AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **51** (2011), Pages 243–257

On twizzler, zigzag and graceful terraces

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Abstract

We present several constructions for terraces for cyclic groups that are zigzag terraces, graceful terraces or both. These allow us to complete the classification of imperfect twizzler terraces, unify several previously unrelated constructions of terraces from the literature, find terraces for some non-cyclic abelian groups, and show that for any given r the Oberwolfach Problem with parameters (r, r, s) has a cyclic 1-rotational solution for all sufficiently large odd s.

1 Introduction

Terraces for cyclic groups were first used to construct quasi-complete Latin squares [17]. Since then they have been generalised to arbitrary groups [2] and become objects of interest in their own right. Our first goal in this paper is to consider "zigzag" terraces for cyclic groups to generalise results of [14] and to provide a unifying view of some terraces in the literature that seem otherwise unrelated. A crucial tool is the "graceful" terrace, derived from a graceful labelling of a path. We then move to apply these methods to two problems. First, we construct terraces for cyclic groups of order a multiple of 4 that may be used to build terraces for some non-cyclic groups. Second, we construct a family of graceful terraces that lead to new cyclic 1-rotational solutions of some instances of the Oberwolfach problem.

Since we only consider terraces for abelian groups, we restrict the definition to this case. The more general definition may be found in [2] or [10]. Let A be an additively-written abelian group of order n and let $\mathbf{a} = (a_1, a_2, \ldots a_n)$ be an arrangement of the elements of A. Define $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ by $b_i = a_{i+1} - a_i$ for $1 \leq i \leq n-1$. If $\mathbf{b}$ contains exactly one occurrence of each involution of A and exactly two occurrences from each set $\{x, -x : 2x \neq 0\}$ then $\mathbf{a}$ is a *terrace* for A and $\mathbf{b}$ is the associated 2-*sequencing* of A. If $\mathbf{b}$ contains every non-zero element of A exactly once then the terrace $\mathbf{a}$ is *directed* and $\mathbf{b}$ is called a *sequencing*.

Example 1 *The sequence*

$$0 \;\; 1 \;\; (n-1) \;\; 2 \;\; (n-2) \;\; \ldots \;\; \lceil n/2 \rceil$$

is a terrace for $\mathbb{Z}_n$ *(the additively written cyclic group of order* n *on the symbols* $\{0, 1, \ldots, n-1\}$*) called the* Lucas-Walecki-Williams terrace*, or* LWW terrace*. See [14] or [3] for a brief summary of its history.*

There are several straightforward methods for transforming one terrace for A to another: a constant can be added to each element, called a *translate* of the terrace; an automorphism of A can be applied to each element; and/or the terrace can be reversed [2]. Each terrace has a translate in which the first element is 0, such a terrace is called *basic*. As a particular example of an automorphism, each element of a terrace for an abelian group can be negated. In the next two sections we frequently use the *negated LWW terrace*:

$$0 \;\; (n-1) \;\; 1 \;\; (n-2) \;\; 2 \;\; \ldots \;\; \lfloor n/2 \rfloor.$$

Preece [14] recently investigated zigzag terraces. The notion of a zigzag terrace provides a unifying theme for previously unlinked terraces. A *zigzag* is a sequence of the form $(x, y, x+c, y-c, x+2c, y-2c, \ldots)$ and a *zigzag terrace* is a terrace that may be divided into zigzag subsequences (with possibly a small number of non-zigzag elements). The negated LWW terrace consists of a single zigzag.

In the next section we consider "twizzler terraces" and "imperfect twizzler terraces" for cyclic groups, concepts introduced by Preece [14], and deduce exactly when imperfect twizzler terraces exist. In Section 3 we consider how terraces that we call "graceful terraces" may be glued together to provide new families of zigzag terraces. These various methods for constructing terraces provide a more general framework for considering terrace constructions. We see that several terraces for $\mathbb{Z}_n$ from the literature may be viewed as special cases of these methods.

In Section 4 we consider the extension theorem for terraces for abelian groups in [12]. There is an error in a minor branch of the construction methods given there. We construct a particular type of terrace for $\mathbb{Z}_{4m}$ that allows us to circumvent this branch of the argument and use the main result of that paper to produce terraces for all of the groups claimed.

Finally, in Section 5, we turn to the Oberwolfach problem. The *Oberwolfach problem* for parameters $(t_1, t_2, \ldots, t_k)$, denoted $\mathrm{OP}(t_1, t_2, \ldots, t_k)$, asks for a decomposition of the complete graph on v vertices, where $v = t_1 + t_2 + \cdots + t_k$, such that each element of the decomposition is a 2-factor with cycles of lengths $t_1, t_2, \ldots, t_k$. The only instances known not to have a solution are $\mathrm{OP}(4,5)$ and $\mathrm{OP}(3,3,5)$; [4] surveys what else is known.

A solution to the Oberwolfach Problem is *1-rotational* under the action of a group G if G is an automorphism group of the solution which acts by fixing one vertex and is sharply transitive on the remainder [6]. We construct a family of graceful zigzag terraces that have a "match-point" in an even position which, when combined with known results, shows that for any given r a 1-rotational solution to $\mathrm{OP}(r, r, s)$ under the action of a cyclic group exists for all sufficiently large odd s.

