AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **49** (2011), Pages 209–215

Bounds on the arboricities of connected graphs

Araya Chaemchan*

Department of Mathematics and Statistics
Thammasat University
Rangsit, Pathumthani 12120
Thailand
araya@mathstat.sci.tu.ac.th

Abstract

The *vertex* [*edge*] *arboricity* $\mathsf{a}(G)$ $[\mathsf{a}_1(G)]$ of a graph G is the minimum number of subsets into which $V(G)$ $[E(G)]$ can be partitioned so that each subset induces an acyclic subgraph.

Let $\mathcal{G}(m,n)$ be the class of connected simple graphs of order n and size m and let $\pi \in \{\mathsf{a}, \mathsf{a}_1\}$. In this paper we determine

$$\pi(m,n) := \{\pi(G) : G \in \mathcal{G}(m,n)\}$$

for integers m, n such that $n-1 \leq m \leq \binom{n}{2}$.

1 Introduction and Overview

We limit our discussion to graphs that are simple and finite. For the most part, our notation and terminology follow that of Chartrand and Lesniak [2]. In addition we use $\nu(G)$ and $\varepsilon(G)$ for the order and size of a graph G, respectively. For a graph G, it is always possible to partition $V(G)$ into subsets V_i, $1 \leq i \leq k$, such that each induced subgraph $\langle V_i \rangle$ contains no cycle. The *vertex arboricity*, $\mathsf{a}(G)$, of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces an acyclic subgraph. The *edge arboricity* or simply the *arboricity* $\mathsf{a}_1(G)$ of a nonempty graph G is the minimum number of subsets into which $E(G)$ can be partitioned so that each subset induces an acyclic subgraph of G. If $\pi \in \{\mathsf{a}, \mathsf{a}_1\}$ then clearly $\pi(G) = 1$ if and only if G is a forest. Furthermore, $\pi(C_n) = 2$ and $\pi(K_n) = \lceil \frac{n}{2} \rceil$. Also $\mathsf{a}(K_{r,s}) = 1$ if $r = 1$ or $s = 1$ and $\mathsf{a}(K_{r,s}) = 2$ otherwise. It should be noted that if H is a subgraph of a graph G then $\pi(H) \leq \pi(G)$. Also if $G = G_1 \cup G_2$ and $\pi(G) = \max\{\pi(G_1), \pi(G_2)\}$. Thus it is reasonable to work on the class of connected graphs.

* This research was financially supported by Thammasat University.

Let $\mathcal{G}(m,n)$ be the class of connected graphs of order n and size m and let $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$. We consider the problem of determining

$$\pi(m,n) := \{\pi(G) : G \in \mathcal{G}(m,n)\}.$$

2 Interpolation theorems

Let G be a graph with $e \in E(\overline{G})$ and $f \in E(G)$. The edge jump operation $\sigma = \sigma(e,f)$ on G produces the graph $G^\sigma = G + e - f$. Thus G and G^σ have the same order and size. The following theorems are proved in [4].

Theorem 2.1 *If $G, H \in \mathcal{G}(m,n)$, and $G \ncong H$, then there exists a finite sequence of edge jumps $\sigma(e_1,f_1), \sigma(e_2,f_2), \ldots, \sigma(e_t,f_t)$ such that for all $i = 1, 2, \ldots, t$, $G^{\sigma(e_1,f_1)\sigma(e_2,f_2)\ldots\sigma(e_i,f_i)} \in \mathcal{G}(m,n)$ and $H = G^{\sigma(e_1,f_1)\sigma(e_2,f_2)\ldots\sigma(e_t,f_t)}$.*

Theorem 2.2 *Let $\pi \in \{\chi, \omega, \alpha, \alpha_1, \beta, \beta_1, \gamma\}$. If $G \in \mathcal{G}(m,n)$ and σ is an edge jump on G, then $|\pi(G) - \pi(G^\sigma)| \le 1$.*

Combining the two results we have:

Theorem 2.3 *Let $\pi \in \{\chi, \omega, \alpha, \alpha_1, \beta, \beta_1, \gamma\}$. Then there exist nonnegative integers a and b such that there exists a $G \in \mathcal{G}(m,n)$ with $\pi(G) = c$ if and only if c is an integer satisfying $a \le c \le b$.*

As a consequence of Theorem 2.3, if $\pi \in \{\chi, \omega, \alpha, \alpha_1, \beta, \beta_1, \gamma\}$ and $\pi(m,n) := \{\pi(G) : G \in \mathcal{G}(m,n)\}$, then

$$\pi(m,n) = \{x \in \mathbb{Z} : a \le x \le b\}.$$

The problem of determining $\chi(m,n)$ and $\omega(m,n)$ in all situations has been completely resolved in [4].

