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Ordering of the signless Laplacian spectral radii of unicyclic graphs

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Abstract

For $n \geq 11$, we determine all the unicyclic graphs on n vertices whose signless Laplacian spectral radius is at least $n-2$. There are exactly sixteen such graphs and they are ordered according to their signless Laplacian spectral radii.

1 Introduction

Throughout the paper, $G = (V, E)$ is a connected undirected simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$, i.e., $|V| = n$ and $|E| = m$. If $m = n$, then G is called a *unicyclic graph*. The symbol $\mathbb{U}_n$ is used to denote the set of unicyclic graphs of order n. The set of neighbors of a vertex v is denoted by $N(v)$. Write $d(v)$ for the degree of vertex v. Specially, Δ denotes the *maximum degree* of G.

The adjacency matrix $A(G) = [a_{ij}]$ of G is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if v_i and v_j are joined by an edge. Suppose the degree of vertex v_i equals $d(v_i)$ for $i = 1, 2, \ldots, n$, and let $D(G)$ be the diagonal matrix whose (i,i)-entry is $d(v_i)$. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. The *signless Laplacian characteristic polynomial* of G is denoted by $\Phi(G, \lambda)$, i.e., $\Phi(G, \lambda) = \det(\lambda I - Q(G))$. The maximum eigenvalue of $Q(G)$, denoted by $\mu(G)$, is called the *signless Laplacian spectral radius* of G. The notation $\lambda(G)$, called the *Laplacian spectral radius* of G, is used to denoted the maximum eigenvalue of $L(G)$.

Our terminology and notation are standard except as indicated. For terminology and notation not defined here, we refer the reader to [1, 2, 20].

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It is well-known that graph spectra have important applications in many fields. Several graph spectra, i.e., spectra of $A(G)$, $L(G)$ and $Q(G)$, have been defined in [2]. The spectra of $A(G)$, $L(G)$ are well studied (for instance, see [1, 2, 5, 20]), but the spectrum of $Q(G)$ seems to be less well known. It is not until recently that some researchers found that the spectrum of $Q(G)$ has a strong connection with the structure of a graph (see [6, 11]). For new results on the signless Laplacian spectrum, one can refer to [3, 4, 21].

The largest spectral radius of $A(G)$ in the class of unicyclic graphs on n vertices was firstly determined in [12]. Following this, Guo [8] determined the first six spectral radii of $A(G)$ in the class of unicyclic graphs on n vertices. After this, the first three largest spectral radii of $A(G)$ in the class of bicyclic graphs on n vertices were given in [9]. Very recently, Liu et al. [17] have obtained the largest spectral radius and the minimal least eigenvalue of $A(G)$ in the class of tricyclic graphs of order n. Similarly, the researchers have investigated the spectral radius of $L(G)$ in the class of unicyclic, bicyclic and tricyclic graphs. Up to now, the first to thirteenth largest spectral radii of $L(G)$ have been obtained in the class of unicyclic graphs on n vertices in [7, 15, 16]. After this, He et al. [10] obtained the first four largest spectral radii of $L(G)$ in the class of bicyclic graphs of order n. In recent work [22], using different methods, we have determined the first eight largest spectral radii of $L(G)$ in the class of bicyclic graphs of order n. Very recently, the first nineteen largest Laplacian spectral radii in the class of tricyclic graphs on n vertices were given in [19]. Motivated by the recent results of spectral radius on $A(G)$ and $L(G)$, we shift our goals to the investigation of the spectral radius on $Q(G)$. Recently, Liu et al. [18] have determined the first four largest spectral radii of $Q(G)$ in the class of unicyclic graphs on n vertices. Moreover, the first two largest spectral radii of $Q(G)$ in the class of bicyclic graphs on n vertices, and the first four largest spectral radii of $Q(G)$ in the class of tricyclic graphs of order n have been identified in [13]. Moreover, Wei et al. [23] obtained the third to eleventh largest spectral radii of signless Laplacian matrix in the class of bicyclic graphs of order n. Following the work of [18], we determine all the unicyclic graphs on n $(n \geq 11)$ vertices whose signless Laplacians spectral radius is at least $n-2$. There are exactly sixteen such graphs and they share the first to sixteenth largest spectral radii of signless Laplacian matrix in the class of unicyclic graphs on $n \geq 11$ vertices.

2 Main results

In the following, let $H_1, H_2, \ldots, H_{16}$ be the unicyclic graphs on $n \geq 11$ vertices as showed in Figure 1. We list some known results which will be used in the sequel.

