AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **51** (2011), Pages 29–40

On local connectivity of $K_{2,p}$-free graphs

Andreas Holtkamp

Lehrstuhl C für Mathematik
RWTH Aachen University
52056 Aachen
Germany
holtkamp@mathc.rwth-aachen.de

Lutz Volkmann

Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen
Germany
volkm@math2.rwth-aachen.de

Abstract

For a vertex v of a graph G, we denote by $d(v)$ the *degree* of v. The *local connectivity* $\kappa(u,v)$ of two vertices u and v in a graph G is the maximum number of internally disjoint u–v paths in G. Clearly, $\kappa(u,v) \leq \min\{d(u), d(v)\}$ for all pairs u and v of vertices in G. We call a graph G *maximally local connected* when $\kappa(u,v) = \min\{d(u), d(v)\}$ for all pairs u and v of distinct vertices in G. Let $p \geq 2$ be an integer. We call a graph $K_{2,p}$-*free* if it contains no complete bipartite graph $K_{2,p}$ as a (not necessarily induced) subgraph. If $p \geq 3$ and G is a connected $K_{2,p}$-free graph of order n and minimum degree δ such that $n \leq 3\delta - 2p + 2$, then G is maximally local connected due to our earlier result on p-diamond-free graphs [*Discrete Math.* 309 (2009), 6065–6069]. Now we present examples showing that the condition $n \leq 3\delta - 2p + 2$ is best possible for $p = 3$ and $p \geq 5$. In the case $p = 4$ we present the improved condition $n \leq 3\delta - 5$ implying maximally local connectivity. In addition, we present similar results for $K_{2,2}$-free graphs.

1 Terminology and introduction

We consider finite graphs without loops and multiple edges. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex

$v \in V(G)$, the *open neighborhood* $N_G(v) = N(v)$ is the set of all vertices adjacent to v, and $N_G[v] = N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v. If $A \subseteq V(G)$, then $N_G[A] = \bigcup_{v \in A} N_G[v]$, and $G[A]$ is the subgraph induced by A. The numbers $|V(G)| = n(G) = n$, $|E(G)| = m(G) = m$ and $|N(v)| = d_G(v) = d(v)$ are called the *order*, the *size* of G and the *degree* of v, respectively. The *minimum degree* of a graph G is denoted by $\delta(G) = \delta$. For an integer $p \geq 2$, we define a *p-diamond* as the graph with $p+2$ vertices, where two adjacent vertices have exactly p common neighbors, and the graph contains no further edges. For $p = 2$, the 2-diamond is the usual *diamond*. A graph is *p-diamond-free* if it contains no p-diamond as a (not necessarily induced) subgraph. The *complete graph* of order n is denoted by K_n. Let $K_{s,t}$ be the *complete bipartite graph* with the bipartition A, B such that $|A| = s$ and $|B| = t$. We call a graph *$K_{s,t}$-free* if it contains no $K_{s,t}$ as a (not necessarily induced) subgraph. Notice that in the special case $s = t = 2$, the graph $K_{2,2}$ is isomorphic to the cycle C_4 of length 4.

The *connectivity* $\kappa(G)$ of a connected graph G is the smallest number of vertices whose deletion disconnects the graph or produces the trivial graph (the latter only applying to complete graphs). The *local connectivity* $\kappa_G(u,v) = \kappa(u,v)$ between two distinct vertices u and v of a connected graph G, is the maximum number of internally disjoint u–v paths in G. It is a well-known consequence of Menger's theorem [11] that

$$\kappa(G) = \min\{\kappa_G(u,v) \,|\, u, v \in V(G)\}. \tag{1}$$

It is straightforward to verify that $\kappa(G) \leq \delta(G)$ and $\kappa(u,v) \leq \min\{d(u), d(v)\}$. We call a graph G *maximally connected* when $\kappa(G) = \delta(G)$ and *maximally local connected* when $\kappa(u,v) = \min\{d(u), d(v)\}$ for all pairs u and v of distinct vertices in G.

Because of $\kappa(G) \leq \delta(G)$, there exists a special interest on graphs G with $\kappa(G) = \delta(G)$. Different authors have presented sufficient conditions for graphs to be maximally connected, as, for example Balbuena, Cera, Diánez, García-Vázquez and Marcote [1], Esfahanian [3], Fàbrega and Fiol [4, 5], Fiol [7], Hellwig and Volkmann [8], Soneoka, Nakada, Imase and Peyrat [12] and Topp and Volkmann [13]. For more information on this topic we refer the reader to the survey articles by Hellwig and Volkmann [9] and Fàbrega and Fiol [6]. However, closely related investigations for the local connectivity have received little attention until recently. In this paper we will present such results for $K_{2,p}$-free graphs. We start with a simple and well-known proposition.