2 Twizzler terraces

Twizzler terraces were described by the first author in private communications while considering graceful labellings of paths in 2005. Preece named and generalised the concept in [14]. In this section we define twizzler terraces and prove some new results about imperfect twizzler terraces (Preece's generalisation). In the next section they play a role in the construction of new families of zigzag terraces.

Let $n = pq$ for any integers p and q. The *p-twizzler terrace* for $\mathbb{Z}_n$ is obtained as follows: take the negated LWW terrace for $\mathbb{Z}_n$; divide it into q sections each of length p; reverse ("twizzle") each of the sections. It is routine to check that this sequence is indeed a terrace for $\mathbb{Z}_n$.

Example 2 *Consider $\mathbb{Z}_{20}$. The 4-twizzler terrace is (we delimit the sections that have been "twizzled" with underbraces):*

$$\underbrace{18\ 1\ 19\ 0}\ \underbrace{16\ 3\ 17\ 2}\ \underbrace{14\ 5\ 15\ 4}\ \underbrace{12\ 7\ 13\ 6}\ \underbrace{10\ 9\ 11\ 8}.$$

The 5-twizzler terrace is:

$$\underbrace{2\ 18\ 1\ 19\ 0}\ \underbrace{15\ 4\ 16\ 3\ 17}\ \underbrace{7\ 13\ 6\ 14\ 5}\ \underbrace{10\ 9\ 11\ 8\ 12}.$$

Let $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ be an arrangement of the integers $\{0, 1, \ldots n-1\}$ and let $\mathbf{h} = (h_1, h_2, \ldots, h_{n-1})$ where $h_i = |g_{i+1} - g_i|$. If the sequence $\mathbf{h}$ contains each of the integers between 1 and $n-1$ inclusive then $\mathbf{g}$ is a *graceful labelling* of the path P_n (Note: the usual definition has the elements of $\mathbf{g}$ in the range $1 \le g_i \le n$, but this definition is naturally equivalent.) The reverse of a graceful labelling of a path is also a graceful labelling of the path. The *complement* of a graceful labelling of a path is obtained by subtracting each element from $n-1$. This too is a graceful labelling of the path.

Taking a graceful labelling of P_n and considering the symbols to be those of $\mathbb{Z}_n$ gives a terrace for $\mathbb{Z}_n$ [11]. We call those terraces that are obtainable in this manner *graceful* terraces. The negated LWW terrace for $\mathbb{Z}_n$ is graceful.

Theorem 1 *Let $n = pq$. The p-twizzler terrace for $\mathbb{Z}_n$ is graceful.*

Proof. When we cut the negated LWW terrace into the q sections we lose the $q-1$ differences that are non-zero multiples of p. These are regained when we rejoin the sections. □

Preece [14] considers extending the concept of p-twizzler terraces to cases with n not a multiple of p. Let $n = pq + r$ with $1 \le r < p$ and set $q' \le q$. Suppose it is possible to obtain a graceful terrace by reversing each of the first q' sections of length p of the negated LWW terrace for $\mathbb{Z}_n$ and to arrange the remaining elements in such a way as to complete this sequence to a graceful terrace for $\mathbb{Z}_n$. Any resulting graceful terrace is an *imperfect* p-twizzler terrace for $\mathbb{Z}_n$.

Example 3 *Let $n = 11$ and $p = 3$. In this case we can set $q' = 3 = q$ and $r = 2$:*

$$\underbrace{1\ 10\ 0}\ \underbrace{8\ 2\ 9}\ \underbrace{4\ 7\ 3}\ 5\ 6.$$

An imperfect p-twizzler terrace must exist for each non-multiple of p (consider $q' = 0$), but the following example shows that we do not have complete freedom of choice for q'.

Example 4 *Let $n = 21$ and $p = 4$. Attempting to set $q' = 5$ does not give a terrace. However, with $q' = 4$ we have:*

$$\underbrace{19\ 1\ 20\ 0}\ \underbrace{17\ 3\ 18\ 2}\ \underbrace{15\ 5\ 16\ 4}\ \underbrace{13\ 7\ 14\ 6}\ 11\ 10\ 8\ 12\ 9.$$

If q' is chosen to be as large as possible for a given n and p then we have a *near-perfect* p-twizzler terrace. Theorem 2 describes exactly when it is possible to set $q' = q$ in the construction of imperfect twizzler terraces. It relies on the following result of Gvozdjak [9] (the result reportedly also appears, in French, in [8] and has been rediscovered in [7]).

Lemma 1 *Let x and n be non-negative integers with $x < n$. There is a graceful terrace for $\mathbb{Z}_n$ that begins with x.*

Gvozdjak's proof involves an intricate argument considering many different cases.

Theorem 2 *Let $n = pq + r$ with $1 \leq r < p$. There is an imperfect p-twizzler terrace for $\mathbb{Z}_n$ with $q' = q$ if and only if $r \geq p/2$.*

Proof. Take the negated LWW terrace and twizzle the first q sections to get a sequence of length pq. This sequence gives all differences that are greater than r. Considered as integers, the largest and smallest of the remaining elements differ by $r - 1$. If we are to extend our sequence to a graceful terrace then the difference of r must be generated by the next element. Such an element is available if and only if $r \geq p/2$.