Lemma 2.1 [5] *$\mu(G) \leq max\{d(v)+m(v) : v \in V\}$, where $m(v) = \sum\limits_{u \in N(v)} d(u)/d(v)$.*

Lemma 2.2 [20] *If G is a graph with at least one edge, then $\Delta + 1 \leq \lambda(G) \leq n$. Moreover, if G is connected, the left equality holds if and only if $\Delta = n-1$.*

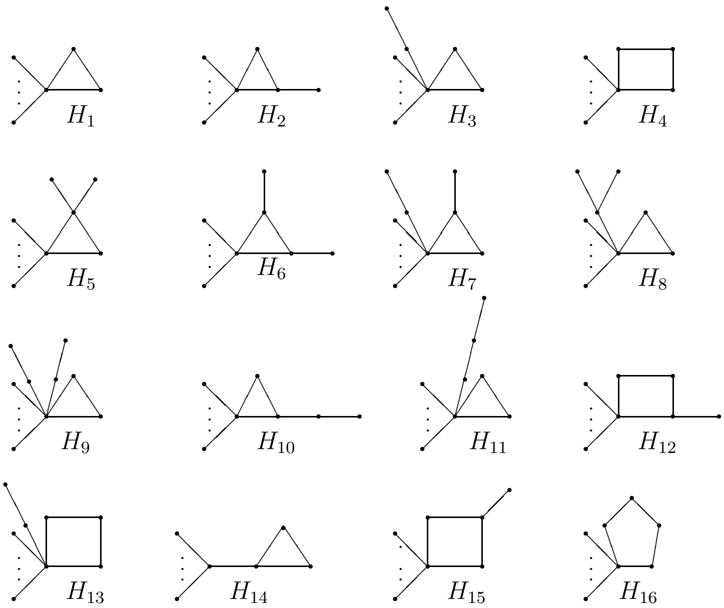

Fig. 1. The unicyclic graphs with $\Delta \geq n-3$.

Lemma 2.3 [20] *$\lambda(G) \leq \mu(G)$, and the equality holds if and only if G is bipartite.*

Lemma 2.4 [18] *If $n \geq 8$, then $\mu(H_1) > \mu(H_2) > \mu(H_3) > \mu(H_4) > n-1$.*

Lemma 2.5 *Suppose G is a unicyclic graph with $\Delta \leq n-4$. If $n > 10$, then $\mu(G) < n-2$.*

Proof. By Lemma 2.1, we only need to prove that $\max\{d(v)+m(v) : v \in V\} < n-2$.

Suppose $\max\{d(v)+m(v) : v \in V\}$ occurs at the vertex u. Three cases arise: $d(u) = 1$, $d(u) = 2$, or $3 \leq d(u) \leq \Delta$.

Case 1. $d(u) = 1$.
Suppose $v \in N(u)$. Since $d(v) \leq \Delta \leq n-4$, we have $d(u)+m(u) = d(u)+d(v) \leq n-3 < n-2$.

Case 2. $d(u) = 2$.
Suppose $v, w \in N(u)$. Note that $G \in \mathbb{U}_n$, so then $|N(v) \cap N(w)| \leq 2$ and $|N(v) \cup N(w)| \leq n$. Therefore

$$d(u)+m(u) \ \leq \ 2+\frac{d(v)+d(w)}{2} \ \leq \ 2+\frac{n+2}{2} \ < \ n-2.$$

Case 3. $3 \le d(u) \le \Delta$.
Note that $\Delta \le n-4$, so that

$$d(u)+m(u) \ \le \ d(u)+\frac{2m-d(u)-3}{d(u)} \ = \ d(u)-1+\frac{2n-3}{d(u)}.$$

Let $f(x)=x-1+(2n-3)/x$, where $3 \le x \le n-4$.

If $3 \le x \le \sqrt{2n-3}$, since $f'(x) \le 0$, then $f(x) \le f(3) = 2+(2n-3)/3 < n-2$. If $\sqrt{2n-3} \le x \le n-4$, since $f'(x) \ge 0$, then $f(x) \le f(n-4) = n-5+\frac{2n-3}{n-4} < n-2$.

Recall that $3 \le d(u) \le \Delta \le n-4$, so that $d(u)+m(u) < n-2$.