Observation 1 *If a graph G is maximally local connected, then it is maximally connected.*

Proof. Since G is maximally local connected, we have $\kappa(u,v) = \min\{d(u), d(v)\}$ for all pairs u and v of vertices in G. Thus (1) implies

$$\kappa(G) = \min_{u,v \in V(G)} \{\kappa(u,v)\} = \min_{u,v \in V(G)} \{\min\{d(u), d(v)\}\} = \delta(G). \qquad \Box$$

2 $K_{2,p}$-free graphs with $p \geq 3$

Recently, Holtkamp and Volkmann [10] gave a sufficient condition for connected p-diamond-free graphs to be maximally local connected.

Theorem 2 (Holtkamp and Volkmann [10] 2009) *Let $p \geq 3$ be an integer, and let G be a connected p-diamond-free graph. If $n(G) \leq 3\delta(G) - 2p + 2$, then G is maximally local connected.*

Since a $K_{2,p}$-free graph is also p-diamond-free, the next corollary is immediate.

Corollary 3 *Let $p \geq 3$ be an integer, and let G be a connected $K_{2,p}$-free graph. If $n(G) \leq 3\delta(G) - 2p + 2$, then G is maximally local connected.*

The next result is a direct consequence of Corollary 3 and Observation 1.

Corollary 4 *Let $p \geq 3$ be an integer, and let G be a connected $K_{2,p}$-free graph. If $n(G) \leq 3\delta(G) - 2p + 2$, then G is maximally connected.*

The following examples will demonstrate that the condition $n(G) \leq 3\delta(G)-2p+2$ in Corollaries 3 and 4 is best possible for $p = 3$ and $p \geq 5$.

Example 5 The connected graph in Figure 1 is $K_{2,3}$-free with minimum degree $\delta = 4$ and order $n = 3\delta - 6 + 3 = 9$. The vertex set S with $|S| = 3$ disconnects the graph, and therefore it is neither maximally connected nor maximally local connected. Thus the condition $n(G) \leq 3\delta(G)-2p+2$ in Corollaries 3 and 4 are best possible for $p = 3$.

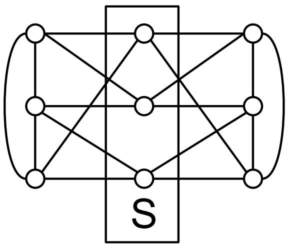

Figure 1: $K_{2,3}$-free graph with $\delta = 4$ and $n = 3\delta - 3 = 9$ vertices which is not maximally (local) connected.

Let G_3, G_4, G_5 and G_6 be the graphs depicted in Figure 2. Each G_p is a connected $K_{2,p}$-free graph with $\delta(G_p) = p$ and $n(G_p) = 3\delta(G_p)-2p+3 = p+3$. The graphs G_5 and G_6 are not maximally connected and therefore not maximally local connected, since the removal of the vertex set S with $|S| = \delta(G_p) - 1 = p - 1$ disconnects the graphs. So Corollaries 3 and 4 are best possible for $p = 5$ and $p = 6$.

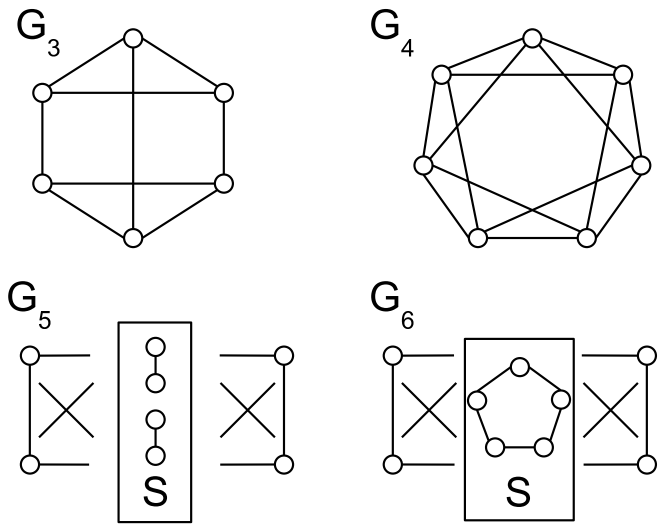

Figure 2: $K_{2,p}$-free graphs G_p ($p \in \{3,4,5,6\}$) with $\delta(G_p) = p$ and $n = 3\delta(G_p) - 2p + 3 = \delta(G_p) + 3 = p + 3$. The graphs G_5 and G_6 are not maximally (local) connected, G_3 and G_4 are.

