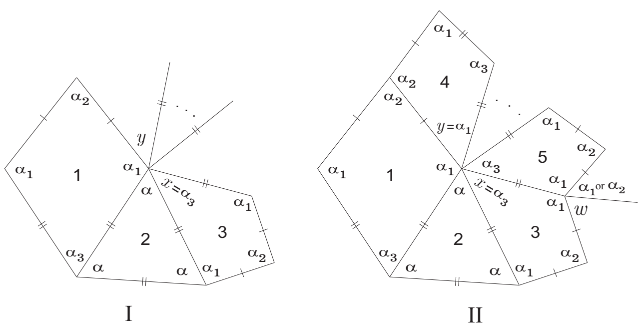


Figure 10: Planar representations.

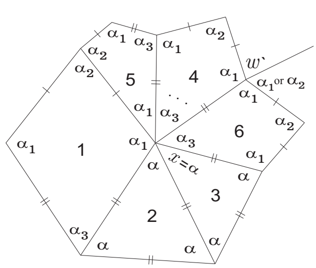
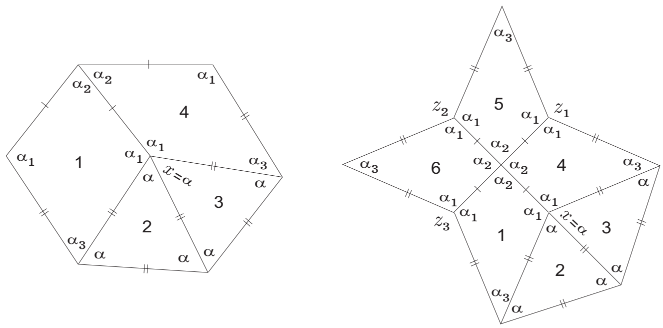


Figure 11: Planar representation.

2.2.2. If $\alpha_1 + \alpha = \pi$, then $\alpha_2 > \alpha$ and we get the configuration illustrated in Figure 12-I. A vertex enclosed by two angles α_2 in adjacent positions takes place. By the analysis of the side lengths and the relation of angles, we must have one vertex surrounded by four angles α_2, i.e., $\alpha_2 = \frac{\pi}{2}$, and the configuration extends a bit more to get the one illustrated in Figure 12-II.



I II

Figure 12: Planar representations.

With the labeling of this figure, at each vertex z_i $(i = 1, 2, 3)$ we have necessarily $\alpha_1 + k\alpha_3 = \pi$, for some $k \geq 1$, or $\alpha_1 + \alpha = \pi$. However, the first situation leads to the appearance of two angles α_1 in adjacent positions as in Figures 10-II and 11, and so we reach a contradiction. Therefore $\alpha_1 + \alpha = \pi$ at any vertex z_i $(i = 1, 2, 3)$, and the configuration illustrated in Figure 12-II is extended to obtain the one in Figure 13.

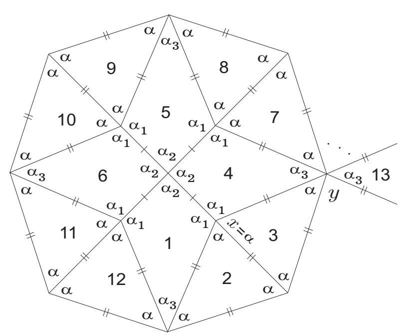


Figure 13: Planar representation.

A vertex enclosed, in cyclic order, by the sequence $(\alpha, \alpha_3, \alpha, \dots)$ must satisfy $2\alpha + k\alpha_3 = \pi$, for some $k \geq 1$ $(\alpha \geq \frac{\pi}{3} > \alpha_3)$; note that at this vertex, and in the planar context, we cannot have a sum of alternate angles containing three angles α, as the remaining sum will be lower than π (note that $\alpha_2 = \frac{\pi}{2}$, $\alpha_1 = \pi - \alpha = \frac{2\pi}{3}$ and $2\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$ implies $\alpha_3 = \frac{\pi}{6}$). Therefore, analyzing Figure 13, we have $y = \alpha_3$ or $y = \alpha$. The first case $(y = \alpha_3)$ implies, once more, two angles α_1 in adjacent positions in a similar situation that has already been analyzed and leads to a contradiction.