By combining the above arguments, this completes the proof. ∎

In a similar fashion we have:

Remark 1. Suppose G is a unicyclic graph with $\Delta \le n-3$. If $n > 8$, then $\mu(G) < n-1$.

Theorem 2.1 *If $G \in \mathbb{U}_n$ and $n \ge 11$, then $n-2 < \mu(G) < n-1$ if and only if $G \cong H_i$, where $5 \le i \le 16$.*

Proof. If $\Delta = n-1$ or $n-2$, then $G \cong H_i$, where $1 \le i \le 4$. By Lemmas 2.2–2.3, we have $\mu(H_i) > n-1$ holding for $1 \le i \le 4$. If $\Delta \le n-4$, by Lemma 2.5 it follows that $\mu(G) < n-2$. Thus, if $n-2 < \mu(G) < n-1$, we have $\Delta(G) = n-3$, i.e., G should be one of the graphs H_5–H_{16}. On the contrary, if G is one of the graphs $\{H_5, \dots, H_{16}\}$, then $n-2 < \mu(G) < n-1$ follows from Remark 1 and Lemmas 2.2–2.3. ∎

Suppose B is a square matrix. Let $a_{ii}(B)$ be the entry appearing in the i-th row and the i-th column of B. The next result gives a method to calculate the signless Laplacian characteristic polynomial of an n-vertex graph.

Lemma 2.6 [14] *Let G be a graph on $n-k$ $(1 \le k \le n-2)$ vertices with $V(G) = \{v_n, v_{n-1}, \dots, v_{k+1}\}$. If G' is obtained from G by attaching k new pendant vertices, say $v_1, \dots, v_k$, to v_{k+1}, then*

$$\Phi(Q(G'), \lambda) = (\lambda-1)^k \cdot \det(\lambda I_{n-k} - Q(G) - B_{n-k}),$$

where $a_{11}(Q(G))$ is corresponding to the vertex v_{k+1}, and $B_{n-k} = diag\{k+\frac{k}{\lambda-1}, 0, \dots, 0\}$.

By Lemma 2.6 and the application of "MATLAB", it easily follows that

(1a) $\Phi(H_5, \lambda) = (\lambda-1)^{n-5}(\lambda^5-(n+5)\lambda^4+(7n-3)\lambda^3-(11n-13)\lambda^2+(3n+8)\lambda-4)$.

(2a) $\Phi(H_6, \lambda) = (\lambda-1)^{n-6}(\lambda^2-3\lambda+1)(\lambda^4-(n+3)\lambda^3+(5n-7)\lambda^2-3n\lambda+4)$.

(3a) $\Phi(H_7, \lambda) = (\lambda-1)^{n-7}(\lambda^7-(n+7)\lambda^6+(9n+10)\lambda^5-(28n-14)\lambda^4+(36n-22)\lambda^3-(18n+18)\lambda^2+(3n+20)\lambda-4)$.

(4a) $\Phi(H_8, \lambda) = (\lambda-1)^{n-5}(\lambda^5-(n+5)\lambda^4+(7n-1)\lambda^3-(13n-17)\lambda^2+(3n+16)\lambda-4)$.

(5a) $\Phi(H_9, \lambda) = (\lambda-1)^{n-7}(\lambda^2-3\lambda+1)(\lambda^5-(n+4)\lambda^4+(6n-2)\lambda^3-(10n-11)\lambda^2+(3n+12)\lambda-4)$.

(6a) $\Phi(H_{10}, \lambda) = (\lambda-1)^{n-6}(\lambda^6-(n+6)\lambda^5+(8n+4)\lambda^4-(20n-18)\lambda^3+(17n-10)\lambda^2-(3n+16)\lambda+4)$.

(7a) $\Phi(H_{11}, \lambda) = (\lambda-1)^{n-6}(\lambda^6-(n+6)\lambda^5+(8n+5)\lambda^4-(21n-18)\lambda^3+(19n-10)\lambda^2-(3n+24)\lambda+4)$.

(8a) $\Phi(H_{12}, \lambda) = (\lambda-1)^{n-5}\lambda(\lambda^4-(n+5)\lambda^3+(7n-1)\lambda^2-(13n-19)\lambda+4n)$.

(9a) $\Phi(H_{13}, \lambda) = (\lambda-1)^{n-7}(\lambda-2)\lambda(\lambda^5-(n+5)\lambda^4+(7n+1)\lambda^3-(15n-17)\lambda^2+(10n-8)\lambda-2n)$.