To prove the result we now need to show that as well as being able to generate a difference of r with the next element, we can generate all of the differences that are less than r with the remaining elements. Suppose that z is the last element of our sequence of length pq. Let y be the integer such that $|z - y| = r$ and let s be the smallest integer not yet used. Take a graceful terrace that of length r that begins with x, where $x = y - s$ (such a terrace exists by Lemma 1), add s to each element and append this sequence to our original sequence. Adding a constant to each element of a graceful terrace does not change the differences, and so we now have our graceful terrace for $\mathbb{Z}_n$. $\square$

While the above argument guarantees the existence of imperfect twizzler terraces with $q' = q$, it does not tell us how to construct them without working through Gvozdjak's complex constructions for constructing graceful terraces with a specified endpoint. We now give examples of some natural constructions that can be achieved without explicit recourse to Gvozdjak's result. Proving their correctness is a straightforward exercise in checking the differences.

Example 5 *Let $n = pq + r$ with $r = p/2$. To obtain a near-perfect p-twizzler terrace for $\mathbb{Z}_n$, twizzle the first q sections of length p and append the remaining r elements in their original order. For instance, taking $18 = 4 \cdot 4 + 2$ gives the following near-perfect 4-twizzler terrace for $\mathbb{Z}_{18}$:*

$$\underbrace{16\ 1\ 17\ 0}\ \underbrace{14\ 3\ 15\ 2}\ \underbrace{12\ 5\ 13\ 4}\ \underbrace{10\ 7\ 11\ 6}\ 8\ 9.$$

Example 6 *Let $n = pq + r$ with $r = p - 1$. To obtain a near-perfect p-twizzler terrace for $\mathbb{Z}_n$, twizzle the first q sections and append the reverse of the last r elements. For instance, taking $19 = 5 \cdot 3 + 4$ gives the following near-perfect 5-twizzler terrace for $\mathbb{Z}_{19}$:*

$$\underbrace{2\ 17\ 1\ 18\ 0}\ \underbrace{14\ 4\ 15\ 3\ 16}\ \underbrace{7\ 12\ 6\ 13\ 5}\ 9\ 10\ 8\ 11.$$

The next result shows that if we cannot set $q' = q$ then the next best value is possible.

Theorem 3 *Let $n = pq + r$ with $1 \leq r < p/2$. There is a near-perfect p-twizzler terrace for $\mathbb{Z}_n$ with $q' = q - 1$.*

Proof. Theorem 2 implies that we cannot take $q' = q$ as $r < p/2$.

Create an initial sequence by twizzling the first $q-1$ sections of the negated LWW terrace. This gives all differences greater than $r + p$. If we were to twizzle one more section we would produce the difference $r + p$ at the join; let y be the integer that produces this difference. Let s be the smallest of the remaining integers and take a graceful terrace of length $r + p$ that starts with $y - s$. As in the proof of Theorem 2, adding s to each integer and appending it to our initial sequence gives the required graceful terrace. $\square$

Theorems 2 and 3 together complete the spectrum of possible near-perfect twizzler terraces. Preece [14] began this result, proving it for the cases $p = 3$ and $p = 4$.

Again, actually producing the near-perfect twizzler terraces described in this theorem requires Gvozdjak's construction. We give examples of more direct constructions. The proofs of their correctness are straightforward.

Example 7 *Let $n = 6q + 1$. A near-perfect 6-twizzler terrace may be obtained as follows. Twizzle the first q sections of the negated LWW terrace to get a new sequence of length n (this is* not *a terrace) and then reverse the last 5 elements of this sequence. For instance, taking $19 = 6 \cdot 3 + 1$ gives the following near-perfect 6-twizzler terrace for $\mathbb{Z}_{19}$:*

$$\underbrace{16\ 2\ 17\ 1\ 18\ 0}\ \underbrace{13\ 5\ 14\ 4\ 15\ 3}\ 10\ 8\ 9\ 6\ 12\ 7\ 11.$$

Example 8 *Let $n = pq + 1$ where p is odd. Twizzle the first $q - 1$ sections of the negated LWW terrace to get a sequence of length $p(q - 1)$. Take the remaining elements and split them into sections of length 2. Reverse the order of these sections (without reversing the sections themselves) and append them to the end of the sequence of length $p(q - 1)$. Taking $22 = 7 \cdot 3 + 1$ gives the following near-perfect 7-twizzler terrace for $\mathbb{Z}_{22}$:*

$$\underbrace{3\ 19\ 2\ 20\ 1\ 21\ 0}\ \underbrace{15\ 6\ 16\ 5\ 17\ 4\ 18}\ 10\ 11\ 9\ 12\ 8\ 13\ 7\ 14.$$

Example 9 *The method of Example 8 applies more generally. Let $n = pq + r$ where $p + r$ is a multiple of $r + 1$. Twizzle the first $q - 1$ sections of the negated LWW terrace to get a sequence of length $p(q - 1)$. Take the remaining $p + r$ elements and split them into sections of length $r + 1$. Reverse the order of these sections (without reversing the sections themselves) and append them to the end of the sequence of length $p(q - 1)$. Taking $23 = 7 \cdot 3 + 2$ gives the following near-perfect 7-twizzler terrace for $\mathbb{Z}_{23}$:*

$$\underbrace{3\ 20\ 2\ 21\ 1\ 22\ 0}\ \underbrace{16\ 6\ 17\ 5\ 18\ 4\ 19}\ 10\ 12\ 11\ 14\ 9\ 13\ 7\ 15\ 8.$$

Of course, each twizzled section of a twizzler terrace is a zigzag. The following result follows immediately from Theorems 2 and 3.