The second case $(y = \alpha)$ leads to the configuration illustrated in Figure 14, with $\alpha = \frac{\pi}{3}$ and $\alpha_3 = \frac{\pi}{6}$. However, since we cannot have two angles α_3 in adjacent positions, we obtain an absurdity at vertex v. □

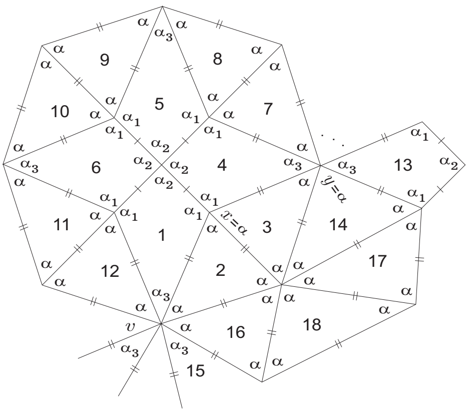


Figure 14: Planar representation.

Proposition 2.2. *If $\alpha_2 > \alpha_1 \geq \alpha_3$, then*

(i) in the spherical case, $\Omega(K,T)$ is the empty set.

(ii) in the planar case, $\Omega(K,T)$ is composed of an infinite family of folding tilings, such that the angles of the prototiles satisfy $\alpha_1 + \alpha_1 = \pi$, $\alpha_2 + \alpha = \pi$ and $\alpha_3 + \alpha + \alpha = \pi$, with $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = \frac{2\pi}{3}$ and $\alpha_3 = \alpha = \frac{\pi}{3}$.

Each tiling of this family is obtained by successive gluing of two patterns composed of triangles and kites, respectively, illustrated in Figure 25, where the only constraint is that it must has at least one of each of the patterns. An example of one tessellation of this family is illustrated in Figure 26.

Proof. **1.** We begin by considering the case of adjacency of type I, i.e., $a = c$. We prove that there is no tiling in this case.

As $2\alpha_2 > \alpha_2 + \alpha_1 > \pi$ and taking into account the side lengths, the configuration illustrated in Figure 2-I extends to the one that follows (Figure 15-I).

It follows that $\alpha_2 + \alpha < \pi$ or $\alpha_2 + \alpha = \pi$.

1.1. If $\alpha_2 + \alpha < \pi$, then necessarily $\alpha_2 + \alpha + k\alpha_3 = \pi$, for some $k \geq 1$ (Figure 15-II). And so $\alpha_2 > \alpha_1 > \alpha \geq \frac{\pi}{3} > \alpha_3$. Taking into account the edge lengths, the other sum of alternate angles around that vertex must be $\alpha_1 + \alpha + \alpha_1 + \ldots > \pi$, which is a contradiction.

1.2. The configuration that illustrates the case $\alpha_2 + \alpha = \pi$ is presented in Figure 16, in which $\alpha_2 > \alpha_1 > \alpha \geq \frac{\pi}{3}$. With the labeling of this figure, one has $x = \alpha_1$ or $x = \alpha$.

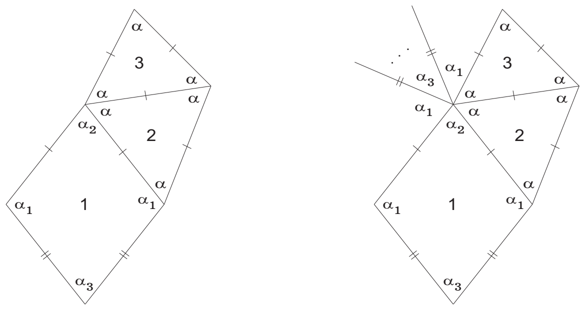


Figure 15: Planar representations.

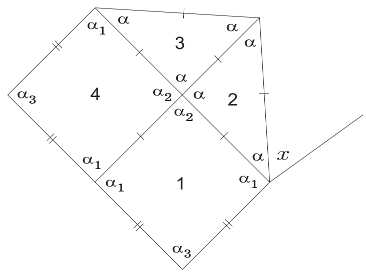


Figure 16: Planar representation.