(10a) $\Phi(H_{14}, \lambda) = (\lambda-1)^{n-4}(\lambda^4-(n+4)\lambda^3+(6n-5)\lambda^2-(7n-12)\lambda+4)$.

(11a) $\Phi(H_{15}, \lambda) = (\lambda-1)^{n-6}\lambda(\lambda-2)(\lambda^4-(n+4)\lambda^3+(6n-4)\lambda^2-(8n-12)\lambda+2n)$.

(12a) $\Phi(H_{16}, \lambda) = (\lambda-1)^{n-6}(\lambda^2-3\lambda+1)(\lambda^4-(n+3)\lambda^3+(5n-5)\lambda^2-(5n-8)\lambda+4)$.

Theorem 2.2 *If $n \geq 11$ and $G \in \mathbb{U}_n \setminus \{H_1, H_2, \ldots, H_{16}\}$, then $\mu(G) < \mu(H_{16}) < \mu(H_{15}) < \cdots < \mu(H_2) < \mu(H_1)$.*

Proof. By Lemmas 2.4–2.5 and Theorem 2.1, we only need to show that $\mu(H_{i+1}) < \mu(H_i)$ for $5 \leq i \leq 15$. We shall divide the proof into the following eleven steps.

(1) $\mu(H_5) > \mu(H_6)$.

Rewrite equality (1a) as $\Phi(H_5, \lambda) = (\lambda-1)^{n-6}f_1(\lambda)$ and (2a) as $\Phi(H_6, \lambda) = (\lambda-1)^{n-6}f_2(\lambda)$, where $f_1(\lambda) = \lambda^6-(n+6)\lambda^5+(8n+2)\lambda^4-(18n-16)\lambda^3+(14n-5)\lambda^2-(3n+12)\lambda+4$, and $f_2(\lambda) = \lambda^6-(n+6)\lambda^5+(8n+3)\lambda^4-(19n-18)\lambda^3+(14n-3)\lambda^2-(3n+12)\lambda+4$. Thus, $\mu(H_5)$ and $\mu(H_6)$ equals the maximum root of the equation $f_1(\lambda) = 0$ and $f_2(\lambda) = 0$, respectively. Since $f_2(\lambda)-f_1(\lambda) = \lambda^2(\lambda^2-(n-2)\lambda+2) > 0$ for $\lambda > n-2$, $\mu(H_5) > \mu(H_6)$.

(2) $\mu(H_6) > \mu(H_7)$.

Rewrite equality (2a) as $\Phi(H_6, \lambda) = (\lambda-1)^{n-7}f_3(\lambda)$, where $f_3(\lambda) = \lambda^7-(n+7)\lambda^6+(9n+9)\lambda^5-(27n-15)\lambda^4+(33n-21)\lambda^3-(17n+9)\lambda^2+(3n+16)\lambda-4$. Then, $\mu(H_6)$ equals the maximum root of the equation $f_3(\lambda) = 0$. Let $f_4(\lambda) = \lambda^7-(n+7)\lambda^6+(9n+10)\lambda^5-(28n-14)\lambda^4+(36n-22)\lambda^3-(18n+18)\lambda^2+(3n+20)\lambda-4$. By equality (3a), $\mu(H_7)$ equals the maximum root of the equation $f_4(\lambda) = 0$. Let $f_4(\lambda)-f_3(\lambda) = \lambda\varphi_1(\lambda)$, where $\varphi_1(\lambda) = \lambda^4-(n+1)\lambda^3+(3n-1)\lambda^2-(n+9)\lambda+4$. Note that $\alpha_1 = (3(n+1)+\sqrt{9n^2-54n+33})/12$ is the maximum root of the equation $\varphi_1''(\lambda) = 0$. Since $\alpha_1 < n-2$ and $\lim_{\lambda \longrightarrow +\infty} \varphi_1''(\lambda) = +\infty$, $\varphi_1''(\lambda) > 0$ for $\lambda > n-2$. Thus, $\varphi_1'(\lambda) > \varphi_1'(n-2) = n^3-9n^2+33n-49 > 0$ for $\lambda > n-2$. This implies that

$\varphi_1(\lambda) > \varphi_1(n-2) = 4n^2 - 27n + 42 > 0$ for $\lambda > n-2$. So $f_4(\lambda) > f_3(\lambda)$, where $\lambda > n-2$. Therefore, $\mu(H_6) > \mu(H_7)$.

(3) $\mu(H_7) > \mu(H_8)$.