Starting with the four graphs G_3, G_4, G_5 and G_6, we are able to construct successively similar graphs G_p for all $p \geq 7$. Each G_p will be connected and $K_{2,p}$-free with $\delta(G_p) = p$ and $n(G_p) = 3\delta(G_p) - 2p + 3 = p + 3$. A vertex set S with $|S| = p - 1$ will separate G_p, showing that neither of the graphs is maximally connected or maximally local connected.

Given a graph G_p with the described properties, we can construct a graph G_{p+4} with the same qualities in the subsequently specified way. For G_{p+4} not to be maximally (local) connected the maximally (local) connectivity of G_p is irrelevant (e.g. G_3 and G_4 are maximally (local) connected). The existence of G_p for all $p \geq 7$ then follows by induction.

So let G_p be a graph with the properties mentioned above. We obtain the graph G_{p+4} by adding four new vertices u, u', v and v', the edges uu' and vv' as well as all possible edges between the four new vertices and the vertices of G_p that means $\{xy | x \in \{u, u', v, v'\}$ and $y \in V(G_p)\}$. Then $n(G_{p+4}) = n(G_p) + 4 = p + 3 + 4 = (p + 4) + 3$ and $\delta(G_{p+4}) = \delta(G_p) + 4 = n(G_p) + 1 = p + 4$. We will now show that G_{p+4} is $K_{2,p+4}$-free. So let w and z be two arbitrary vertices of G_{p+4}. We distinguish three different cases.

Case 1. Assume that $w, z \in \{u, u', v, v'\}$. Then w and z can only have common neighbors in G_p. Because $n(G_p) = p + 3$, the vertices w and z have at most $p + 3$ common neighbors.

Case 2. Assume that $w \in \{u, u', v, v'\}$ and $z \in V(G_p)$. Without loss of generality, we can assume that $w = u$. Therefore w and z only have $|\{u'\} \cup (V(G_i) - \{z\})| = p+3$ common neighbors.

Case 3. Assume that $w, z \in V(G_p)$. Since G_p is $K_{2,p}$-free, w and z again have at most $(p-1)+4 = p+3$ common neighbors.

We have seen that no two vertices in G_{p+4} could have more than $p+3$ common neighbors. Therefore G_{p+4} is $K_{2,p+4}$-free. Since $G_{p+4} - V(G_p)$ is disconnected with $n(G_p) = p+3$ and $\delta(G_{p+4}) = p+4$, the graph G_{p+4} is not maximally connected and therefore not maximally local connected. □

Next we will present an improved condition on maximally local connectivity for $K_{2,4}$-free graphs. For the proof we use the following result.

Theorem 6 (Holtkamp and Volkmann [10] 2009) *Let $p \geq 2$ be an integer, and let G be a connected p-diamond-free graph. In addition, let $u, v \in V(G)$ be two vertices of G and define $r = \min\{d_G(u), d_G(v)\} - \delta(G)$.*

(1) If $uv \notin E(G)$ and $n(G) \leq 3\delta(G) + r - 2p + 2$, then $\kappa_G(u,v) = \delta(G) + r$.

(2) If $uv \in E(G)$ and $n(G) \leq 3\delta(G) + r - 2p + 1$, then $\kappa_G(u,v) = \delta(G) + r$.

Theorem 7 *Let G be a connected $K_{2,4}$-free graph with minimum degree $\delta(G) \geq 3$. If $n(G) \leq 3\delta(G) - 5$, then G is maximally local connected.*

Proof. If $n(G) \leq 3\delta(G) - 6$, then the maximally local connectivity of G follows from Corollary 3. Thus let now $n(G) = 3\delta(G) - 5$. If $\delta(G) = 3$, then $n(G) = 4$ and therefore G is isomorphic to the complete graph K_4, which is maximally local connected. In the case $\delta(G) \geq 4$, we suppose to the contrary that G is not maximally local connected. This means that there are two vertices $u, v \in V(G)$ with $\kappa_G(u,v) \leq \delta(G) + r - 1$ for $r = \min\{d_G(u), d_G(v)\} - \delta(G)$. Next we distinguish two cases.