Let $\mathcal{G}$ be a class of graphs, $\mathcal{J} \subseteq \mathcal{G}$, and π be a graph parameter. Then π is called an *interpolation graph parameter over* $\mathcal{J}$ if there exist nonnegative integers a and b such that

$$\pi(\mathcal{J}) = \{\pi(G) : G \in \mathcal{J}\} = \{x \in \mathbb{Z} : a \le x \le b\}.$$

We will prove in the next theorem that $\mathtt{a}$ and $\mathtt{a}_1$ are interpolation graph parameters over $\mathcal{G}(m,n)$.

Theorem 2.4 *Let $\pi \in \{\mathtt{a}, \mathtt{a}_1\}$. If $G \in \mathcal{G}(m,n)$ and $\sigma = \sigma(e,f)$ is an edge jump on G, then $|\pi(G) - \pi(G^\sigma)| \le 1$.*

Proof. Clearly $\pi(G) \le \pi(G+e) \le \pi(G)+1$ and $\pi(G+e)-1 \le \pi(G+e-f) \le \pi(G+e)$. Therefore $\pi(G) - 1 \le \pi(G + e - f) \le \pi(G) + 1$ and hence $|\pi(G) - \pi(G^\sigma)| \le 1$, as required. ∎

By Theorem 2.4 we have that if $\pi \in \{\mathsf{a}, \mathsf{a}_1\}$, then π is an interpolation graph parameter over $\mathcal{G}(m, n)$. Therefore the bounds of $\pi(G)$ are uniquely determined by $\min(\pi; m, n) := \min\{\pi(G) : G \in \mathcal{G}(m, n)\}$ and $\max(\pi; m, n) := \max\{\pi(G) : G \in \mathcal{G}(m, n)\}$.

Thus it follows that the bounds for $\pi(G)$ are uniquely determined. For $G \in \mathcal{G}(m, n)$, we denote the upper and lower bounds for $\pi(G)$ by $\max(\pi; m, n)$ and $\min(\pi; m, n)$ respectively.

3 Extremal results

Let m and n be positive integers with $n-1 \le m \le \binom{n}{2}$ and $\pi \in \{\mathsf{a}, \mathsf{a}_1\}$. We consider in this section the problem of determining $\min(\pi; m, n)$ and $\max(\pi; m, n)$.

3.1 Vertex arboricity

N. Achuthan, N.R. Achuthan and L. Caccetta proved in [1] the following theorem.

Theorem 3.1 *[1] Let G be a graph of order n and size m. If $\mathsf{a}(G) = p \ge 2$, then $m \ge \binom{2p-1}{2}$. Furthermore, if $m = \binom{2p-1}{2}$, then $G \cong K_{2p-1} \cup \overline{K}_{n-2p+1}$.*

A graph G is said to be *critical with respect to vertex-arboricity* if $\mathsf{a}(G-v) < \mathsf{a}(G)$ for all vertices v of G. It is clear that if $\mathsf{a}(G) = p$ and G is a critical graph with respect to vertex-arboricity then $p \ge 2$, $\mathsf{a}(G - v) = p - 1$ for all vertices v of G, and G is connected. In this subsection, a graph G is *p-critical* if G is a critical graph with respect to vertex-arboricity and $\mathsf{a}(G) = p$. Thus the graph K_{2p-1}, where $p \ge 2$, is p-critical while each cycle is 2-critical.

Let $G \in \mathcal{G}(m, n)$ and $\mathsf{a}(G) = p \ge 2$. Then either G is p-critical or there is a vertex v of G such that $\mathsf{a}(G - v) = p$. Therefore every graph G with $\mathsf{a}(G) = p \ge 2$ contains an induced p-critical subgraph.

The following theorem is cited from [2].