1.2.1. If $x = \alpha_1$ and $\alpha_1 + \alpha_1 = \pi$, i.e., $\alpha_1 = \frac{\pi}{2}$, then we would also have $\alpha + \alpha_3 = \pi$, by analysis of the side lengths. This is an impossibility since $\alpha < \alpha_1$ and $\alpha_3 \leq \alpha_1$. Therefore $\alpha_1 < \frac{\pi}{2}$, and consequently $2\alpha_1 + k\alpha_3 = \pi$, for some $k \geq 1$ (Figure 17-I). It follows that the remaining sum of alternate angles around such vertex is $\alpha + (k+1)\alpha_3 = \alpha + k\alpha_3 + \alpha_3 < \alpha_1 + k\alpha_3 + \alpha_1 = \pi$, leading us to an incompatibility (note that $\alpha_3 < \alpha_1$ in this case).

1.2.2. Consider now that If $x = \alpha$ (Figure 17-II).

(i) In the spherical context, we have $\frac{\pi}{3} < \alpha < \alpha_1$ and at vertex v we have necessarily $2\alpha + k\alpha_3 = \pi$, $k \geq 1$. The remaining sum of alternate angles around v must be $\alpha_1 + \alpha + \alpha_1 + (k-1)\alpha_3 > 3\,\frac{\pi}{3} = \pi$, which is an absurdity.

(ii) In the planar context, we have $\alpha = \frac{\pi}{3}$ and $\alpha_2 = 2\alpha$; $\alpha_1 + \alpha < \alpha_2 + \alpha = \pi$ implies that $\alpha_1 + \alpha + t\alpha_3 = \pi$, $t \geq 1$. Therefore $\alpha_2 + 2\alpha_1 + \alpha_3 = 2\pi$ (sum of the internal angles of K) and $2\alpha_1 + 2\alpha + 2t\alpha_3 = 2\pi$, and so $\alpha_3 = 2t\alpha_3$, i.e., $t = \frac{1}{2}$, which is an impossibility.

2. Consider now the case of adjacency of type II ($b = c$). With the labeling of Figure 18-I, we have $\theta = \alpha_1$ or $\theta = \alpha_2$.

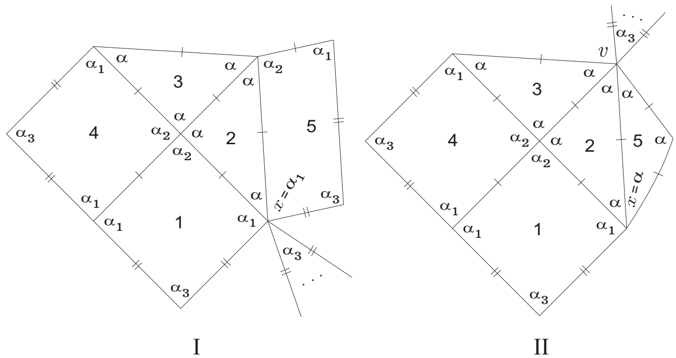


Figure 17: Planar representations.

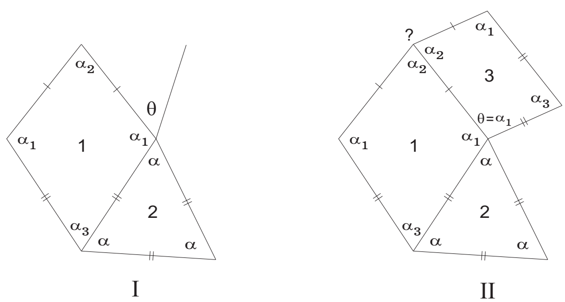


Figure 18: Planar representations.

2.1. If $\theta = \alpha_1$, then we get two angles α_2 in adjacent positions as represented in Figure 18-II. However, any attempt to place new tiles around such vertex in order to full fill the angle folding relation fails (note that $2\alpha_2 > \alpha_2 + \alpha_1 > \pi$).

2.2. It follows that $\theta = \alpha_2$ (Figure 19).

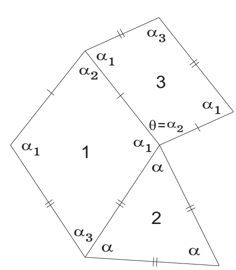


Figure 19: Planar representation.