Case 1. Assume that $uv \in E(G)$. As a $K_{2,4}$-free graph is also 4-diamond-free, Theorem 6(2) implies $0 \leq r \leq 1$. If we define the graph H by $H = G - uv$, then there exists a vertex set $S \subset V(H) = V(G)$ with $|S| \leq \delta(G) + r - 2$ that separates u and v in H. Because $d_H(u) \geq \delta + r - 1$ and $d_H(v) \geq \delta + r - 1$, there is a vertex $u' \in V(H) - S$ adjacent to u as well as a vertex $v' \in V(H) - S$ adjacent to v in H. Since H is also $K_{2,4}$-free, we deduce that $|N_H[\{u,u'\}]| \geq 2\delta(G) + r - 4$ as well as $|N_H[\{v,v'\}]| \geq 2\delta(G) + r - 4$. Combining these two bounds with $|S| \leq \delta(G) + r - 2$, we obtain

$$\begin{aligned} n(G) &= 3\delta(G) - 5 \\ &\geq |N_H[\{u,u'\}]| + |N_H[\{v,v'\}]| - |S| \\ &\geq 4\delta(G) + 2r - 8 - |S| \\ &\geq 4\delta(G) + 2r - 8 - (\delta(G) + r - 2) \\ &= 3\delta(G) + r - 6. \end{aligned}$$

In view of $0 \le r \le 1$, this inequality chain shows that $H - S$ consists of exactly two components with vertex sets W_u and W_v such that $u \in W_u$ and $v \in W_v$. In addition, the inequality

$$3\delta(G) - 5 \ge 4\delta(G) + 2r - 8 - |S|$$

leads to $|S| = \delta(G) - 1$ when $r = 1$ and $\delta(G) - 3 \le |S| \le \delta(G) - 2$ when $r = 0$.

Subcase 1.1. Assume that $r = 1$. Then $|S| = \delta(G) - 1$ and therefore $|W_u| = |W_v| = \delta(G) - 2$.

Subcase 1.1.1. Assume that $\delta(G) = 4$. Then $|S| = 3$, $W_u = \{u, u'\}$ and $W_v = \{v, v'\}$. Because $\delta(H) \ge 4$, each vertex of $\{u, u', v, v'\}$ is adjacent to each vertex in S. Hence G contains a $K_{2,4}$ as a subgraph, a contradiction to the hypothesis.

Subcase 1.1.2. Assume that $\delta(G) = 5$. Then $|S| = 4$ and $|W_u| = |W_v| = 3$. Because $\delta(H) \ge 5$, each vertex of $W_u \cup W_v$ is adjacent to at least three vertices in S. Hence there exist at least two vertices w and z in S such that w has 6 neighbors in $W_u \cup W_v$ and z has 4 neighbors in $W_u \cup W_v$ or w has 5 neighbors in $W_u \cup W_v$ and z has 5 neighbors in $W_u \cup W_v$. In both cases G contains a $K_{2,4}$ as a subgraph, a contradiction.

Subcase 1.1.3. Assume that $\delta(G) = 6$. Then $|S| = 5$ and $|W_u| = |W_v| = 4$.

Assume first that W_u contains a vertex w adjacent to all vertices in S. If there exists a vertex $w' \in W_u - \{w\}$ with 4 neighbors in S, then G contains a $K_{2,4}$ as a subgraph, a contradiction. If each vertex in $W_u - \{w\}$ has at most 3 neighbors in S, then $G[W_u]$ is isomorphic to the complete graph K_4. Now an arbitrary vertex $w' \in W_u - \{w\}$ and w have two common neighbors in W_u and at least 3 common neighbors in S, a contradiction.

Assume secondly that each vertex of W_u has at most 4 neighbors in S. Then $G[W_u]$ is either a cycle C_4, a diamond or a K_4. In the first two cases there are two vertices w and z in W_u sharing two neighbors in W_u and at least 3 in S, a contradiction. In the last case every two vertices in W_u have two common neighbors in W_u, and since every vertex of W_u has at least 3 neighbors in S, it is easy to see that G contains a $K_{2,4}$ as a subgraph, a contradiction.