Theorem 3.2 *Let $G \in \mathcal{G}(m, n)$. If G is p-critical, then $m \ge (p - 1)n$.*

Theorem 3.3 *If $G \in \mathcal{G}(m, n)$ with $\mathsf{a}(G) = p \ge 2$, then $m \ge \binom{2p-1}{2} + n - 2p + 1$. Furthermore, the bound is sharp.*

Proof. Let G be a connected graph of order n and size m. Let H be a p-critical subgraph of G. If H is of order h and size ℓ, then, by Theorem 3.2, $\ell \ge (p - 1)h$.

Since $\mathsf{a}(H) = p$, it follows that $h \geq 2p - 1$ and $\ell \geq (p-1)h$. Therefore

$$\begin{aligned} m &\geq \ell + (n-h) \geq (p-1)h + n - h \\ &= (p-1)(2p-1) + (p-1)(h-2p+1) + n - h \\ &\geq \binom{2p-1}{2} + (h-2p+1) + n - h \\ &= \binom{2p-1}{2} + n - 2p + 1. \end{aligned}$$

Let G be a graph of order n obtained from K_{2p-1} and $\overline{K}_{n-2p+1}$ by joining a fixed vertex v of K_{2p-1} to every vertex of $\overline{K}_{n-2p+1}$. Then G is connected graph of order n and size $m = \binom{2p-1}{2} + n - 2p + 1$ with $\mathsf{a}(G) = p$, as required. ∎

It is clear that $\max(\mathsf{a}; m, n) = 1$ if and only if $m = n-1$. Let $p \geq 2$ be an integer. Then we have the following elementary facts:

1. Let e be an edge of K_{2p+1}. Then $\mathsf{a}(K_{2p+1} - e) = p$.

2. Let G be a proper subgraph of K_{2p+1} containing K_{2p-1} as its subgraph. Then $\mathsf{a}(G) = p$.

We have the following result.

Theorem 3.4 *Let $\ell \geq 2$ be an integer and $f(n,\ell) = \binom{2\ell-1}{2} + n - 2\ell + 1$. If $p \geq 2$ is an integer, then* $\max(\mathsf{a}; m, n) = p$ *if and only if* $f(n,p) \leq m < f(n,p+1)$.

We now turn to the problem of determining the maximum number of edges that a graph G of order n and $\mathsf{a}(G) = p$ can have. Let n and p be positive integers with $n \geq 2p+1$.

Let G be a graph of order n, $\mathsf{a}(G) = p \geq 2$ and there is a partition $V_1, V_2, \ldots, V_p$ of $V(G)$ such that the induced subgraph $\langle V_i \rangle$ is acyclic. If $|V_i| = n_i$ for all $i = 1, 2, \ldots, p$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_p$, then

1. $n_i \geq 2$ for all $i = 2, 3, \ldots, p$, and

2. $\varepsilon(G) \leq \varepsilon(K_{n_1,n_2,\ldots,n_p}) + n - p$.

3. Let $T_{n,p}$ be the Turán graph of order n containing no $(p+1)$-clique and $t_{n,p} = \varepsilon(T_{n,p})$ be the size of $T_{n,p}$. Then $\varepsilon(G) \leq t_{n,p} + n - p$.

4. Put $n = pq + t$, $0 \leq t < p$ and $a = \lfloor \frac{n}{p} \rfloor$. Then the Turán graph, $T_{n,p}$, is the complete p-partite graph of order n with t partite sets of cardinality $a+1$ and $p-t$ partite sets of cardinality a. We can construct a graph H of order n with $\mathsf{a}(H) = p$ from $T_{n,p}$ by adding $a-1$ edges to each partite set of cardinality a and a edges to each of cardinality $a+1$. Therefore $\varepsilon(H) = t_{n,p} + n - p$.

We have the following theorems.

Theorem 3.5 *Let $G \in \mathcal{G}(m,n)$. If $\mathtt{a}(G) = p \geq 2$, then $m \leq t_{n,p} + n - p$. Furthermore, the bound is sharp.*

Theorem 3.6 *Let $\ell \geq 2$ be an integer. Put $g(n,1) = n-1$ and $g(n,\ell) = t_{n,\ell}+n-\ell$. If $p \geq 2$ is an integer, then $\min(\mathtt{a}; m,n) = p$ if and only if $g(n,p-1) < m \leq g(n,p)$.*

3.2 Edge arboricity

Unlike vertex-arboricity there is a formula for the arboricity of a graph, which was discovered by Nash-Williams [3], as we state in the following theorems.