As usually we separate the cases $\alpha_2 + \alpha < \pi$ and $\alpha_2 + \alpha = \pi$.

2.2.1. Firstly consider that $\alpha_2 + \alpha < \pi$ (Figure 20-I). It follows that $\alpha_2 + \alpha + k\alpha_3 = \pi$, $k \geq 1$. Note that it is impossible to have $\alpha_2 + 2\alpha < \pi$. And so $\alpha_2 > \alpha_1 > \alpha \geq \frac{\pi}{3} > \alpha_3$. On the other hand, analyzing the side lengths, we also conclude that $2\alpha_1 + k\alpha_3 = \pi$.

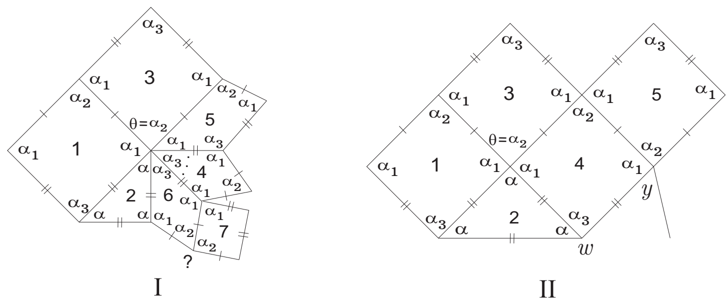


Figure 20: Planar representations.

Once again we obtain two angles α_2 in adjacent positions, which is an impossibility (as seen in 2.1 ($\theta = \alpha_1$)).

2.2.2. If $\alpha_2 + \alpha = \pi$ (Figure 19), then also $\alpha_1 + \alpha_1 = \pi$, i.e., $\alpha_1 = \frac{\pi}{2}$, as illustrated in Figure 20-II. With the labeling used in this configuration, we have necessarily $y = \alpha_3$ or $y = \alpha$.

2.2.2.1. Considering $y = \alpha_3$, at vertex w we have $\alpha_1 + \alpha < \alpha_2 + \alpha = \pi$, and consequently $\alpha_1 + \alpha + k\alpha_3 = \pi$, for some $k \geq 1$. Now, a vertex surrounded by two adjacent angles α_2 takes place, leading us to an impossibility (Figure 21-I).

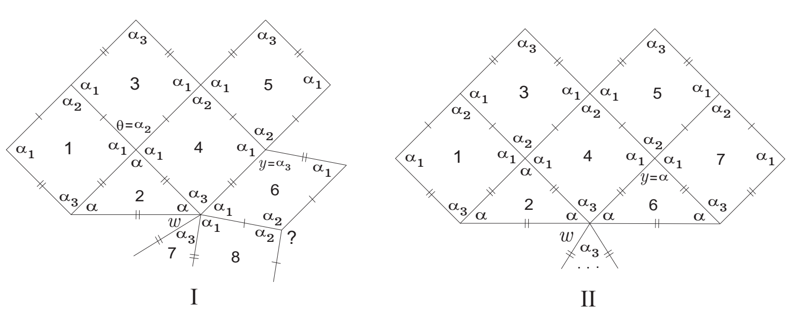


Figure 21: Planar representations.

2.2.2.2. Suppose now that $y = \alpha$ in Figure 20-II.

(i) In the spherical context, the sum of alternate angles containing α at vertex w is of the form $2\alpha + k\alpha_3 = \pi$, $k \geq 1$, and also $\alpha_2 > \alpha_1 = \frac{\pi}{2} > \alpha > \frac{\pi}{3} > \alpha_3$ (Figure 21-II).

Firstly, consider that $k = 1$, i.e., $2\alpha + \alpha_3 = \pi$ (in vertex w). Then the previous configuration extends in a unique way to get the one that follows (Figure 22). The thick dark lines are three spherical geodesics that never meet, which is an absurdity.