Rewrite equality (4a) as $\Phi(H_8, \lambda) = (\lambda-1)^{n-7} f_5(\lambda)$, where $f_5(\lambda) = \lambda^7 - (n+7)\lambda^6 + (9n+10)\lambda^5 - (28n-14)\lambda^4 + (36n-19)\lambda^3 - (19n+19)\lambda^2 + (3n+24)\lambda - 4$. Thus, $\mu(H_8)$ equals the maximum root of the equation $f_5(\lambda) = 0$. Let $f_5(\lambda) - f_4(\lambda) = \lambda\varphi_2(\lambda)$, where $\varphi_2(\lambda) = 3\lambda^2 - (n+1)\lambda + 4$. Note that $\alpha_2 = (n+1)/6$ is the root of the equation $\varphi_2'(\lambda) = 0$ and $\lim\limits_{\lambda \longrightarrow +\infty} \varphi_2'(\lambda) = +\infty$. Hence, $\varphi_2'(\lambda) > 0$ for $\lambda > (n+1)/6$. Since $n-2 > (n+1)/6$, $\varphi_2(\lambda) > \varphi_2(n-2) = 2n^2 - 11n + 18 > 0$ for $\lambda > n-2$. Thus, $f_5(\lambda) > f_4(\lambda)$, where $\lambda > n-2$. This implies that $\mu(H_7) > \mu(H_8)$.

(4) $\mu(H_8) > \mu(H_9)$.

Rewrite equality (5a) as $\Phi(H_9, \lambda) = (\lambda-1)^{n-7} f_6(\lambda)$, where $f_6(\lambda) = \lambda^7 - (n+7)\lambda^6 + (9n+11)\lambda^5 - (29n-13)\lambda^4 + (39n-23)\lambda^3 - (19n+29)\lambda^2 + (3n+24)\lambda - 4$. Thus, $\mu(H_9)$ equals the maximum root of the equation $f_6(\lambda) = 0$. Let $f_6(\lambda) - f_5(\lambda) = \lambda^2\varphi_3(\lambda)$, where $\varphi_3(\lambda) = \lambda^3 - (n+1)\lambda^2 + (3n-4)\lambda - 10$. Note that $\alpha_3 = (n+1+\sqrt{n^2-7n+13})/3$ is the maximum root of the equation $\varphi_3'(\lambda) = 0$. Since $\alpha_3 < n-2$ and $\lim\limits_{\lambda \longrightarrow +\infty} \varphi_3'(\lambda) = +\infty$, $\varphi_3'(\lambda) > 0$ for $\lambda > n-2$. Thus, $\varphi_3(\lambda) > \varphi_3(n-2) = 2n - 14 > 0$ for $\lambda > n-2$. This implies that $f_6(\lambda) > f_5(\lambda)$, where $\lambda > n-2$. Therefore, $\mu(H_8) > \mu(H_9)$.

(5) $\mu(H_9) > \mu(H_{10})$.

When $n = 11, 12$, it is straightforward to check that $\mu(H_9) > \mu(H_{10})$. Thus, we suppose $n \geq 13$ in the following. Let $f_7(\lambda) = \lambda^6 - (n+6)\lambda^5 + (8n+4)\lambda^4 - (20n-18)\lambda^3 + (17n-10)\lambda^2 - (3n+16)\lambda + 4$. By equality (6a), $\mu(H_{10})$ equals the maximum root of the equation $f_7(\lambda) = 0$. Let $f_8(\lambda) = \lambda^5 - (n+4)\lambda^4 + (6n-2)\lambda^3 - (10n-11)\lambda^2 + (3n+12)\lambda - 4$. By equality (5a), $\mu(H_9)$ equals the maximum root of the equation $f_8(\lambda) = 0$. It is easy to see that $f_7(\lambda) = f_8(\lambda)(\lambda-2) + \varphi_4(\lambda)$, where $\varphi_4(\lambda) = -2\lambda^4 + (2n+3)\lambda^3 - 6n\lambda^2 + (3n+12)\lambda - 4$. Suppose α_4 is the maximum root of the equation $\varphi_4(\lambda) = 0$. Since $\varphi_4(0) = -4 < 0$, $\varphi_4(\frac{1}{2}) = \frac{n+9}{4} > 0$, $\varphi_4(1) = 9 - n < 0$, $\varphi_4(n-2) = n^3 - 15n^2 + 66n - 84 > 0$, $\varphi_4(n-1) = -n^3 + 18n - 21 < 0$, thus $n-2 < \alpha_4 < n-1$. Then, $\varphi_4(\lambda) < 0$ for $\lambda > \alpha_4$ and $\varphi_4(\lambda) > 0$ for $n-2 < \lambda < \alpha_4$.