Corollary 1 *Let $n = pq + r$ with $0 \leq r < p$. There is a terrace for $\mathbb{Z}_n$ that begins with at least $q - 1$ zigzags, each of length p.*

To conclude this section we note that to be able to construct a near-perfect p-twizzler terrace with $q' = q$ for $\mathbb{Z}_n$ for all values of n where this is possible, it is sufficient to have to hand a graceful terrace of length r that begins with each possible value for all $r < p$. To achieve the equivalent for the case $q' = q - 1$ it is sufficient to have a graceful terrace of length $p + r$ that begins with each possible value, where r is at most $p/2$.

When there are at most 4 elements to be filled after twizzling, the unique (up to reversal and complementation) graceful terraces $(0, 1)$, $(0, 2, 1)$ and $(0, 3, 1, 2)$ suffice. Similarly, Table 1 gives graceful terraces of lengths from 5 up to 12 from which graceful terraces with all possible starting values may be constructed by reversal and/or complementation. This enables the speedy construction of near-perfect p-twizzler terraces for every n with $q' = q - 1$ when $3 \leq p \leq 8$, and of near-perfect p-twizzler terraces with $q' = q$ for all n for which they exist when $3 \leq p \leq 13$. For $p \in \{3, 4\}$ the resulting terraces are equivalent to those of [14].

Table 1: Some short graceful terraces

Length	Graceful terraces
5	0, 4, 1, 3, 2
	1, 4, 0, 2, 3
6	0, 5, 1, 4, 2, 3
	1, 5, 0, 3, 2, 4
7	0, 6, 1, 5, 2, 4, 3
	1, 6, 0, 4, 3, 5, 2
8	0, 7, 1, 6, 2, 5, 3, 4
	2, 5, 3, 4, 0, 7, 1, 6
9	0, 8, 1, 7, 2, 6, 3, 5, 4
	1, 8, 0, 6, 2, 7, 4, 5, 3
	2, 7, 1, 8, 0, 4, 5, 3, 6
10	0, 9, 1, 8, 2, 7, 3, 6, 4, 5
	1, 9, 0, 7, 2, 8, 4, 5, 3, 6
	2, 8, 1, 9, 0, 5, 4, 6, 3, 7
11	0, 10, 1, 9, 2, 8, 3, 7, 4, 6, 5
	1, 10, 0, 8, 2, 9, 4, 7, 3, 5, 6
	2, 9, 1, 10, 0, 6, 4, 5, 8, 3, 7
12	0, 11, 1, 10, 2, 9, 3, 8, 4, 7, 5, 6
	1, 11, 0, 9, 2, 10, 4, 8, 3, 6, 5, 7
	2, 9, 1, 10, 0, 11, 5, 6, 4, 7, 3, 8

3 More terraces from graceful terraces

In this section we deploy the following result to construct many new zigzag terraces.

Theorem 4 [11] *Given a graceful terrace for $\mathbb{Z}_m$ that ends with y and a graceful terrace for $\mathbb{Z}_n$ that starts with y we can construct a terrace for $\mathbb{Z}_{m+n}$.*

Proof Construction. Add m to each element (considered as integers) of the graceful terrace for $\mathbb{Z}_n$ and append this sequence to the graceful sequence for $\mathbb{Z}_m$. This is a terrace (but not a graceful one) for $\mathbb{Z}_{m+n}$. □

Example 10 *Letting $m = 5$ and $n = 9$, and taking the graceful terraces $(0, 4, 1, 3, 2)$ and $(2, 7, 1, 8, 0, 4, 5, 3, 6)$ from Table 1, gives the following terrace for $\mathbb{Z}_{14}$:*

$$0\ 4\ 1\ 3\ 2\ 7\ 12\ 6\ 13\ 5\ 9\ 10\ 8\ 11.$$

If we put graceful zigzag terraces into the construction of Theorem 4 then a zigzag terrace emerges. We have three classes of graceful zigzag terraces at our disposal: the negated LWW terraces, the twizzler terraces and the imperfect twizzler terraces.

Further, suppose we have a graceful terrace where the difference between the first and last element is x. We may split the terrace at the internal point where the difference x is generated and write the second part before the first part to get a new graceful terrace. We call the new terrace the *translation* of the first, as it is analogous to Bailey's "translation" for arbitrary terraces [2]. Every graceful terrace has exactly one translation to another graceful terrace.

Theorem 4 provides too many zigzag terraces to catalogue, so we content ourselves with describing some elegant examples to complement Preece's examples [14]. It is interesting to note that several terraces in the literature can be viewed as special cases of this method.

As with the examples from the previous section, proofs of correctness are a straightforward matter of checking differences.