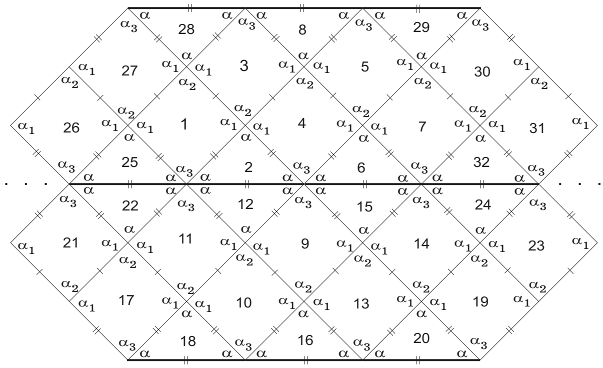


Figure 22: Planar representation.

Consider now that $k \geq 2$. Then, we obtain the configuration that follows (Figure 23). Since $\alpha_1 + \alpha_2 > \pi$, we obtain an absurdity, as illustrated in the planar representation.

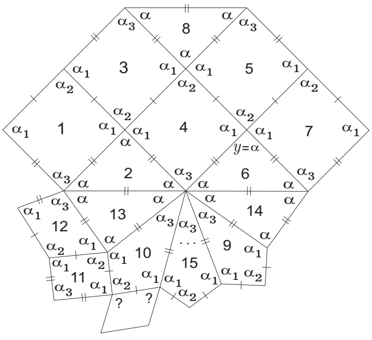


Figure 23: Planar representation.

(ii) In the planar context, the sum of alternate angles containing α at vertex w could be $2\alpha + k\alpha_3 = \pi$, $k \geq 1$, or $3\alpha = \pi$. The case $2\alpha + k\alpha_3 = \pi$, with $k \geq 2$, is analogous to the spherical case and also leads to an absurdity. The remaining cases lead to $\alpha_2 = \frac{2\pi}{3}$, $\alpha_1 = \frac{\pi}{2}$ and $\alpha = \alpha_3 = \frac{\pi}{3}$ (Figure 24). It follows that any folding tiling in this case is obtained by combining the two patterns illustrated in Figure 25 (the planar representation of Figure 24 is obtained by fitting the two patterns in Figure 25), where the only constraint is that it must has at least one of each of the patterns. An example of one tessellation of this family is illustrated in Figure 26.

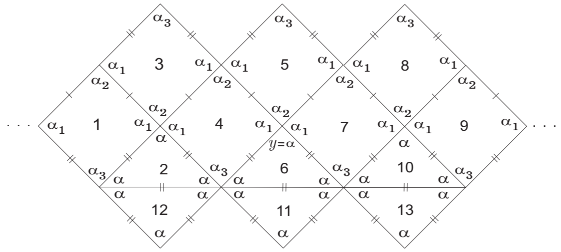


Figure 24: Planar representation.

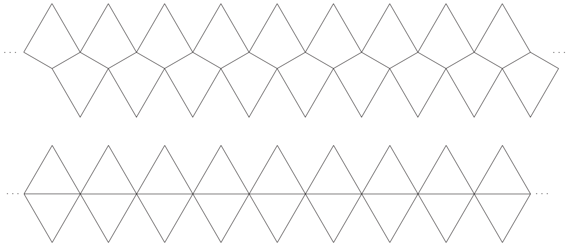


Figure 25: Patterns.

□

Proposition 2.3. *If $\alpha_2 > \alpha_3 > \alpha_1$, then $\Omega(K, T)$ is the empty set for both spherical and planar cases.*

Proof. We have $2\alpha_2 > \alpha_2 + \alpha_3 > \alpha_2 + \alpha_1 > \pi$, and so $\alpha_2 + \alpha = \pi$ (note that $\alpha_2 + 2\alpha > \frac{\pi}{2} + \frac{2\pi}{3} > \pi$), in which $\alpha_2 > \alpha_3 > \alpha_1 > \alpha \geq \frac{\pi}{3}$. Now, if a spherical folding tiling satisfying these conditions exists, then $2\alpha_3 = \pi$ or $\alpha_3 + \alpha_1 = \pi$. However, if the first situation occurs then $\alpha_1 + \rho = \pi$ has no solution with $\rho \in \{\alpha_1, \alpha_2, \alpha_3, \alpha\}$, which is an impossibility. Therefore $\alpha_3 + \alpha_1 = \pi$ as illustrated in Figure 27. Observe that, without loss of generality, the side lengths of the angle α_1 in tile 2 can be exchanged.