Moreover, we have $f_8(\lambda) = \varphi_4(\lambda)(-\frac{\lambda}{2} + \frac{5}{4}) + \varphi_5(\lambda)$, where $\varphi_5(\lambda) = (\frac{n}{2} - \frac{23}{4})\lambda^3 - (n-17)\lambda^2 - (\frac{3n}{4} + 5)\lambda + 1$. Let α_5 be the maximum root of the equation $\varphi_5(\lambda) = 0$. Since $\varphi_5(-8) = -314n + 4073 < 0$, $\varphi_5(0) = 1 > 0$, $\varphi_5(2) = \frac{-3n+26}{2} < 0$, $\varphi_5(3) = \frac{9n-65}{4} > 0$, thus $\alpha_5 < n-2$. Hence, $\varphi_5(\lambda) > 0$ for $\lambda > n-2$. This implies that $f_8(\lambda) > 0$ and $f_7(\lambda) = f_8(\lambda)(\lambda-2) + \varphi_4(\lambda) = \frac{-2\lambda^2+9\lambda-6}{4}\varphi_4(\lambda) + (\lambda-2)\varphi_5(\lambda) > 0$ for $\lambda \geq \alpha_4$. Thus, $n-2 < \mu(H_9)$, $\mu(H_{10}) < \alpha_4$. If $n-2 < \lambda < \alpha_4$, $\varphi_4(\lambda) > 0$, then $f_7(\lambda) = f_8(\lambda)(\lambda-2) + \varphi_4(\lambda) > 0$ for $\lambda \in [\mu(H_9), \alpha_4]$. This implies that $\mu(H_9) > \mu(H_{10})$.

(6) $\mu(H_{10}) > \mu(H_{11})$.

Let $f_9(\lambda) = \lambda^6 - (n+6)\lambda^5 + (8n+5)\lambda^4 - (21n-18)\lambda^3 + (19n-10)\lambda^2 - (3n+24)\lambda + 4$. By equality (7a), $\mu(H_{11})$ equals the maximum root of the equation $f_9(\lambda) = 0$. Let $f_9(\lambda) - f_7(\lambda) = \lambda\varphi_6(\lambda)$, where $\varphi_6(\lambda) = \lambda^3 - n\lambda^2 + 2n\lambda - 8$. Suppose α_6 is the maximum root of the equation $\varphi_6'(\lambda) = 0$. Since $\alpha_6 = (n + \sqrt{n^2 - 6n})/3 < n - 2$ and $\lim_{\lambda \longrightarrow +\infty} \varphi_6'(\lambda) = +\infty$, it follows that $\varphi_6'(\lambda) > 0$ for $\lambda > n - 2$. Then $\varphi_6(\lambda) > \varphi_6(n-2) = 4n - 16 > 0$, and hence $f_9(\lambda) > f_7(\lambda)$ for $\lambda > n - 2$. Therefore, $\mu(H_{10}) > \mu(H_{11})$.

(7) $\mu(H_{11}) > \mu(H_{12})$.

Rewrite equality (8a) as $\Phi(H_{12}, \lambda) = (\lambda - 1)^{n-6} f_{10}(\lambda)$, where $f_{10}(\lambda) = \lambda^6 - (n+6)\lambda^5 + (8n+4)\lambda^4 - (20n-20)\lambda^3 + (17n-19)\lambda^2 - 4n\lambda$. Thus, $\mu(H_{12})$ equals the maximum root of the equation $f_{10}(\lambda) = 0$. Suppose $f_{10}(\lambda) - f_9(\lambda) = -\varphi_7(\lambda)$, where $\varphi_7(\lambda) = \lambda^4 - (n+2)\lambda^3 + (2n+9)\lambda^2 + (n-24)\lambda + 4$. Let α_7 denote the maximum root of the equation $\varphi_7''(\lambda) = 0$. Since $\alpha_7 = (3n + 6 + \sqrt{9n^2 - 12n - 180})/12 < n - 2$ and $\lim_{\lambda \longrightarrow +\infty} \varphi_7''(\lambda) = +\infty$, we have $\varphi_7''(\lambda) > 0$ for $\lambda > n - 2$. Thus, $\varphi_7'(\lambda) > \varphi_7'(n-2) = n^3 - 14n^2 + 71n - 116 > 0$ for $\lambda > n - 2$. This implies that $\varphi_7(\lambda) < \varphi_7(n-1) = -n^3 + 15n^2 - 50n + 40 < 0$. Thus, $f_{10}(\lambda) - f_9(\lambda) = -\varphi_7(\lambda) > 0$ for $\lambda \in (n-2, n-1)$. From the fact that $n - 2 < \mu(H_{11}),\ \mu(H_{12}) < n - 1$, we have $\mu(H_{11}) > \mu(H_{12})$.