Example 11 *The p-twizzler terrace for $\mathbb{Z}_{3p}$ is a graceful terrace with endpoint p and the reverse of the negated LWW terrace for $\mathbb{Z}_{2p}$ starts with p. Combining these using Theorem 4 gives a terrace for $\mathbb{Z}_{5p}$ consisting of three zigzags of length p followed by a zigzag of length $2p$. Here is the resulting terrace for $\mathbb{Z}_{25}$ obtained by taking $p = 5$:*

$$\underbrace{2\ 13\ 1\ 14\ 0}\ \underbrace{10\ 4\ 11\ 3\ 12}\ \underbrace{7\ 8\ 6\ 9\ 5}\ \underbrace{20\ 19\ 21\ 18\ 22\ 17\ 23\ 16\ 24\ 15}.$$

As an alternative to using $\mathbb{Z}_{2p}$ we could also take the negated LWW terrace for $\mathbb{Z}_{p+1}$ or $\mathbb{Z}_{2p+1}$ to give zigzag terraces for $\mathbb{Z}_{4p+1}$ and $\mathbb{Z}_{5p+1}$ respectively.

Example 12 *$\mathbb{Z}_{3p}$ and $\mathbb{Z}_{4q}$ always provide twizzler terraces with an endpoint of 1 when twizzling by 3 and 4 respectively (use the complement of the twizzler terrace for $\mathbb{Z}_{4q}$). Theorem 4 gives a zigzag terrace for $\mathbb{Z}_{3p+4q}$ with p zigzags of length 3 followed by q zigzags of length 4. With $p = 5$ and $q = 4$ we get the following terrace for $\mathbb{Z}_{31}$:*

$$\underbrace{6\ 8\ 7}\ \underbrace{10\ 5\ 9}\ \underbrace{3\ 11\ 4}\ \underbrace{13\ 2\ 12}\ \underbrace{0\ 14\ 1}$$
$$\underbrace{16\ 29\ 15\ 30}\ \underbrace{18\ 27\ 17\ 28}\ \underbrace{20\ 25\ 19\ 26}\ \underbrace{22\ 23\ 21\ 24}.$$

More generally, let t be odd. Then $\mathbb{Z}_{pt}$ and $\mathbb{Z}_{q(t+1)}$ both have twizzler terraces with $(t+1)/2$ as an endpoint. These may be used to construct a zigzag terrace for $\mathbb{Z}_{(pt+q(t+1))}$ with p zigzags of length t followed by q zigzags of length $t+1$.

The next three examples describe terraces that appeared in the literature independently as direct constructions but are all in fact easy consequences of the work we have done to this point.

Example 13 The Ringel-Owens terrace. *Consider the negated LWW terrace for $\mathbb{Z}_s$ and the reverse of the complement of the negated LWW terrace for $\mathbb{Z}_{s+1}$. We can take $m = s$, $n = s+1$ and $y = \lfloor s/2 \rfloor$ in Theorem 4 to construct a terrace for $\mathbb{Z}_{2s+1}$ with two zigzags:*

$$\underbrace{0\ \ (s-1)\ \ 1\ \ (s-2)\ldots\lfloor s/2\rfloor}\ \ \underbrace{(\lfloor s/2\rfloor+s)\ldots s\ \ 2s}.$$

The reverse of this terrace is a translate of the Ringel-Owens terrace for $\mathbb{Z}_{2s+1}$ *as described in [14].*

Example 14 The Ollis terrace. *Similarly to the previous example, using the negated LWW terrace for* $\mathbb{Z}_s$*, with* s *odd, and the reverse of the complement of the negated LWW terrace for* $\mathbb{Z}_s$ *gives the following zigzag terrace for* $\mathbb{Z}_{2s}$*:*

$$\underbrace{0 \;\; (s-1) \;\; 1 \;\; (s-2) \ldots (s-1)/2}\;\; \underbrace{3(s-1)/2 \ldots (s-1) \;\; (2s-1)}.$$

This is a reverse of a translate of the Ollis terrace as described in [14].

Example 15 Tripartite terraces. *In Theorem 4, taking the negated LWW terrace for* $\mathbb{Z}_{2q}$ *as the first terrace and the reverse of the translation of the negated LWW terrace for* $\mathbb{Z}_{4q}$ *as the second gives:*

$$\underbrace{0 \;\; (2q-1) \;\; 1 \ldots q}\;\; \underbrace{5q \;\; (3q-1) \;\; (5q+1) \ldots 2q}\;\; \underbrace{4q \;\; (4q-1) \;\; (4q+1) \ldots 3q}.$$

This is the tripartite terrace for $\mathbb{Z}_{6q}$ *as described in [14]. The other 5 cases of tripartite terraces (as* n *varies modulo 6) can be obtained in similar fashion.*

Another consequence of Theorem 4 is that we may write down the negated LWW terrace for $\mathbb{Z}_m$, change our minds and decide to terrace $\mathbb{Z}_l$ instead, and carry on from what is already written (provided that l is large enough):

Corollary 2 *Let* l *and* m *be integers with* $l > 3m/2$*. There is a terrace for* $\mathbb{Z}_l$ *that begins*

$$0 \;\; (m-1) \;\; 1 \;\; (m-2) \ldots \lfloor m/2 \rfloor.$$

Proof. Apply Theorem 4 with the negated LWW terrace for $\mathbb{Z}_m$ as the first terrace and a graceful terrace for $\mathbb{Z}_{l-m}$ that begins with $\lfloor m/2 \rfloor$ as the second. Such a terrace is guaranteed to exist by Lemma 1. □

4 Terraces for abelian groups

Bailey's Conjecture [2] is that all groups have a terrace, except for the non-cyclic elementary abelian 2-groups (which are known not to have terraces). The results of [12] attempt to address the conjecture for abelian groups, claiming to construct terraces for all possibly terraceable groups except those of order coprime to 3 whose Sylow 2-subgroup is of the form $\mathbb{Z}_2^{2k+1}$ for $k \geq 1$. However, there is an error in one case of the main theorem which then invalidates the claim that some of the 2-groups with a large cyclic component are terraced. In this section we show that a particular type of terrace exists for $\mathbb{Z}_{4m}$ and that this is sufficient to circumnavigate the error. Hence the claimed results of [12] hold.