Theorem 3.7 *A graph G has $\mathtt{a}_1(G) = p$ if and only if every non-trivial subgraph H has at most $p(\nu(H)-1)$ edges.*

Theorem 3.8 *For every nonempty graph G,*

$$\mathtt{a}_1(G) = \max \left\lceil \frac{\varepsilon(H)}{\nu(H)-1} \right\rceil,$$

where the maximum is taken over all nontrivial induced subgraphs H of G.

It follows as a consequence of Theorem 3.7 that $\mathtt{a}_1(K_n) = \lceil \frac{n}{2} \rceil$ and $\mathtt{a}_1(K_{r,s}) = \lceil \frac{rs}{r+s+1} \rceil$.

In order to obtain the values of $\min(\mathtt{a}_1; m,n)$ and $\max(a_1; m,n)$, we first note that $\min(\mathtt{a}_1; m,n) = 1$ if and only if $m = n-1$, also $\max(\mathtt{a}_1; m,n) = 1$ if and only if $m = n-1$. Thus we assume from now on that $m \geq n$ and therefore $\mathtt{a}_1(G) = p \geq 2$ for $G \in \mathcal{G}(m,n)$.

Any spanning subgraph of a graph G is referred to as a *factor* of G. A k-regular factor is called a *k-factor*. If the order of G is odd, then G can not have a 1-factor. A *near 1-factor* of G of order $2n+1$ is a factor of G having n independent edges and an isolated vertex. A graph G is said to be *factorable* into the factors $G_1, G_2, \ldots, G_t$ if these factors are pairwise edge-disjoint and $\bigcup_{i=1}^{t} E(G_i) = E(G)$. If G is factored into $G_1, G_2, \ldots, G_t$, then we write $G = G_1 \oplus G_2 \oplus \cdots \oplus G_t$, which is called a *factorization* of G. If there exists a factorization of a graph G such that each factor is a k-factor of G, for a fixed k, then G is *k-factorable*. If a graph G is factorable into $G_1, G_2, \ldots, G_t$, where each $G_i \cong H$ for some graph H, then we say that G is *H-factorable*. A *Hamiltonian factorization* of a graph G is a factorization of a graph G such that every factor is a Hamiltonian cycle of G. The following facts are well known (see [2], for example).

Theorem 3.9 *Let k be a positive integer. Then*

1. *the complete graph K_{2k+1} is Hamiltonian factorable,*

2. *the complete graph* K_{2k} *can be factored into* k *Hamiltonian paths,*

3. *the complete graph* K_{2k} *can be factored into* $k-1$ *Hamiltonian cycles and a* 1*-factor, and*

4. *the complete graph* K_{2k-1} *can be factored into* $k-1$ *Hamiltonian paths and a near* 1*-factor.*

Lemma 3.10 *Let* G *be a graph of order* n *and size* m *with* $\mathsf{a}_1(G) = p \geq 2$*. Then* $n \geq 2p-1$ *and* $m \geq 2(p-1)^2+1$*. If* G *is connected, then* $m \geq 2(p-1)^2-2p+n+2$*. Furthermore the bounds are sharp.*

Proof. Clearly $p = \mathsf{a}_1(G) \leq \mathsf{a}_1(K_n) = \lceil \frac{n}{2} \rceil$. Therefore $n \geq 2(p-1)+1 = 2p-1$. By Theorem 3.7, there is a subgraph H of G such that $\mathsf{a}_1(G) = p = \lceil \frac{\varepsilon(H)}{\nu(H)-1} \rceil$. Thus $\nu(H) \geq 2p-1$ and hence $m \geq \varepsilon(H) \geq (p-1)(\nu(H)-1)+1 \geq (p-1)(2p-2) = 2(p-1)^2+1$. Next suppose that G is connected. Then there are at least $n-\nu(H)$ edges from H to $G-H$. Therefore

$$\begin{aligned} m &\geq \varepsilon(H) + n - \nu(H) \\ &\geq (p-1)(\nu(H)-1)+1+n-\nu(H) \\ &\geq p\nu(H) - p - \nu(H) + 1 + 1 + n - \nu(H) \\ &= (p-2)\nu(H) - p + 2 + n. \end{aligned}$$

By replacing $\nu(H)$ by $2p-1$ we get $m \geq 2(p-1)^2 - 2p + n + 2$, as required.