(8) $\mu(H_{12}) > \mu(H_{13})$.

By equalities (8a) and (9a), we have $\Phi(H_{12}, \lambda) = (\lambda-1)^{n-7} f_{11}(\lambda)$ and $\Phi(H_{13}, \lambda) = (\lambda-1)^{n-7} f_{12}(\lambda)$, where $f_{11}(\lambda) = \lambda^7 - (n+7)\lambda^6 + (9n+10)\lambda^5 - (28n-16)\lambda^4 + (37n-39)\lambda^3 - (21n-19)\lambda^2 + 4n\lambda$ and $f_{12}(\lambda) = \lambda^7 - (n+7)\lambda^6 + (9n+11)\lambda^5 - (29n-15)\lambda^4 + (40n-42)\lambda^3 - (22n-16)\lambda^2 + 4n\lambda$. Thus, $\mu(H_{12})$ and $\mu(H_{13})$ equals the maximum root of the equation $f_{11}(\lambda) = 0$ and $f_{12}(\lambda) = 0$, respectively. Let $f_{12}(\lambda) - f_{11}(\lambda) = \lambda^2 \varphi_8(\lambda)$, where $\varphi_8(\lambda) = \lambda^3 - (n+1)\lambda^2 + (3n-3)\lambda - (n+3)$. Suppose α_8 is the maximum root of the equation $\varphi_8'(\lambda) = 0$. Since $\alpha_8 = (n + 1 + \sqrt{n^2 - 7n + 10})/3 < n - 2$ and $\lim_{\lambda \longrightarrow +\infty} \varphi_8'(\lambda) = +\infty$, we have $\varphi_8'(\lambda) > 0$ for $\lambda > n - 2$. Then, $\varphi_8(\lambda) > \varphi_8(n-2) = 2n - 9 > 0$ for $\lambda > n - 2$. This implies that $f_{12}(\lambda) > f_{11}(\lambda)$ for $\lambda > n - 2$. Therefore, $\mu(H_{12}) > \mu(H_{13})$.

(9) $\mu(H_{13}) > \mu(H_{14})$.

By equalities (9a) and (10a), we have $\Phi(H_{13}, \lambda) = (\lambda - 1)^{n-7} \lambda(\lambda - 2) f_{13}(\lambda)$ and $\Phi(H_{14}, \lambda) = (\lambda - 1)^{n-5} f_{14}(\lambda)$, where $f_{13}(\lambda) = \lambda^5 - (n+5)\lambda^4 + (7n+1)\lambda^3 - (15n-17)\lambda^2 + (10n-8)\lambda - 2n$ and $f_{14}(\lambda) = \lambda^5 - (n+5)\lambda^4 + (7n-1)\lambda^3 - (13n-17)\lambda^2 + (7n-8)\lambda - 4$. Thus, $\mu(H_{13})$ and $\mu(H_{14})$ equals the maximum root of the equation $f_{13}(\lambda) = 0$ and $f_{14}(\lambda) = 0$, respectively. Then $f_{14}(\lambda) - f_{13}(\lambda) = -\varphi_9(\lambda)$, where $\varphi_9(\lambda) = 2\lambda^3 - 2n\lambda^2 + 3n\lambda - 2n + 4$. Suppose α_9 is the maximum root of the equation $\varphi_9'(\lambda) = 0$. Since $\alpha_9 = (2n + \sqrt{4n^2 - 18n})/6 < n - 2$ and $\lim_{\lambda \longrightarrow +\infty} \varphi_9'(\lambda) = +\infty$, $\varphi_9'(\lambda) > 0$ for $\lambda > n - 2$. Let α_{10} be the maximum root of the equation $\varphi_9(\lambda) = 0$. Note that $\varphi_9(n-2) = -n^2 + 8n - 12 < 0$ and $\varphi_9(n-1) = n^2 - n + 2 > 0$. Thus, $\alpha_{10} \in (n-2, n-1)$. Then $\varphi_9(\lambda) > 0$ for $\lambda \in (\alpha_{10}, n-1)$ and $\varphi_9(\lambda) < 0$ for $\lambda \in (n-2, \alpha_{10})$