We start with some preliminary concepts. Let A be an abelian group of order n and let $\mathbf{a} = [a_1, a_2, \dots, a_{n-1}]$ be a circular arrangement of the non-zero elements of A (we follow the convention of earlier papers and write circular lists in square brackets and consider the subscripts to be calculated modulo the length of the list). Define $\mathbf{b} = [b_1, b_2, \dots, b_{n-1}]$ by $b_i = a_{i+1} - a_i$ for each i. If $\mathbf{b}$ contains exactly one occurrence of each involution of A and exactly two occurrences from each set $\{x, -x : 2x \neq 0\}$ then $\mathbf{a}$ is a *rotational terrace* or *R-terrace* for A and $\mathbf{b}$ is the associated *rotational 2-sequencing* or *R-2-sequencing* of A. If $\mathbf{b}$ contains every non-zero element of A exactly once then the rotational terrace $\mathbf{a}$ is *directed* and $\mathbf{b}$ is called an *R-sequencing.*

Further, if $a_i = a_{i-1} + a_{i+1}$ for some i then $\mathbf{a}$ is an *R*-terrace* for A and if $i = 1$ then the R*-terrace is *standard.* If $b_j = -a_{j+1}$ for some j then j is a *right match-point* of $\mathbf{b}$.

The terminology "R-sequencing" is well-established in the literature, but the extension of this to "R-2-sequencings" and "R-terraces" is fairly recent [12]. The more descriptive "rotational" is suggested by Ahmed et al [1]. While the present authors agree that rotational is the better term, we here usually use the "R-" vocabulary to make clearer the connections to results in [12].

Right match-points allow us to convert rotational terraces to terraces:

Lemma 2 [12] *Let A be an abelian group of order n and let $[a_1, a_2, \dots, a_{n-1}]$ be a rotational terrace with associated rotational 2-sequencing* $\mathbf{b}$. *If* $\mathbf{b}$ *has a right match-point then A has a terrace.*

Proof. If j is a right match-point of $\mathbf{b}$ then

$$(0, a_{j+1}, a_{j+2}, \dots, a_{n-1}, a_1, a_2, \dots, a_j)$$

is a terrace for A. We lose the difference $-a_{j+1} = b_j = a_{j+1} - a_j$ from the rotational 2-sequencing and replace it with $a_{j+1} - 0 = a_{j+1}$. This preserves the required difference properties. □

The following result is the crux of the construction. In [12] it was mistakenly claimed that the cyclic group of order 4 could be used in place of V, the elementary abelian group of order 4.

Theorem 5 *Let A be an abelian group of order n with a subgroup V isomorphic to $\mathbb{Z}_2^2$. If A/V has a standard R*-terrace with right match-point j for some $j \leq n/4 - 3$ then A also has a standard R*-terrace with right match-point j.*

See [12] for the details of the construction.

The only place in [12] where the theorem was used with the cyclic group in place of V was in the construction of terraces for non-elementary abelian 2-groups of order greater than 16 for which it is impossible to reach a non-cyclic non-elementary abelian group of order 8 or 16 by repeatedly factoring out copies of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Starting from such 2-groups and repeatedly factoring out copies for $\mathbb{Z}_2 \times \mathbb{Z}_2$ one will reach a cyclic group of order 8 or more. Therefore to complete the proof that non-elementary abelian 2-groups have a terrace it is sufficient to construct a standard R*-terrace for $\mathbb{Z}_{2^k}$ with right match-point $j \leq 2^k - 3$ for each $k \geq 3$. In fact, we are able to construct a zigzag terrace for $\mathbb{Z}_{4m}$ for all $m \geq 2$ using the methods of Sections 2 and 3 that leads immediately to the required standard R*-terrace.

Theorem 6 *The cyclic group $\mathbb{Z}_{4m}$ has a standard R^*-terrace with right match-point $j \leq 4m - 3$ for all $m \geq 2$.*

Proof. The construction of Lemma 2 is reversible: $\mathbb{Z}_n$ has a standard R*-terrace if and only if it has a basic terrace with $a_n = 2a_2$ and $a_{j-1} + a_{j+1} = a_j$. To ensure that the right match-point occurs in a valid position of the R*-terrace we require in addition that $j \geq 5$. To prove the result we construct such terraces for $\mathbb{Z}_{4m}$.

The negated LWW graceful terrace for $\mathbb{Z}_p$ has $a_{j-1} + a_{j+1} = a_j$ for a suitable value of j when $p \geq 8$ and $p \equiv 2 \pmod 3$ [13]. The construction takes such a terrace as a start and appends another graceful terrace to it using Theorem 4. We break the proof for $m > 4$ into three cases according to the value of m modulo 3. There are also two small exceptional terraces needed for $m = 3$ and $m = 4$.