Subcase 1.1.4. Assume that $\delta(G) \ge 7$. Then $|W_u| \ge 5$. Let $w_1, w_2, w_3 \in W_u$ be three pairwise distinct vertices. Since G is $K_{2,4}$-free and $\delta(H) = \delta(G)$, it is straightforward to verify that $|N_H[\{w_1, w_2, w_3\}]| \ge 3\delta(G) - 9$. We deduce that

$$3\delta(G) - 5 = n(G) \ge |N_H[\{w_1, w_2, w_3\}]| + |W_v| \ge 4\delta(G) - 11,$$

and we obtain the contradiction $\delta(G) \le 6$.

Subcase 1.2. Assume that $r = 0$ and $|S| = \delta(G) - 3$. Then $|W_u| = |W_v| = \delta(G) - 1$.

Subcase 1.2.1. Assume that $\delta(G) = 4$. Then $|S| = 1$ and $|W_u| = 3$. However, this is impossible, since $d_H(u') \ge 4$ for $u' \in (W_u - \{u\})$.

Subcase 1.2.2. Assume that $\delta(G) = 5$. Then $|S| = 2$ and $|W_u| = |W_v| = 4$. Hence every vertex in $(W_u \cup W_v) - \{u, v\}$ is adjacent to every vertex in S. So G contains a $K_{2,4}$ as a subgraph, a contradiction.

Subcase 1.2.3. Assume that $\delta(G) \geq 6$. Then $|W_u| \geq 5$. Let $w_1, w_2, w_3 \in (W_u - \{u\})$ be three pairwise distinct vertices. Since G is $K_{2,4}$-free and $d_H(w_i) \geq \delta(G) \geq 6$ for $1 \leq i \leq 3$, we conclude that $|N_H[\{w_1, w_2, w_3\}]| \geq 3\delta(G) - 9$. This yields the contradiction

$$3\delta(G) - 5 = n(G) \geq |N_H[\{w_1, w_2, w_3\}]| + |W_v| \geq 4\delta(G) - 10 \geq 3\delta(G) - 4.$$

Subcase 1.3. Assume that $r = 0$ and $|S| = \delta(G) - 2$. Then, without loss of generality, $|W_u| = \delta(G) - 2$ and $|W_v| = \delta(G) - 1$.

Subcase 1.3.1. Assume that $\delta(G) = 4$. Then $|S| = 2$ and $|W_u| = 2$. However, this is impossible, since $d_H(u') \geq 4$ for $u' \in (W_u - \{u\})$.

Subcase 1.3.2. Assume that $\delta(G) = 5$. Then $|S| = |W_u| = 3$. If $W_u = \{u, u', u''\}$, then u' as well as u'' is adjacent to every vertex in $S \cup \{u\}$. So G contains a $K_{2,4}$ as a subgraph, a contradiction.

Subcase 1.2.3. Assume that $\delta(G) \geq 6$. Then $|W_u| \geq 4$. Let $w_1, w_2, w_3 \in (W_u - \{u\})$ be three pairwise distinct vertices. Since G is $K_{2,4}$-free and $d_H(w_i) \geq \delta(G) \geq 6$ for $1 \leq i \leq 3$, it follows that $|N_H[\{w_1, w_2, w_3\}]| \geq 3\delta(G) - 9$. Therefore we obtain the contradiction

$$3\delta(G) - 5 = n(G) \geq |N_H[\{w_1, w_2, w_3\}]| + |W_v| \geq 4\delta(G) - 10 \geq 3\delta(G) - 4.$$

Case 2. Assume that $uv \notin E(G)$. Now Theorem 6(1) implies $r = 0$. So there exists a vertex set $S \subset V(G)$ with $|S| \leq \delta(G) - 1$ that separates u and v in G. Hence there is a vertex $u' \in V(G) - S$ adjacent to u as well as a vertex $v' \in V(G) - S$ adjacent to v. Since G is $K_{2,4}$-free, we deduce that $|N_G[\{u, u'\}]| \geq 2\delta(G) - 3$ as well as $|N_G[\{v, v'\}]| \geq 2\delta(G) - 3$. Thus we obtain

$$\begin{aligned} n(G) &= 3\delta(G) - 5 \\ &\geq |N_G[\{u, u'\}]| + |N_G[\{v, v'\}]| - |S| \\ &\geq 4\delta(G) - 6 - |S| \\ &\geq 4\delta(G) - 6 - (\delta(G) - 1) \\ &= 3\delta(G) - 5. \end{aligned}$$

This shows that $G - S$ consists of exactly two components with vertex sets W_u and W_v such that $u \in W_u$ and $v \in W_v$, $|S| = \delta(G) - 1$ and $|W_u| = |W_v| = \delta(G) - 2$.