It is easy to see that $f_{13}(\lambda) = \frac{1}{2}(\lambda^2 - 5\lambda + \frac{n}{2} + 1)\varphi_9(\lambda) + \varphi_{10}(\lambda)$ and $f_{14}(\lambda) = \frac{1}{2}(\lambda^2 - 5\lambda + \frac{n}{2} - 1)\varphi_9(\lambda) + \varphi_{10}(\lambda)$, where

$$\varphi_{10}(\lambda) = \frac{1}{2}(n^2 - 11n + 30)\lambda^2 + \frac{1}{4}(-3n^2 + 14n + 8)\lambda + \frac{1}{2}(n^2 - 4n - 4).$$

Note that the unique root of $\varphi'_{10}(\lambda) = 0$ is $\dfrac{3n^2 - 14n - 8}{4(n^2 - 11n + 30)} < n - 2$. Thus,

$$\varphi_{10}(\lambda) \ > \ \varphi_{10}(n-2) \ = \ \frac{1}{2}n^4 - \frac{33}{4}n^3 + \frac{89}{2}n^2 - 89n + 54 \ > \ 0$$

for $\lambda > n-2$. Therefore, when $\lambda \geq \alpha_{10} > n-2$, we have $f_{13}(\lambda) > 0$ and $f_{14}(\lambda) > 0$. This implies that $\mu(H_{13})$, $\mu(H_{14}) \in (n-2, \alpha_{10})$. Moreover, since $f_{14}(\lambda) - f_{13}(\lambda) = -\varphi_9(\lambda) > 0$ for $\lambda \in (n-2, \alpha_{10})$, it follows that $\mu(H_{13}) > \mu(H_{14})$.

(10) $\mu(H_{14}) > \mu(H_{15})$.

Let $f_{15}(\lambda) = \lambda^4 - (n+4)\lambda^3 + (6n-5)\lambda^2 - (7n-12)\lambda + 4$ and $f_{16}(\lambda) = \lambda^4 - (n+4)\lambda^3 + (6n-4)\lambda^2 - (8n-12)\lambda + 2n$. By equalities (10a) and (11a), $\mu(H_{14})$ and $\mu(H_{15})$ equals the maximum root of the equation $f_{15}(\lambda) = 0$ and $f_{16}(\lambda) = 0$, respectively. Then $f_{16}(\lambda) - f_{15}(\lambda) = \varphi_{11}(\lambda)$, where $\varphi_{11}(\lambda) = \lambda^2 - n\lambda + 2n - 4$. Since $\varphi'_{11} = 2\lambda - n > 0$, we have $\varphi_{11}(\lambda) > \varphi_{11}(n-2) = 0$ for $\lambda > n-2$. Therefore, $\mu(H_{14}) > \mu(H_{15})$.

(11) $\mu(H_{15}) > \mu(H_{16})$.

Let $f_{17}(\lambda) = \lambda^4 - (n+3)\lambda^3 + (5n-5)\lambda^2 - (5n-8)\lambda + 4$. By equality (12a), $\mu(H_{16})$ equals the maximum root of the equation $f_{17}(\lambda) = 0$. Then $f_{17}(\lambda) - f_{16}(\lambda) = \varphi_{12}(\lambda)$, where $\varphi_{12}(\lambda) = \lambda^3 - (n+1)\lambda^2 + (3n-4)\lambda - 2(n-2)$. Suppose α_{11} is the maximum root of the equation $\varphi'_{12}(\lambda) = 0$. Note that $\alpha_{11} = (n+1+\sqrt{n^2-7n+13})/3 < n-2$ and $\lim_{\lambda \longrightarrow +\infty} \varphi'_{12}(\lambda) = +\infty$. Then, $\varphi'_{12}(\lambda) > 0$ for $\lambda > n-2$. Thus, $\varphi_{12}(\lambda) > \varphi_{12}(n-2) = 0$ for $\lambda > n-2$. This implies that $f_{17}(\lambda) > f_{16}(\lambda)$, where $\lambda > n-2$. Therefore, $\mu(H_{15}) > \mu(H_{16})$.

By combining the above arguments, the result follows. ∎

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