Case 1: $\mathbb{Z}_{12k+4}$ for $k \geq 1$. Start with the negated LWW terrace for $\mathbb{Z}_{6k+2}$. As $6k + 2 \equiv 2 \pmod 3$, we have the sum condition satisfied. The second element of this terrace is $6k + 1$ and last element is $3k + 1$. We extend this with a graceful labelling of length $6k+2$. To satisfy the conditions, this graceful labelling must have first element $3k + 1$ and last element $2(6k + 1) - (6k + 2) = 6k$. The reverse of the complement of the near-perfect 3-twizzler for $\mathbb{Z}_{6k+2}$ suffices, see Example 6.

Case 2: Z_{12k+8} for $k \geq 1$. As in Case 1, we start with the negated LWW terrace for $\mathbb{Z}_{6k+2}$. We extend this with a graceful labelling of length $6k + 6$ that, as before, is required to have first element $3k + 1$ and last element $6k$.

The construction varies slightly depending on the parity of k. For odd k, 12-twizzle $(k - 1)/2$ sections of the negated LWW terrace. This moves $6k$ to the first position and leaves 12 elements remaining. To get the last 12 elements add $3k - 3$ to a graceful terrace of length 12 that starts with 6 and ends with 4. A possible terrace is:

$$6 \;\; 1 \;\; 10 \;\; 0 \;\; 11 \;\; 3 \;\; 8 \;\; 2 \;\; 5 \;\; 7 \;\; 8 \;\; 4.$$

Reverse.

For even k, 12-twizzle $k/2 - 1$ sections of the negated LWW terrace. This moves $6k$ to the first position and leaves 18 elements remaining. To get the last 18 elements add $3k - 6$ to a graceful labelling of length 18 that starts with 12 and ends with 7. A possible terrace is:

$$12 \;\; 3 \;\; 13 \;\; 6 \;\; 10 \;\; 5 \;\; 11 \;\; 9 \;\; 8 \;\; 16 \;\; 1 \;\; 17 \;\; 0 \;\; 14 \;\; 2 \;\; 15 \;\; 4 \;\; 7.$$

Reverse.

Case 3: $\mathbb{Z}_{12k}$ for $k \geq 2$. Begin with the negated LWW terrace for $\mathbb{Z}_{6k-1}$; to complete this requires a graceful labelling of length $6k+1$ with endpoints $6k-3$ and $3k-1$. Now, 8-twizzling gives us a starting element of $6k-3$; keep 8-twizzling until at most 13 elements remain (there will be 7, 9, 11 or 13 depending on the value of k modulo 4). The graceful labelling of the appropriate length from this list successfully completes the imperfect twizzler terrace in the same manner as the earlier cases:

$$3 \;\; 1 \;\; 4 \;\; 5 \;\; 0 \;\; 6 \;\; 2,$$

$$5 \;\; 6 \;\; 1 \;\; 4 \;\; 2 \;\; 8 \;\; 0 \;\; 7,$$

$$7 \;\; 1 \;\; 8 \;\; 3 \;\; 6 \;\; 5 \;\; 9 \;\; 0 \;\; 10 \;\; 2 \;\; 4,$$

$$9 \;\; 1 \;\; 10 \;\; 3 \;\; 8 \;\; 2 \;\; 12 \;\; 0 \;\; 11 \;\; 7 \;\; 4 \;\; 6.$$

Finally, we need to exhibit suitable terraces for $\mathbb{Z}_8$ and $\mathbb{Z}_{12}$:

$$0 \;\; 5 \;\; 3 \;\; 4 \;\; 1 \;\; 7 \;\; 6 \;\; 2,$$

$$0 \;\; 1 \;\; 6 \;\; 8 \;\; 11 \;\; 5 \;\; 3 \;\; 7 \;\; 4 \;\; 9 \;\; 10 \;\; 2.$$

This completes the proof. □

As described earlier, we now have all the ingredients to have fully proved the following corollary.

Corollary 3 *The non-cyclic elementary abelian 2-groups excepted, each abelian 2-group has a terrace.*

5 The Oberwolfach problem

A solution to the Oberwolfach problem (that is, a 2-factorisation of the complete graph K_v, where v is odd, with mutually isomorphic 2-factors) is 1-*rotational* under the action of a group G if G is an automorphism group of the solution which acts by fixing one vertex and is sharply transitive on the remainder. The group G must have order $v-1$. Further, if G is cyclic then in [11] such a solution is called *cyclic*; we here modify this terminology and call such solutions *cyclic* 1-*rotational* to avoid confusion with solutions that have a cyclic automorphism group acting sharply transitively on all v vertices (such solutions are possible; see, for example, [5]). Recent work on finding 1-rotational solutions to the Oberwolfach problem for various parameter sets and choices of G may be found in [6, 15, 16].

Here is the result that links graceful terraces to the Oberwolfach problem.

Theorem 7 [11] *Let* $\mathbf{g} = (g_1, g_2, \ldots, g_m)$ *be a graceful terrace for* $\mathbb{Z}_m$ *with* $g_{r+1} - g_r = g_r - g_1$ *for some* r. *There is a cyclic* 1*-rotational solution to* $\mathrm{OP}(r, r, s)$ *for* $s = 2(m-r)+1$ *and for all odd* s *with* $s > 2(m + g_m - r)$.

A graceful terrace as required by the theorem was constructed for each odd value of r in [11]. Our goal here is to construct a graceful terrace as required for each even r.