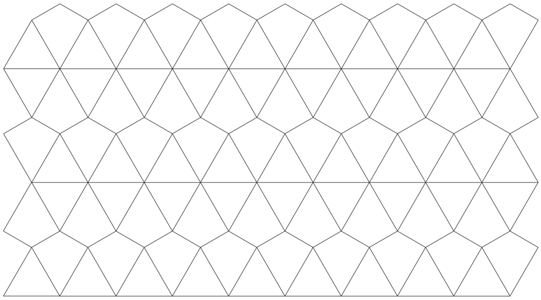


Figure 26: Planar f-tiling.

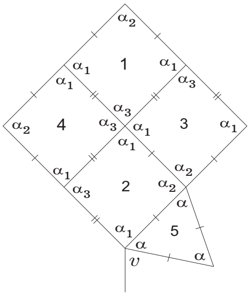


Figure 27: Planar representation.

Nevertheless, taking into account the side lengths of the prototiles, we obtain an absurdity as indicated in the planar configuration (vertex v). □

The symmetries of the spherical folding tiling with prototiles a kite and an equilateral triangle (Figure 9) that fix a dark vertex v (and its antipode) are generated by the rotation through an angle $\frac{\pi}{2}$ around the axis by $\pm v$. On the other hand, for any two dark vertices v_1 and v_2, there is a symmetry of this tiling sending v_1 into v_2. It follows that the symmetry group has exactly 24 elements, and they form the group of orientation preserving symmetries (rotations) of the cube, sometimes referred as S_4. Finally, such tiling is 2-isohedral (one transitivity class with 24 kites and one transitivity class with 8 triangles) and 2-isogonal.

Concerning to the example of the planar tiling illustrated in Figure 26, one has reflections in two perpendicular directions and two rotations of order two whose centers are on a reflection axis. It also has rotations of order two whose centers are not on a reflection axis. Following the notation used in [11], the group of symmetries of this planar tiling is the wallpaper group *cmm*.

References

[1] A.M. Breda, A class of tilings of S^2, *Geometriae Dedicata* 44 (1992), 241–253.

[2] A.M. Breda and A.F. Santos, Dihedral f-tilings of the sphere by rhombi and triangles, *Discrete Math. Theor. C.* 7 (2005), 123–140.

[3] A.M. Breda, P.S. Ribeiro and A.F. Santos, A Class of Spherical Dihedral F-Tilings, *Europ. J. Combin.* 30(1) (2009), 119–132.

[4] R.J. Dawson, Tilings of the sphere with isosceles triangles, *Discrete Comput. Geom.* 30 (2003), 467–487.

[5] R.J. Dawson and B. Doyle, Tilings of the sphere with right triangles I: the asymptotically right families, *Electron. J. Combin.* 13 (1) (2006), #R48.

[6] R.J. Dawson and B. Doyle, Tilings of the sphere with right triangles II: the $(1,3,2)$, $(0,2,n)$ family, *Electron. J. Combin.* 13 (1) (2006), #R49.

[7] B. Grünbaum and G.C. Shephard, The 2-homeotoxal tilings of the plane and the 2-sphere, *J. Combin. Theory Ser. B* 34 (2) (1983), 113–150.

[8] B. Grünbaum and G.C. Shephard, *Tilings and Patterns*, W.H. Freeman and Company, New York, 1986.

[9] D. Renault, The uniform locally finite tilings of the plane, *J. Combin. Theory Ser. B* 98 (4) (2008), 651–671.

[10] S.A. Robertson, Isometric folding of Riemannian manifolds, *Proc. Royal Soc. Edinburgh Sect. A* 79 (1977), 275–284.

[11] D. Schattschneider, The plane symmetry groups: their recognition and notation, *Amer. Math. Monthly* 85 (6) (1978), 439–450.

[12] Y. Ueno and Y. Agaoka, Classification of tilings of the 2-dimensional sphere by congruent triangles, *Hiroshima Math. J.* 32 (2002), 463–540.

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