Subcase 2.1. Assume that $\delta(G) = 4$. Then $|S| = 3$ and $|W_u| = |W_v| = 2$. This implies that each vertex of $W_u \cup W_v$ is adjacent to each vertex in S. Hence G contains a $K_{2,4}$ as a subgraph, a contradiction.

Subcase 2.2. Assume that $\delta(G) = 5$. Then $|S| = 4$ and $|W_u| = |W_v| = 3$. Now we have the same situation as in Subcase 1.1.2. Hence G contains a $K_{2,4}$ as a subgraph, a contradiction.

Subcase 2.3. Assume that $\delta(G) = 6$. Then $|S| = 5$ and $|W_u| = |W_v| = 4$. Now we have the same situation as in Subcase 1.1.3. Hence G contains a $K_{2,4}$ as a subgraph, a contradiction.

Subcase 2.4. Assume that $\delta(G) \geq 7$. Then $|W_u| \geq 5$. Let $w_1, w_2, w_3 \in W_u$ be three pairwise distinct vertices. Since G is $K_{2,4}$-free, we observe that

$$|N_G[\{w_1, w_2, w_3\}]| \geq 3\delta(G) - 9,$$

and we arrive at the contradiction

$$3\delta(G) - 5 = n(G) \geq |N_G[\{w_1, w_2, w_3\}]| + |W_v| \geq 4\delta(G) - 11 \geq 3\delta - 4. \qquad \square$$

Combining Theorem 7 with Observation 1, we obtain the next result immediately.

Corollary 8 *Let G be a connected $K_{2,4}$-free graph with minimum degree $\delta \geq 3$. If $n(G) \leq 3\delta(G) - 5$, then G is maximally connected.*

The example in Figure 3 demonstrates that the bound given in Theorem 7 as well as in Corollary 8 is best possible, at least for $\delta = 4$.

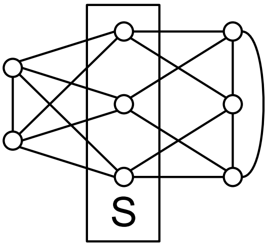

Figure 3: $K_{2,4}$-free graph with $\delta = 4$ and $n = 3\delta - 4 = 8$ vertices which is not maximally (local) connected.

3 C_4-free graphs

In 2007, Dankelmann, Hellwig and Volkmann [2] presented the following sufficient condition for C_4-free graphs to be maximally connected.

Theorem 9 (Dankelmann, Hellwig and Volkmann [2] 2007) *Let G be a connected C_4-free graph of order n and minimum degree $\delta \geq 2$. If*

$$n \leq \begin{cases} 2\delta^2 - 3\delta + 2 & \text{if } \delta \text{ is even,} \\ 2\delta^2 - 3\delta + 4 & \text{if } \delta \text{ is odd,} \end{cases}$$

then G is maximally connected.

Using Theorem 9, we will prove a similar result for C_4-free graphs to be maximally local connected.

Theorem 10 *Let G be a connected C_4-free graph of order n, minimum degree $\delta \geq 3$, $u, v \in V(G)$ and $r = \min\{d(u), d(v)\} - \delta$. If*

$$n \leq \begin{cases} 2\delta^2 - 5\delta + 6 - r & \text{if } uv \notin E(G), \\ 2\delta^2 - 5\delta + 7 - r & \text{if } uv \in E(G), \end{cases}$$

then $\kappa(u, v) = \delta + r$.

Proof. Case 1. Assume that $uv \notin E(G)$. Suppose to the contrary that $\kappa(u, v) \leq \delta + r - 1$. Then there exists a vertex set $S \subset V(G)$ with $|S| \leq \delta + r - 1$ that separates u and v. Let W_u and W_v be the vertex sets of the components of $G - S$ such that $u \in W_u$ and $v \in W_v$.

Suppose that $|N(z) \cap W_u| \leq \delta - 2$ for all vertices $z \in W_u$. Then $|N(u) \cap S| \geq r + 2$ and $|N(z) \cap S| \geq 2$ for all $z \in W_u - \{u\}$. Now we choose a vertex $w \in W_u - \{u\}$ such that $|N(w) \cap S| = x$ is minimal. Since G is C_4-free, each vertex in $W_u - \{u\}$ can have at most one neighbor in $N(u) \cap S$. Hence $2 \leq x \leq \delta - 2$.