Our terrace consists of two zigzags, the second approximately twice as long as the first. Unlike the zigzag terraces presented earlier the step-size (c in the notation of the introduction) is ± 3 rather than ± 1. There are two variants, one for $m = 6k+2$ and one for $m = 6k+5$. Here is the terrace for $m = 6k+2$:

$$\underbrace{3k\ \ (3k+3)\ \ (3k-3)\ \ (3k+6)\ \ldots\ 3\ \ 6k\ \ 0}\qquad \underbrace{(6k+1)\ \ 2\ \ (6k-2)\ \ 5\ \ldots\ 4\ \ (6k-1)\ \ 1},$$

and here is the terrace for $m = 6k+5$:

$$\underbrace{(3k+3)\ \ 3k\ \ (3k+6)\ \ (3k-3)\ \ldots\ 3\ \ 6k+3\ \ 0}\qquad \underbrace{(6k+4)\ \ 2\ \ (6k+1)\ \ 5\ \ldots\ 4\ \ (6k+2)\ \ 1}.$$

It is straightforward to check that these are indeed graceful terraces: the first zigzag generates the differences that are a multiple of 3; the second zigzag generates the others, with the exception of $m-1$ which is created at the join.

Further, in the $m = 6k+2$ case we can set $r = 4k+2$ and we have $g_{r+1} = 3k+2$, $g_r = 3k+1$ and $g_1 = 3k$ giving

$$g_{r+1} - g_r = 1 = g_r - g_1.$$

Similarly, in the $m = 6k+5$ case, we can set $r = 4k+4$ to give

$$g_{r+1} - g_r = -1 = g_r - g_1.$$

In both cases the final element of the terrace is 1.

Theorem 8 *Let r be even. Then* OP(r, r, s) *has a cyclic 1-rotational solution for all odd $s \geq r-1$.*

Proof. The conditions of Theorem 7 apply to our new graceful terrace and in both cases we find that $2(m + g_m - r) = r$. □

Combining this with Theorem 3.3 of [11], which shows that for each odd r there is a cyclic 1-rotational solution to OP(r, r, s) for all odd s with $s \geq 2\lfloor(3r-1)/4\rfloor + r + 2$, we have:

Corollary 4 *For any given r there is a cyclic 1-rotational solution to* OP(r, r, s) *for all sufficiently large odd s.*

Acknowledgements

We are grateful to Donald Preece (Queen Mary, University of London and University of Kent), Marco Buratti (Universita degli Studi di Perugia) and an anonymous referee for helpful comments on this work.

References

[1] A. Ahmed, M. I. Azimli, I. Anderson and D. A. Preece, Rotational terraces from rectangular arrays, *Bull. Inst. Combin. Appl.* (to appear).

[2] R. A. Bailey, Quasi-complete Latin squares: construction and randomisation, *J. Royal Statist. Soc. Ser. B* **46** (1984), 323–334.

[3] R. A. Bailey, M. A. Ollis and D. A. Preece, Round-dance neighbour designs from terraces, *Discrete Math.* **266** (2003), 69–86.

[4] D. Bryant and C. Rodger, Cycle decompositions, in *CRC Handbook of Combinatorial Designs (C.J. Colbourn and J.H. Dinitz, Eds.)*, CRC Press, Boca Raton, FL, 2007, 373–382.

[5] M. Buratti and A. Del Fra, Cyclic Hamiltonian cycle systems of the complete graph, *Discrete Math.* **279** (2004), 107–119.

[6] M. Buratti and G. Rinaldi, 1-rotational k-factorizations of the complete graph and new solutions to the Oberwolfach problem, *J. Combin. Des.* **16** (2008), 87–100

[7] R. Cattell, Graceful labellings of paths, *Discrete Math.* **307** (2007), 3161–3176.

[8] E. Flandrin, I. Fourier and A. Germa, Numerotations gracieuses des chemins, *Ars Combin.* **16** (1983), 149–181.

[9] P. Gvözdjak, *On the Oberwolfach problem for cycles with multiple lengths (PhD thesis)*, Simon Fraser University, (2004).

[10] M. A. Ollis, Sequenceable groups and related topics, *Electron. J. Combin.*, Dynamic Survey 10 (2002), 34pp.

[11] M. A. Ollis, Some cyclic solutions to the three table Oberwolfach problem, *Electron. J. Combin* **12** R58 (2005), 7pp.

[12] M. A. Ollis, On terraces for abelian groups, *Discrete Math.* **305** (2005), 250–263.

[13] M. A. Ollis and D. A. Preece, Sectionable terraces and the (generalised) Oberwolfach problem, *Discrete Math.* **266** (2003), 399–416.

[14] D. A. Preece, Zigzag and foxtrot terraces for $\mathbb{Z}_n$, *Australas. J. Combin.* **42** (2008), 261–278.

[15] G. Rinaldi and T. Traetta, Graph products and new solutions to Oberwolfach problems, *Electron. J. Combin.*, **18(1)** P52 (2011), 17pp.

[16] T. Traetta, Some new results on 1-rotational 2-factorizations of the complete graph, *J. Combin. Des.* **18** (2010), 237–247.

[17] E. J. Williams, Experimental designs balanced for the estimation of residual effects of treatments, *Aust. J. Scient. Res. A*, **2** (1949), 149–168.

(Received 24 Mar 2011)