To prove the sharpness, we start with a factorization of K_{2p-1} into $p-1$ Hamiltonian paths $H_1, H_2, \ldots, H_{p-1}$ and a near 1-factor F. Put $H = H_1 \oplus H_2 \oplus \cdots \oplus H_{p-1}$. Then $\mathsf{a}_1(H) = p-1$ and for any e in F we see that $\mathsf{a}_1(H+e) = p$. Thus $H+e$ is a graph of order $2p-1$ and size $2(p-1)^2+1$ with $\mathsf{a}_1(H+e) = p$. By adding $n-2p+1$ isolated vertices to $H+e$ we obtain a graph G of order n, size $2(p-1)^2+1$ and $\mathsf{a}_1(G) = p$. Again by adding $n-2p+1$ edges from the isolated vertices to a fixed vertex of H we obtain a connected graph G' of order n, size $2(p-1)^2-2p+n+2$ and $\mathsf{a}_1(G') = p$. This shows that the bounds are sharp. ∎

Theorem 3.11 *Let* $\ell \geq 2$ *be an integer and* $f_1(n,\ell) = 2(\ell-1)^2 - 2\ell + n + 2$*. If* $p \geq 2$ *is an integer, then* $\max(\mathsf{a}_1; m, n) = p$ *if and only if* $f_1(n,p) \leq m < f_1(n,p+1)$*.*

From Theorem 3.7, it is easy to see that if a graph $G \in \mathcal{G}(m,n)$ then $\mathsf{a}_1(G) = p \geq \lceil \frac{m}{n-1} \rceil$. To show that the bound is sharp, we will show that there exists a graph $H \in \mathcal{G}(m,n)$ such that $\mathsf{a}_1(H) = \lceil \frac{m}{n-1} \rceil$.

We now consider the problem of determining the maximum number of edges that a graph G of order n with $\mathsf{a}_1(G) = p$ can have for two cases according to the parity of n as follows.

Case 1. Let $n = 2k$ be an even integer. The complete graph K_{2k} can be factored into k Hamiltonian paths $H_1, H_2, \ldots, H_k$. Thus $K_{2k} = H_1 \oplus H_2 \oplus \cdots \oplus H_k$.

For $p = 1, 2, \ldots, k$, put $G_p = H_1 \oplus H_2 \oplus \cdots \oplus H_p$. Then G_p is a graph of order n, size $p(n-1)$ and $\mathsf{a}_1(G_p) \leq p$. Since a forest of order n contains at most $n-1$ edges, it follows that $\mathsf{a}_1(G_p) = p$.

Case 2. If $n = 2k - 1$ is an odd integer, then K_{2k-1} can be factored into $k - 1$ Hamiltonian paths $H_1, H_2, \ldots, H_{k-1}$ and a near 1-factor F. Thus $K_{2k-1} = H_1 \oplus H_2 \oplus \cdots \oplus H_{k-1} \oplus F$. For $p = 1, 2, \ldots, k-1$, put $G_p = H_1 \oplus H_2 \oplus \cdots \oplus H_p$. Then G_p is a graph of order n, size $p(n-1)$ and therefore $\mathsf{a}_1(G_p) = p$.

This means that the graph G_p of order n with $\mathsf{a}_1(G_p) = p$ has size $m = p(n-1)$. Equivalently, $p = \lceil \frac{m}{n-1} \rceil$.

Then we have the following theorem.

Theorem 3.12 $$\min(\mathsf{a}_1; m, n) = \left\lceil \frac{m}{n-1} \right\rceil.$$

Acknowledgment

The author would like to thank the referees for their useful comments.

References

[1] N. Achuthan, N.R. Achuthan and L. Caccetta, On the vertex arboricity of graphs with prescribed size, *Australas. J. Combin.* **19** (1999), 17–36.

[2] G. Chartrand and L. Lesniak, "Graphs & Digraphs", 4^{th} Edition, Chapman & Hall / CRC, A CRC Press Company, 2005.

[3] St. J. A. Nash-Williams, Decomposition of finite graphs into forests, *J. London Math. Soc.* **39** (1964), 12.

[4] N. Punnim, Interpolation theorems in jump graphs, *Australas. J. Combin.* **39** (2007), 103–114.

(Received 2 May 2010; revised 18 Nov 2010)