Assume first that $uw \in E(G)$. Then w has at least $\delta - x - 1$ neighbors in $W_u - \{u\}$, and at least $x - 1$ neighbors in $S - (N(u) \cap S)$. In addition, each neighbor of w in $W_u - \{u\}$ has no neighbor in $N(u) \cap S$ and at least $x - 1$ further neighbors in $S - (N(u) \cap S)$. Therefore we obtain

$$(\delta - x) \cdot (x - 1) \leq |S| - |N(u) \cap S| \leq \delta + r - 1 - (r + 2) = \delta - 3.$$

We deduce that

$$\delta(x - 2) \leq x^2 - x - 3, \tag{2}$$

a contradiction for $x = 2$. If $x \geq 3$, then (2) leads to the contradiction

$$\delta \leq \frac{x^2 - x - 3}{x - 2} = x + 1 - \frac{1}{x - 2} \leq x + 1 \leq \delta - 1.$$

Assume secondly that $uw \notin E(G)$. Then w has at least $\delta - x$ neighbors in $W_u - \{u\}$, and at least $x - 1$ neighbors in $S - (N(u) \cap S)$. In addition, each neighbor of w in $W_u - \{u\}$ has at least $x - 2$ further neighbors in $S - (N(u) \cap S)$. This leads to

$$(\delta - x + 1) \cdot (x - 2) + 1 \leq |S| - |N(u) \cap S| \leq \delta + r - 1 - (r + 2) = \delta - 3.$$

We deduce that

$$\delta(x - 3) \leq x^2 - 3x - 2. \tag{3}$$

If $x \geq 4$, then (3) yields the contradiction

$$\delta \leq \frac{x^2 - 3x - 2}{x - 3} = x - \frac{2}{x - 3} \leq x \leq \delta - 2.$$

In the case $x = 2$, we observe that w has at least 1 neighbor in $S - (N(u) \cap S)$, and each neighbor of w in $W_u - \{u\}$ has at least 1 further neighbor in $S - (N(u) \cap S)$, with at most one possible exception. So we obtain the contradiction

$$\delta - 2 = \delta - x \leq |S| - |N(u) \cap S| \leq \delta + r - 1 - (r + 2) = \delta - 3.$$

In the remaining case $x = 3$, we see that w has at least 2 neighbors in $S - (N(u) \cap S)$, and each neighbor of w in $W_u - \{u\}$ has at least 2 further neighbors in $S - (N(u) \cap S)$, with at most two possible exceptions, where there only exists at least 1 further neighbor. It follows that

$$2(\delta - x + 1) - 2 = 2(\delta - 2) - 2 \leq |S| - |N(u) \cap S| \leq \delta + r - 1 - (r + 2) = \delta - 3,$$

and we arrive at the contradiction $2\delta - 6 \leq \delta - 3$ when $\delta \geq 4$. If $\delta = 3$, then we obtain the contradiction

$$2(\delta - x + 1) = 2 \leq |S| - |N(u) \cap S| \leq \delta + r - 1 - (r + 2) = \delta - 3.$$

Consequently, there exists a vertex $w \in W_u$ such that $|N(w) \cap W_u| \geq \delta - 1$. Since G is C_4-free, each vertex in $N(w) \cap W_u$ can have at most one neighbor in $N(w)$. This leads to

$$\begin{aligned} |N[N[w] \cap W_u]| &\geq |N(w) \cap W_u| \cdot (\delta - 2) + |N[w]| \\ &\geq (\delta - 1) \cdot (\delta - 2) + \delta + 1 \\ &= \delta^2 - 2\delta + 3. \end{aligned}$$

Analogously, we obtain $|N[N[w'] \cap W_v]| \geq \delta^2 - 2\delta + 3$ for a vertex $w' \in W_v$ and therefore we arrive at the contradiction

$$n \geq |N[N[w] \cap W_u]| + |N[N[w'] \cap W_v]| - |S| \geq 2\delta^2 - 5\delta + 7 - r.$$

Case 2. Assume that $uv \in E(G)$. If $r = 0$, then the result follows directly from Theorem 9, since $n \leq 2\delta^2 - 5\delta + 7 \leq 2\delta^2 - 3\delta + 2$ for $\delta \geq 3$.

If $r \geq 1$, then we define the graph H by $H = G - uv$. We note that $\delta(H) = \delta(G) = \delta$ and $s = \min\{d_H(u), d_H(v)\} - \delta = r - 1$. Therefore the hypothesis leads to $n \leq 2\delta^2 - 5\delta + 7 - r = 2\delta^2 - 5\delta + 6 - s$. Applying Case 1, we deduce that $\kappa_H(u, v) = \delta + s$, and hence we finally obtain $\kappa_G(u, v) = \delta + s + 1 = \delta + r$. □

Theorem 11 *Let G be a connected C_4-free graph of order n and minimum degree $\delta \geq 3$. If*

$$n \leq 2\delta^2 - 6\delta + 10 - \frac{5}{\delta},$$

then G is maximally local connected.

Proof. Let Δ be the maximum degree of G, and let w be a vertex with $d(w) = \Delta$. Since G is C_4-free, the neighbors of w cannot have common neighbors. Hence $n \geq |N[N(w)]| \geq \Delta(\delta - 2) + \Delta + 1$ and thus $\Delta \leq \frac{n-1}{\delta-1}$. In order to ensure the maximally local connectivity of G, we will show that $\kappa(u, v) = \delta + r$ with $r = \min\{d(u), d(v)\} - \delta$ for all distinct vertices u and v in G. We observe that

$$r \leq \Delta - \delta = \Delta - \frac{\delta^2 - \delta}{\delta - 1} \leq \frac{n - \delta^2 + \delta - 1}{\delta - 1},$$

and this leads to

$$\begin{aligned} 2\delta^2 - 5\delta + 6 - r &\geq 2\delta^2 - 5\delta + 6 + \frac{\delta^2 - \delta - n + 1}{\delta - 1} \\ &= \frac{2\delta^3 - 6\delta^2 + 10\delta - 5}{\delta - 1} - \frac{n}{\delta - 1}. \end{aligned}$$

Now

$$\frac{2\delta^3 - 6\delta^2 + 10\delta - 5}{\delta - 1} - \frac{n}{\delta - 1} \geq n$$

is equivalent with the hypothesis

$$n \leq 2\delta^2 - 6\delta + 10 - \frac{5}{\delta},$$

and therefore Theorem 10 shows that G is maximally local connected. □

References

[1] C. Balbuena, M. Cera, A. Diánez, P. García-Vázquez and X. Marcote, Connectivity of graphs with given girth pair, *Discrete Math.* **307** (2007), 155–162.

[2] P. Dankelmann, A. Hellwig and L. Volkmann, On the connectivity of diamond-free graphs, *Discrete Appl. Math.* **155** (2007), 2111–2117.

[3] A.H. Esfahanian, Lower bounds on the connectivity of a graph, *J. Graph Theory* **9** (1985), 503–511.

[4] J. Fàbrega and M.A. Fiol, Maximally connected digraphs, *J. Graph Theory* **13** (1989), 657–668.

[5] J. Fàbrega and M.A. Fiol, Bipartite graphs and digraphs with maximum connectivity. *Discrete Appl. Math.* **69** (1996), 271–279.

[6] J. Fàbrega and M.A. Fiol, Further topics in connectivity, in "Handbook of Graph Theory" (J.L. Gross, J. Yellen, Eds.), CRC Press, Boca Raton, FL (2004), 300–329.

[7] M.A. Fiol, The connectivity of large digraphs and graphs, *J. Graph Theory* **17** (1993), 31–45.

[8] A. Hellwig and L. Volkmann, On connectivity in graphs with given clique number, *J. Graph Theory* **52** (2006), 7–14.

[9] A. Hellwig and L. Volkmann, Maximally edge-connected and vertex-connected graphs and digraphs: A survey, *Discrete Math.* **308** (2008), 3265–3296.

[10] A. Holtkamp and L. Volkmann, On the connectivity of p-diamond-free graphs, *Discrete Math.* **309** (2009), 6065–6069. doi: `10.1016/j.disc.2009.05.009`.

[11] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* **10** (1927), 96–115.

[12] T. Soneoka, H. Nakada, M. Imase and C. Peyrat, Sufficient conditions for maximally connected dense graphs, *Discrete Math.* **63** (1987), 53–66.

[13] J. Topp and L. Volkmann, Sufficient conditions for equality of connectivity and minimum degree of a graph, *J. Graph Theory* **17** (1993), 695–700.

(Received 18 Jan 2010; revised 20 May 2011)